Pushing Back the Infrared Frontier: The Two-Loop Infrared Structure of Mixed Gauge Groups

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Introduction

The infrared structure of gauge theory amplitudes is governed by a set of anomalous dimensions. The anomalous dimensions at a particular loop-level can be computed directly or extracted from a small number of relatively simple amplitude calculations. Once determined, these anomalous dimensions allow one to predict, for any amplitude, no matter how complex, the complete infrared structure to the given loop level.

Because of the many diagrams involved in multi-loop calculations and the complexity of the resulting amplitudes, foreknowledge of the infrared structure is extremely valuable. This knowledge was an important guide for the ground-breaking calculations of two-loop parton scattering.

Introduction

Precision measurements in particle physics often involve the interaction of more than one gauge group. In particular, at hadron colliders, nominally electroweak processes always involve some interaction with QCD. Precision calculations of such processes, therefore, require the computation of higher-order corrections in mixed gauge groups.

The origin of this project was the calculation of two-loop QCD $\times$ QED corrections to Drell-Yan production. Sturm and I needed to develop checks on our result. For that project, we determined the anomalous dimensions for quarks in QCD $\times$QED. I now return to the subject and present the results for the general case of mixed gauge groups.

Setup

I consider a theory with the following structure:

- 3 gauge symmetries: $SU(N) \times SU(M) \times U(1)$
- An arbitrary number of fermions in one of four representations:

	$SU(N)$	$SU(M)$	$U(1)$
F_l	1	1	Q_l
F_n	N	1	Q_n
F_m	1	M	Q_m
F_b	N	M	Q_b

Infrared Structure and Factorization

The infrared structure of gauge theory amplitudes is governed by anomalous dimensions and is therefore completely universal and can be predicted entirely in terms of the identities and momenta of the external states.

In the language of Sterman and Tejeda-Yeomans, an amplitude factorizes into three functions: the Jet function, the Soft function and the Hard Scattering Function.

$$\left|\mathcal{M}_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right)\right\rangle = \mathscr{J}_{\mathbf{f}}\left(\alpha_s(\mu^2), \varepsilon\right) \mathbf{S}_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) \left|H_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)\right\rangle .$$

The Jet Function

The jet function $\mathcal{J}_{\mathbf{f}}$ describes the collinear evolution of the amplitude's external legs and is found to be the product of individual jet functions $\mathcal{J}_{f_i}$, for each external leg, which are naturally defined in terms of their Sudakov form factors,

The jet function in QCD is:

$$\ln \mathcal{J}_i\left(\alpha_s(\mu^2),\varepsilon\right) = -\left(\frac{\alpha_s}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(1)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(1)}(\varepsilon)\right]$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\left\{\frac{\beta_{QCD}^{(2)}}{8}\frac{1}{\varepsilon^2}\left[\frac{3}{4\,\varepsilon}\gamma_{K\,i}^{(1)} + \mathcal{G}_i^{(1)}(\varepsilon)\right] - \frac{1}{8}\left[\frac{\gamma_{K\,i}^{(2)}}{4\,\varepsilon^2} + \frac{\mathcal{G}_i^{(2)}(\varepsilon)}{\varepsilon}\right]\right\} + \ldots$$

Because the infrared structure is universal, the cusp (γ_K) and $\mathcal{G}$ anomalous dimensions can be extracted from the direct calculation of QCD amplitudes.

The Soft Function

The soft function, $\mathbf{S_f}$, describes soft exchanges between the external legs. It is a matrix in color space because soft gluon exchanges can rearrange the color flow that took place in the hard scattering.

Like the jet function, the soft function can be defined in terms of eikonal amplitudes but can be extracted from Feynman diagram calculations of particular scattering amplitudes. It is determined entirely by the soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}_{S_f} \propto \frac{1}{2} \sum_{i\in\mathbf{f}} \sum_{j\neq i} \mathbf{T}_i \cdot \mathbf{T}_j \ln\left(\frac{\mu^2}{-s_{ij}}\right)$.

