Four-Loop On-shell Integrals: $\overline{\text{MS}}$–onshell relation and $g-2$

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in collaboration with

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Outline

Introduction

- At four-loop level two classes of integrals have been studied extensively: massive tadpoles and massless propagators.
- Both classes have many phenomenological applications
- We are now ready for the treatment of a new class of integrals: on-shell integrals!
- This new class of integrals has many phenomenological applications, too!
- In the following: Calculation organized by number of massless fermion loops.
- Note: Only n_l^3 and n_l^2 part finished!

Application I: $\overline{\mathrm{MS}}$ – on-shell relation

- Fundamental relation between different renormalization schemes.
- Last missing renormalization constant at four loops in QCD.
- Improved precision needed e.g. for the measurement of the top quark mass (the PS mass) at a linear collider
  - Aim: top-mass measurement with $\Delta M_t \approx 100\,\mathrm{MeV}$
  - three-loop correction in the $\overline{\mathrm{MS}}$–on-shell relation $\approx 300\,\mathrm{MeV}$

Application II: lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

$$a_e|_{\text{exp}} = 0.00115965218073(28)$$
$$a_e|_{\text{theo}} = 0.00115965218178(6)(4)(3)(77)$$

Application II: lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

$$\begin{aligned} a_e|_{\text{exp}} &= 0.00115965218073(28) \\ a_e|_{\text{theo}} &= 0.00115965218178(6)(4)(3)(77) \end{aligned}$$

$$\begin{aligned} a_\mu|_{\text{exp}} &= 1.16592080(54)(33)[63] \cdot 10^{-3} \\ a_\mu|_{\text{theo}} &= 1.16591790(65) \cdot 10^{-3} \qquad 3.2\sigma \text{ diff.} \end{aligned}$$

- QED contributions known numerically up to 5 loops but starting from four loops not checked by an independent calculation

Review: Jegerlehner, Nyffeler, Phys.Rept. 477 (2009) 1-110

Technicalities

Why discussing $\overline{\mathrm{MS}}$ – on-shell relation and $g-2$ together?

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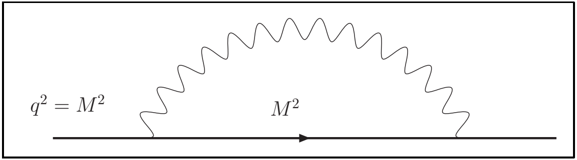
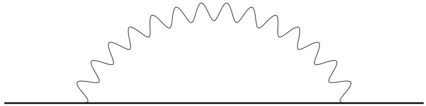
Both lead to the same type of topologies / integrals : on-shell integrals!



Order of Complexity

$Z_m : \Sigma(q^2, M^2)|_{q^2=M^2}$



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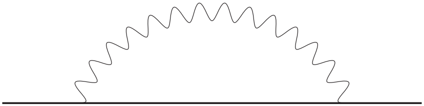
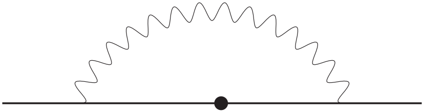


$Z_2 : \frac{\mathrm{d}}{\mathrm{d}q^2}\Sigma(q^2, M^2)|_{q^2=M^2}$



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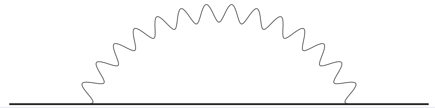

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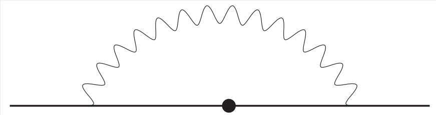
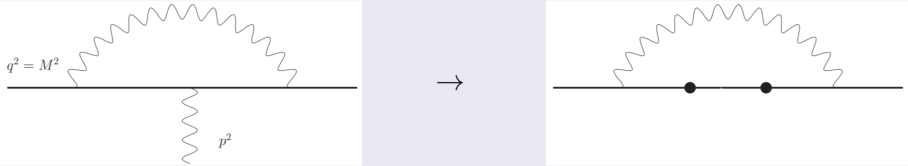


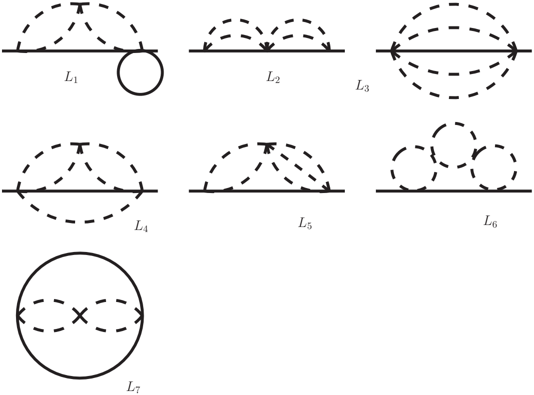
$g-2 : \frac{\mathrm{d}}{\mathrm{d}p^2}\Gamma(p^2, q^2 = M^2, M^2)|_{p^2=0}$



