Multiloop integrals in dimensional regularization made simple

based on arXiv:1304.1806

Johannes M. Henn
Institute for Advanced Study

supported in part by the Department of Energy grant DE-FG02-90ER40542 and the IAS AMIAS fund
Analytic computation of (Feynman) loop integrals

- what functions are needed?
- how do they depend on the kinematical variables?
- how do they depend on $D=4-2\,\epsilon$?
- how can we compute the integrals?
Integral functions

Experience shows: many processes described by iterated integrals

- simple case: logarithms, polylogarithms $\text{Li}$
- generalization: harmonic polylogarithms
- more general: Goncharov polylogarithms
- multiple masses $\rightarrow$ Elliptic functions
- ...

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Harmonic polylogarithms

- defined iteratively
  \[ H_1(x) = -\log(1 - x), \quad H_0(x) = \log(x), \quad H_{-1}(x) = \log(1 + x). \]
  \[ H_{a_1,a_2,...a_n}(x) = \int_0^x f_{a_1}(y)H_{a_2,...,a_n}(y)dy \]
  kernels: \[ f_1(y) = \frac{1}{1 - y}, \quad f_0(y) = \frac{1}{y}, \quad f_{-1}(y) = \frac{1}{1 + y} \]

- naturally arise in differential equations
- weight: number of integrations
  \text{``transcendentality''}
- more general integration kernels: Goncharov polylogarithms

[Remiddi, Vermaseren, 1999]
Integration by parts identities (IBP)

[Chetyrkin, Tkachov, 1981]  public computer codes [Anastasiou, Lazopoulos]
[Smirnov, Smirnov] [Studerus, von Manteuffel]

- IBP relates integrals with different indices

\[
\int d^{4-2\epsilon}k \frac{\partial}{\partial k^\mu} q^\mu \frac{1}{[k^2]^{a_1}[(k + p_1)^2]^{a_2}[(k + p_1 + p_2)^2]^{a_3}[(k - p_4)^2]^{a_4}} = 0
\]

- for a given topology, finite number of master integrals needed

- how to choose a ‘good’ integral basis?

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Differential equation technique

[Kotikov, 1991] [Gehrmann, Remiddi, 1999]
[Bern, Dixon, Kosower, 1993]

• differentiate master integrals w.r.t. momenta and masses
• use IBP to re-express RHS in terms of master integrals
• system of differential equations
  \[ \partial_i f(x_j, \epsilon) = A_i(x_j, \epsilon) f(x_j, \epsilon) \]
• change of integral basis:
  \[ f \rightarrow B f \]
  \[ A_j \rightarrow B^{-1} A_j B - B^{-1}(\partial_j B) \]
Pure functions of uniform weight

- uniform weight ("transcendentality") \( T \)
  \[ f_1(x) = \text{Li}_3(x) + \frac{1}{2} \log^3 x \quad T(f_1) = 3 \]
  \[ f_2(x, y) = \text{Li}_4(x/y) + 3 \log x \text{Li}_3(1 - y) \quad T(f_2) = 4 \]
- pure functions: derivative reduces weight
  \[ T(f) = n \quad \rightarrow \quad T(df) = n - 1 \]
  \( f_1, f_2 \) are pure functions of uniform weight
  \[ f_3 = \frac{1}{x} \log^2 x + \frac{1}{1 + x} \text{Li}_2(1 - x) \] has uniform weight 2, but is not pure functions with unique normalization
- dimensional regularization
  \[ x^\epsilon = 1 + \epsilon \log(x) + \ldots \] assign weight -1 to \( \epsilon \)
Optimal choice of integral basis

• idea: use transcendentality as guiding principle

• how to find such integrals?
  - unitarity cuts, leading singularities
  - ‘d-log’ representations
  - explicit parameter integrals

Conjecture

• all basis integrals can be chosen to be pure functions of uniform weight

• leads to simplified form of differential equations

\[ \partial_i f(x_j, \epsilon) = \epsilon A_i(x_j) f(x_j, \epsilon) \]

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Example: massless 2->2 scattering

[Smirnov, 1999][Gehrmann, Remiddi, 1999]

- good choice of master integrals

- Knizhnik-Zamolodchikov equations

\[
\partial_x f = \epsilon \left[ \frac{a}{x} + \frac{b}{1 + x} \right] f
\]

\[x = \frac{t}{s}\]

- singular points

\[s = 0, \quad t = 0, \quad u = -s - t = 0\]
Solution of differential equations

- equation(s) in differential form
  \[ df(\epsilon, x_n) = \epsilon \, d \tilde{A}(x_n) \, f(\epsilon, x_n) \]

- specifies class of functions (symbol alphabet can be read off)
  massless 2->2 case: harmonic polylogarithms

- solution in terms of iterated integrals
  \[ f = P \, e^\epsilon \int_C d \tilde{A} \, f(\epsilon = 0) \]
  here \( C \) is a contour in the kinematical space
  boundary conditions at base point from physical limits

- transcendentality properties manifest
  \[ f = f^{(0)} + \epsilon f^{(1)} + \ldots \]
  higher orders in \( \epsilon \) trivial to obtain
Generalizations

• all planar 2->2 three-loop master integrals

\[ \partial_x f = \epsilon \left[ \frac{a}{x} + \frac{b}{1 + x} \right] f \quad x = t/s \]

boundary conditions from \( x = -1 \) and Mellin-Barnes

• non-planar integrals, masses

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Conclusions

• criteria for finding optimal integral basis
• pure functions of uniform weight
• simplified diff. eqs. trivial to solve

Outlook

• further applications: masses, non-planar integrals, phase space integrals, ...
• multi-leg processes: study new classes of functions
• other dimensions