In QCD, the soft function is

$$\mathbf{S_f}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = 1 + \frac{1}{2\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)\mathbf{\Gamma}^{(1)}_{S_f} - \frac{\beta_0}{4\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(1)}_{S_f} + \frac{1}{8\,\varepsilon^2}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(1)}_{S_f}\times\mathbf{\Gamma}^{(1)}_{S_f} + \frac{1}{4\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(2)}_{S_f} + \ldots$$

The Infrared anomalous dimensions in QCD

$$\gamma_{K\,i}^{(1)} = 2\,C_i, \quad \gamma_{K\,i}^{(2)} = C_i\,K = C_i\left[C_A\left(\frac{67}{18} - \zeta_2\right) - \frac{10}{9}T_f\,N_f\right], \quad C_q \equiv C_F, \quad C_g \equiv C_A\,,$$

$$\mathbf{\Gamma}_{S_f}^{(1)} = \frac{1}{2}\sum_{i\in\mathbf{f}}\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j\,\ln\left(\frac{\mu^2}{-s_{ij}}\right), \qquad \mathbf{\Gamma}_{S_f}^{(2)} = \frac{K}{2}\mathbf{\Gamma}_{S_f}^{(1)}\,,$$

$$\mathcal{G}_q^{(1)} = \frac{3}{2}C_F + \frac{\varepsilon}{2}C_F\,(8-\zeta_2)\,, \qquad \mathcal{G}_g^{(1)} = 2\,\beta_0 - \frac{\varepsilon}{2}C_A\,\zeta_2\,,$$

$$\mathcal{G}_q^{(2)} = C_F^2\left(\frac{3}{16} - \frac{3}{2}\zeta_2 + 3\,\zeta_3\right) + C_F\,C_A\left(\frac{2545}{432} + \frac{11}{12}\zeta_2 - \frac{13}{4}\zeta_3\right) - C_F\,T_f\,N_f\left(\frac{209}{108} + \frac{1}{3}\zeta_2\right),$$

$$\mathcal{G}_g^{(2)} = 4\,\beta_1 + C_A^2\left(\frac{10}{27} - \frac{11}{12}\zeta_2 - \frac{1}{4}\zeta_3\right) + C_A\,T_f\,N_f\left(\frac{13}{27} + \frac{1}{3}\zeta_2\right) + \frac{1}{2}C_F\,T_f\,N_f\,,$$

$$\beta_0 = \frac{11}{12}C_A - \frac{1}{3}T_f\,N_f\,, \qquad \beta_1 = \frac{17}{24}C_A^2 - \frac{5}{12}C_A\,T_f\,N_f - \frac{1}{4}C_F\,T_f\,N_f$$

The Hard-Scattering Function

The hard-scattering function, $|H_{\mathbf{f}}\rangle$ describes the short-distance scattering process and is a vector in the color-space representation. As with any factorization, there is considerable freedom to move terms about from one function to the others. I adopt the convention of Sterman et al. that the logarithms of the jet and soft functions contain only infrared poles, while all infrared finite terms are absorbed into the hard scattering function.

$$\begin{aligned}
\left|\mathscr{M}^{(0)}\right\rangle &= \left|H^{(0)}\right\rangle \\
\left|\mathscr{M}^{(1)}\right\rangle &= \left[\left(\mathscr{J}^{(1)} + \mathbf{S}^{(1)}\right)\left|H^{(0)}\right\rangle + \left|H^{(1)}\right\rangle\right] \\
\left|\mathscr{M}^{(2)}\right\rangle &= \left[\left(\mathscr{J}^{(2)} + \mathscr{J}^{(1)}\mathbf{S}^{(1)} + \mathbf{S}^{(2)}\right)\left|H^{(0)}\right\rangle \right. \\
&\qquad \left. + \left(\mathscr{J}^{(1)} + \mathbf{S}^{(1)}\right)\left|H^{(1)}\right\rangle + \left|H^{(2)}\right\rangle\right]
\end{aligned}$$

The Infrared Structure of mixed gauge groups

When one includes additional gauge symmetries, the dominant effect on the infrared structure is to replicate the structure found in QCD, with appropriate changes accounting for the size of the gauge group and the Abelian character of the $U(1)$. There are, however, new terms that correspond to intrinsically mixed gauge interactions.

It is these mixed terms I am interested in computing. Some of the anomalous dimensions for QCD $\times$QED amplitudes have previously been determined (WK, Sturm), while the forms of others, particularly those involving external gauge bosons, were merely conjectured. The calculations I will describe explicitly determine all of the anomalous dimensions at two loops.