Setup

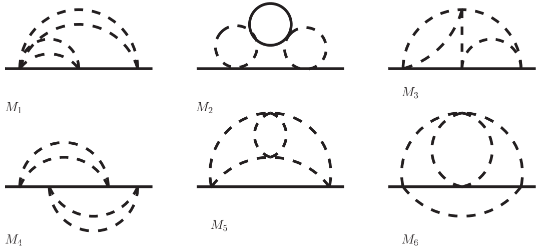
- common setup for both calculations
- tools used include
  - qgraf [Nogueira]
    Generation of Feynman diagrams
  - q2e,exp [Harlander, Seidensticker, Steinhauser]
    Expansion / Mapping to topologies
  - FORM [Vermaseren]
    Algebra
  - CRUSHER,FIRE [Seidel,PM / Smirnov,Smirnov]
    Reduction to master integrals
  - FIESTA [Smirnov,Smirnov]
    Calculation of master integrals
- two independent calculations

Master Integrals: simple



- Expressible through Gamma functions for arbitrary dimension *D*!

Master Integrals: difficult

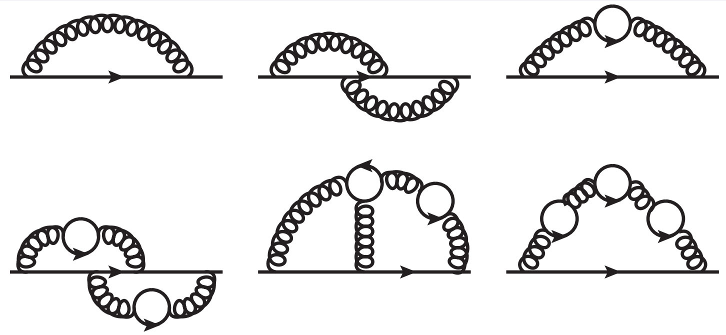


- Calculated analytically in expansion in $\epsilon = (4 - D)/2$ using the DRA (dimensional recurrence and analyticity) method and checked using `FIESTA`!
- Calculated up to $\mathcal{O}(\epsilon^3)$

Outline

$\overline{\text{MS}}$ – on-shell relation

Only n_l^3 and n_l^2 part



Previous works

The $\overline{\text{MS}}$ – on-shell relation has been studied extensively

- two loop [Broadhurst, Grafe, Gray, Schilcher 1990]
- three loop
  - numerical [Chetyrkin, Steinhauser 1999]
  - analytical [Melnikov, van Rittbergen 2000; PM, Mihaila, Piclum, Steinhauser]
- large β_0 approximation [Beneke, Braun]

Results: analytical

$$z_m^{\text{OS}} = \frac{\bar{m}(\mu)}{M} = 1 + \ldots + \left(\frac{\alpha_s(\mu)}{\pi}\right)^4 \delta z_m^{(4)} + \mathcal{O}(\alpha_s^5)$$

$$\delta z_m^{(4)} = \delta z_m^{(40)} + n_l \delta z_m^{(41)} + n_l^2 \delta z_m^{(42)} + n_l^3 \delta z_m^{(43)}$$

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$$\delta z_m^{(4)} = \delta z_m^{(40)} + n_l \delta z_m^{(41)} + n_l^2 \delta z_m^{(42)} + n_l^3 \delta z_m^{(43)}$$

$$\begin{aligned}
\delta z_m^{(43)} &= C_F T^3 \left(\frac{\ell_M^4}{144} + \frac{13\ell_M^3}{216} + \left(\frac{89}{432} + \frac{\pi^2}{36}\right)\ell_M^2 \right. \\
&\quad + \ell_M \left(\frac{\zeta_3}{3} + \frac{1301}{3888} + \frac{13\pi^2}{108}\right) \\
&\quad \left. + \frac{317\zeta_3}{432} + \frac{71\pi^4}{4320} + \frac{89\pi^2}{648} + \frac{42979}{186624} \right), \ell_M = \log\frac{\mu^2}{M^2} \\
\delta z_m^{(42)} &= \ldots
\end{aligned}$$

Results: Numerics

$$
\begin{aligned}
z_m^{\mathrm{OS}} &= 1 - A_s 1.333 + A_s^2 \left(-14.229 - 0.104\, n_h + 1.041\, n_l\right) \\
&+ A_s^3 \left(-197.816 - 0.827\, n_h - 0.064\, n_h^2 \right. \\
&\qquad \left. +26.946\, n_l - 0.022\, n_h n_l - 0.653\, n_l^2\right) \\
&+ A_s^4 \left(-43.465\, n_l^2 - 0.017\, n_h n_l^2 + 0.678\, n_l^3 + \ldots\right) + \mathcal{O}\left(A_s^5\right) ,
\end{aligned}
$$

with $n_l = 5$, $n_h = 1$:

$$
\begin{aligned}
z_m^{\mathrm{OS}} &= 1 - A_s 1.333 + A_s^2 \left(-14.332 + 5.207_{n_l}\right) \\
&+ A_s^3 \left(-198.707 + 134.619_{n_l} - 16.317_{n_l^2}\right) \\
&+ A_s^4 \left(-1087.060_{n_l^2} + 84.768_{n_l^3} + \ldots\right) + \mathcal{O}\left(A_s^5\right) .
\end{aligned}
$$

large β_0 approximation

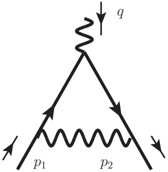
$$\left.\frac{M_q}{\bar{m}_q(\bar{m}_q)}\right|_{\text{large}-\beta_0} = 1 + a_s 1.333 + a_s^2 (17.186 - 1.041 n_l) + a_s^3 (177.695 - 21.539 n_l + 0.653 n_l^2) + a_s^4 (3046.294 - 553.872 n_l + 33.568 n_l^2 - 0.678 n_l^3)$$

