Multiloop integrals in dimensional regularization made simple

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Analytic computation of (Feynman) loop integrals

- what functions are needed?
- how do they depend on the kinematical variables?
- how do they depend on D=4 - 2 eps?
- how can we compute the integrals?

Integral functions

Experience shows: many processes described by iterated integrals

- simple case: logarithms, polylogarithms Li
- generalization: harmonic polylogarithms
- more general: Goncharov polylogarithms
- multiple masses -> Elliptic functions
- ...

Harmonic polylogarithms

[Remiddi, Vermaseren, 1999]

- defined iteratively

$$H_1(x) = -\log(1-x)\,, \qquad H_0(x) = \log(x)\,, \qquad H_{-1}(x) = \log(1+x)\,.$$

$$H_{a_1,a_2,\ldots a_n}(x) = \int_0^x f_{a_1}(y) H_{a_2,\ldots,a_n}(y) dy$$

kernels: $f_1(y) = \frac{1}{1-y}\,, \qquad f_0(y) = \frac{1}{y}\,, \qquad f_{-1}(y) = \frac{1}{1+y}$

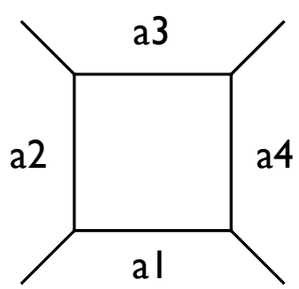
- naturally arise in differential equations
- weight: number of integrations
  ``transcendentality"
- more general integration kernels:
  Goncharov polylogarithms

Integration by parts identities (IBP)

[Chetyrkin, Tkachov, 1981]

public computer codes [Anastasiou, Lazopoulos] [Smirnov, Smirnov] [Studerus, von Manteuffel]

- IBP relates integrals with different indices



$$\int d^{4-2\epsilon}k \frac{\partial}{\partial k^\mu} q^\mu \frac{1}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}} = 0$$

- for a given topology, finite number of master integrals needed
- how to choose a 'good' integral basis?

Differential equation technique

[Kotikov, 1991] [Gehrmann, Remiddi, 1999]
[Bern, Dixon, Kosower, 1993]

- differentiate master integrals w.r.t. momenta and masses
- use IBP to re-express RHS in terms of master integrals
- system of differential equations

$$\partial_i f(x_j, \epsilon) = A_i(x_j, \epsilon) f(x_j, \epsilon)$$

- change of integral basis:

$$f \longrightarrow B\, f$$
$$A_j \longrightarrow B^{-1} A_j B - B^{-1}(\partial_j B)$$

Pure functions of uniform weight

- uniform weight (``transcendentality") $\mathcal{T}$

$$f_1(x) = \mathrm{Li}_3(x) + \frac{1}{2}\log^3 x \qquad \mathcal{T}(f_1) = 3$$

$$f_2(x,y) = \mathrm{Li}_4(x/y) + 3\log x\,\mathrm{Li}_3(1-y) \qquad \mathcal{T}(f_2) = 4$$

- pure functions: derivative reduces weight

$$\mathcal{T}(f) = n \quad \longrightarrow \quad \mathcal{T}(d\,f) = n-1$$

f_1, f_2 are pure functions of uniform weight

$f_3 = \frac{1}{x}\log^2 x + \frac{1}{1+x}\mathrm{Li}_2(1-x)$ has uniform weight 2, but is not pure

functions with unique normalization

- dimensional regularization

$x^\epsilon = 1 + \epsilon\log(x) + \ldots$ assign weight -1 to ϵ

Optimal choice of integral basis

- idea: use transcendentality as guiding principle
- how to find such integrals?
  - unitarity cuts, leading singularities
  - 'd-log' representations
  - explicit parameter integrals

[Cachazo]
[Arkani-Hamed et al.]

Conjecture

[J.M.H., 2013]

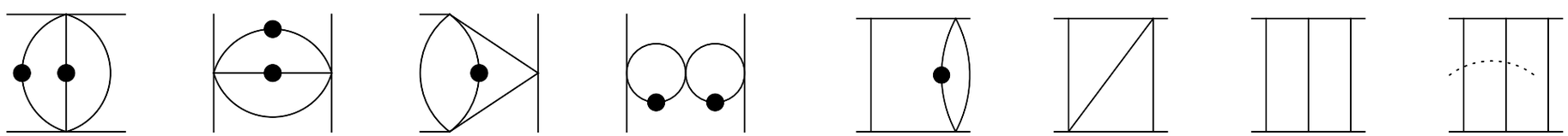
- all basis integrals can be chosen to be pure functions of uniform weight
- leads to simplified form of differential equations

$$\partial_i f(x_j, \epsilon) = \epsilon\, A_i(x_j) f(x_j, \epsilon)$$

Example: massless 2->2 scattering

[Smirnov, 1999][Gehrmann, Remiddi, 1999]

- good choice of master integrals [J.M.H., 2013]



- Knizhnik-Zamolodchikov equations

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f$$

$$x = t/s$$

$$a == \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix} \quad b == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}$$

- singular points

$$s = 0\,, \quad t = 0\,, \quad u = -s - t = 0$$

Solution of differential equations

- equation(s) in differential form

$$d\,f(\epsilon, x_n) = \epsilon\, d\,\tilde{A}(x_n) f(\epsilon, x_n)$$

- specifies class of functions (symbol alphabet can be read off)

  massless 2->2 case: harmonic polylogarithms

- solution in terms of iterated integrals

$$f = P\, e^{\epsilon \int_{\mathcal{C}} d\,\tilde{A}}\, f(\epsilon = 0)$$

  here $\mathcal{C}$ is a contour in the kinematical space

  boundary conditions at base point from physical limits

- transcendentality properties manifest

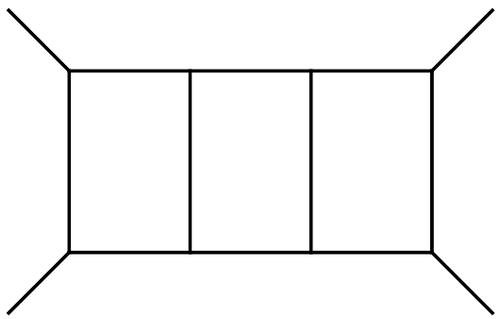
$$f = f^{(0)} + \epsilon f^{(1)} + \ldots$$

  higher orders in ϵ trivial to obtain

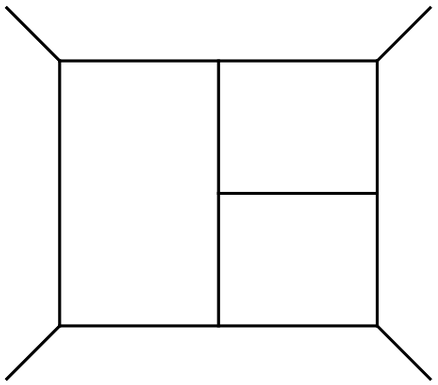
Generalizations

- all planar 2->2 three-loop master integrals

[J.M.H., A.V. Smirnov, V.A. Smirnov, to appear]



(26 master integrals)



(41 master integrals)

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \qquad x = t/s$$

boundary conditions from $x = -1$ and Mellin-Barnes

- non-planar integrals, masses

Conclusions

- criteria for finding optimal integral basis
- pure functions of uniform weight
- simplified diff. eqs. trivial to solve

Outlook

- further applications: masses, non-planar integrals, phase space integrals, ...
- multi-leg processes: study new classes of functions
- other dimensions