The Jet Function for $SU(N) \times SU(M) \times U(1)$

$$
\begin{aligned}
\ln \mathcal{J}_i = & -\left(\frac{\alpha_N}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(100)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(100)}(\varepsilon)\right] + \left(\frac{\alpha_N}{\pi}\right)^2\left\{\ldots\right\} \\
& -\left(\frac{\alpha_M}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(010)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(010)}(\varepsilon)\right] + \left(\frac{\alpha_M}{\pi}\right)^2\left\{\ldots\right\} \\
& -\left(\frac{\alpha_U}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(001)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(001)}(\varepsilon)\right] + \left(\frac{\alpha_U}{\pi}\right)^2\left\{\ldots\right\} \\
& -\left(\frac{\alpha_N}{\pi}\right)\left(\frac{\alpha_M}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(110)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(110)}(\varepsilon)\right] \\
& -\left(\frac{\alpha_N}{\pi}\right)\left(\frac{\alpha_U}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(101)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(101)}(\varepsilon)\right] \\
& -\left(\frac{\alpha_M}{\pi}\right)\left(\frac{\alpha_U}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(011)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(011)}(\varepsilon)\right] \\
& + \ldots .
\end{aligned}
$$

The Soft Function for $SU(N) \times SU(M) \times U(1)$

$$
\begin{aligned}
\mathbf{S_f} = 1 &+ \left(\frac{\alpha_N}{\pi}\right)\frac{1}{2\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(100)} + \left(\frac{\alpha_N}{\pi}\right)^2\left(\frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(200)} + \ldots\right) \\
&+ \left(\frac{\alpha_M}{\pi}\right)\frac{1}{2\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(010)} + \left(\frac{\alpha_M}{\pi}\right)^2\left(\frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(020)} + \ldots\right) \\
&+ \left(\frac{\alpha_U}{\pi}\right)\frac{1}{2\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(001)} + \left(\frac{\alpha_U}{\pi}\right)^2\left(\frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(002)} + \ldots\right) \\
&+ \left(\frac{\alpha_N}{\pi}\right)\left(\frac{\alpha_M}{\pi}\right)\left(\frac{1}{4\,\varepsilon^2}\mathbf{\Gamma}_{S_f}^{(100)} \times \mathbf{\Gamma}_{S_f}^{(010)} + \frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(110)}\right) \\
&+ \left(\frac{\alpha_N}{\pi}\right)\left(\frac{\alpha_U}{\pi}\right)\left(\frac{1}{4\,\varepsilon^2}\mathbf{\Gamma}_{S_f}^{(100)} \times \mathbf{\Gamma}_{S_f}^{(001)} + \frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(101)}\right) \\
&+ \left(\frac{\alpha_M}{\pi}\right)\left(\frac{\alpha_U}{\pi}\right)\left(\frac{1}{4\,\varepsilon^2}\mathbf{\Gamma}_{S_f}^{(010)} \times \mathbf{\Gamma}_{S_f}^{(001)} + \frac{1}{4\,\varepsilon}\mathbf{\Gamma}_{S_f}^{(011)}\right)
\end{aligned}
$$

Extracting the Anomalous Dimensions

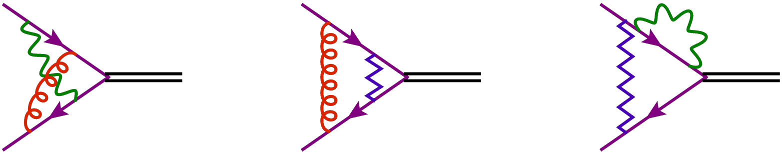
Sudakov-type processes, like Drell-Yan, are the cleanest way to extract the infrared anomalous dimensions. I extract the anomalous dimensions associated with fermions by computing the two-loop corrections to the production of a massive neutral vector boson, $\bar{f}_x f_x \longrightarrow X$.

For bosons, the natural Sudakov-type process would be like Higgs production. But the scalar must either carry the quantum numbers of the vector boson or it must couple to the vectors through an effective interaction. So, either the charged scalar complicates the extraction of the vector boson anomalous dimensions or one must determine the renormalization properties and Wilson coefficients of the effective operators.

Instead, I first determine the fermion anomalous dimensions and then extract the gauge boson anomalous dimensions from calculations of the $\bar{f}_x f_x \longrightarrow V_1 V_2$ amplitudes.

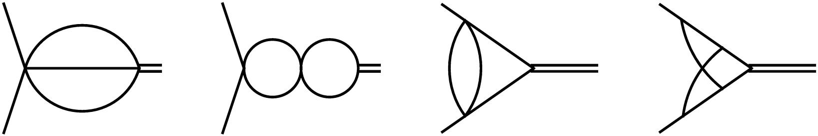
Extracting the Fermion Anomalous Dimensions

I extract the fermion anomalous dimensions from $\bar{f}_x f_x \longrightarrow X$ amplitudes.