[Beneke, Braun 1995]

$$\frac{M_q}{\bar{m}_q(\bar{m}_q)} = 1 + a_s 1.333 + a_s^2 (13.443 - 1.041 n_l) + a_s^3 (190.595 - 26.655 n_l + 0.653 n_l^2) + a_s^4 (c_0 + c_1 n_l + 43.396 n_l^2 - 0.678 n_l^3)$$

Outline

Definition: anomalous magnetic moment a_μ



$$= \quad (-ie)\overline{u}(p_2)\left\{\gamma^\mu F_E(q^2) + i\frac{\sigma^{\mu\nu}q^\nu}{2m}F_M(q^2)\right\}u(p_1)$$

$$a_\mu = F_M(0)$$

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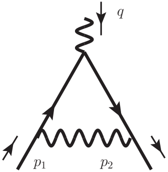

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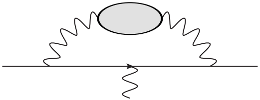
$$a_\mu = F_M(0)$$

Important!

- The calculation is performed with a massive muon and a massless electron
- Therefore only correct up to power correction
- Logarithmic contributions can be obtained using renormalization group arguments

Previous works

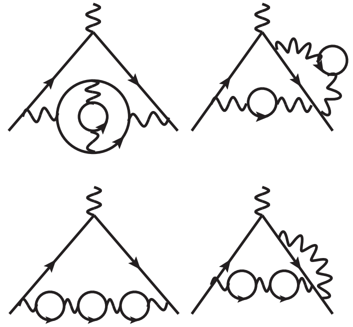
- analytical results
  - one loop: $a_\mu^{(1)} = \frac{1}{2}$ [Schwinger 1948]
  - two loop [Petermann; Sommerfeld 1957]
  - three loop [Laporta, Remiddi 1996]
  - four loops: only partial results, mainly contributions due to corrections to the vacuum polarization function of the photon



- numerical results
  - four loop [Kinoshita et al]
  - five loop [Aoyama,Hayakawa,Kinoshita,Nio 2012]

Diagrams

Only n_l^3 and n_l^2 part



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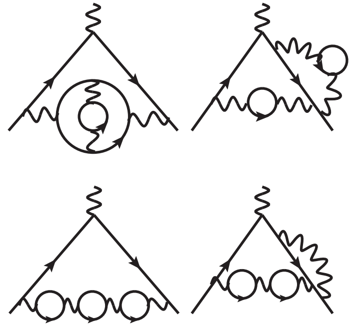
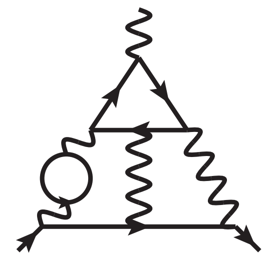


light-by-light



- calculated for massless electrons!
- light-by-light contribution not finite in this approximation!

Results

$$
\begin{aligned}
a_\mu^{(43)} &= \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324}+\frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\
&\approx 7.196\,66\,,
\end{aligned}
$$

[Laporta; Aguilar,Greynat,De Rafael]

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$$a_\mu^{(43)} = \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324} + \frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832}$$
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[Laporta; Aguilar,Greynat,De Rafael]

$$a_\mu^{(42)} = a_\mu^{(42)a} + n_h a_\mu^{(42)b}$$

$$a_\mu^{(42)a} = L_{\mu e}^2\left[\pi^2\left(\frac{5}{36} - \frac{\log 2}{6}\right) + \frac{\zeta_3}{4} - \frac{13}{24}\right] + \ldots \approx -3.624\,27\,,$$

$$\left.a_\mu^{(42)a}\right|_{\rm num} = -3.642\,04(1\,12)\,,$$

[Aoyama,Hayakawa,Kinoshita,Nio 2012]

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$$a_\mu^{(42)b} = \left(\frac{119}{108} - \frac{\pi^2}{9}\right)L_{\mu e}^2 + \left(\frac{\pi^2}{27} - \frac{61}{162}\right)L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944}$$
$$\approx 0.494\,05$$

[Laporta; Aguilar,Greynat,De Rafael]

Conclusions and Outlook

- First steps towards the calculation of $g-2$ and the $\overline{\mathrm{MS}}$–on-shell relation at four loops.
- Setup for the full calculation established
- Main obstacle: Calculation of $\mathcal{O}(600)$ master integrals
  - Numerical solution for needed master integrals
  - Analytical solution of needed master integrals still missing