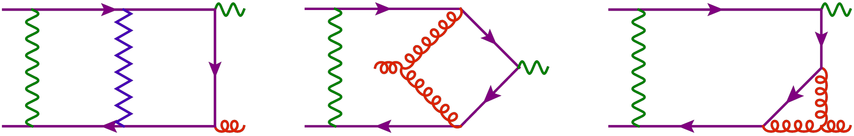
I generate the Feynman diagrams with QGRAF, implement the Feynman rules with FORM and perform IBP reduction to master integrals with REDUZE2.

Such Drell-Yan type calculations are relatively simple and can be expressed in terms of four master integrals

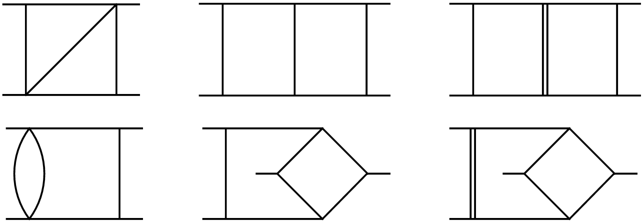


Extracting the Boson Anomalous Dimensions

Once I have the fermion anomalous dimensions, I extract the boson anomalous dimensions from $\bar{f}_x f_x \longrightarrow V_1 V_2$ amplitudes.



These are somewhat more complicated than the Drell-Yan type calculation and require 6 new master integrals in addition to the 4 already shown.



Results agree with expectations

I have now determined all of the two-loop anomalous dimensions through explicit calculation. There are no surprises and results agree with the simplest expectations.

- There are no non-Abelian contributions to mixed anomalous dimensions at two-loops.
- Explicit factors of N_f should be absorbed into β-function coefficients.
- Mixing enters gauge theory β-functions at second order, so leading β-function coefficients do not contribute to mixed anomalous dimensions.

Results for the Cusp and Soft Anomalous Dimensions

The two-loop unmixed cusp and soft anomalous dimensions are either non-Abelian in origin or depend on first-order β-function coefficients. It is no surprise that the mixed cusp and soft anomalous dimensions vanish.

$$\gamma_{K\,i}^{(200)} = \frac{K^{(200)}}{2}\gamma_{K\,i}^{(100)} \qquad \Gamma_{S_f}^{(200)} = \frac{K^{(200)}}{2}\Gamma_{S_f}^{(100)}$$

$$K^{(200)} = C_{A_N}\left(\frac{2}{3} - \zeta_2\right) + \frac{10}{3}\beta_{200}^{N}$$

$$\gamma_{K\,x}^{(110)} = \gamma_{K\,x}^{(101)} = \gamma_{K\,x}^{(011)} = 0\,, \qquad \Gamma_{S_f}^{(200)} = \Gamma_{S_f}^{(200)} = \Gamma_{S_f}^{(200)} = 0$$

Results for the $\mathscr{G}$ Anomalous Dimensions

The two-loop unmixed $\mathscr{G}$ anomalous dimensions receive both Abelian and non-Abelian contributions as well as contributions from both first- and second-order β-function coefficients. One would expect the mixed terms to receive both Abelian and second-order β-function contributions, and indeed, these are the only contributions to the mixed $\mathscr{G}$ anomalous dimensions.

$$\mathscr{G}_{f_n}^{(200)} = \mathscr{G}_{f_b}^{(200)} = C_{F_N}^2 \left(\frac{3}{16} - \frac{3}{2}\zeta_2 + 3\zeta_3 \right) + C_{F_N}\,\beta_{200}^N \left(\frac{209}{36} + \zeta_2 \right) + C_{F_N}\,C_{A_N} \left(\frac{41}{72} - \frac{13}{4}\zeta_3 \right),$$

$$\mathscr{G}_{A_N}^{(200)} = 2\,\beta_{300}^N + C_{A_N}\,\beta_{200}^N \left(\frac{19}{18} - \zeta_2 \right) + C_{A_N}^2 \left(\frac{177}{216} - \frac{1}{4}\zeta_3 \right),$$

$$\mathscr{G}_{f_b}^{(110)} = C_{F_N}\,C_{F_M} \left(\frac{3}{16} - \frac{3}{2}\zeta_2 + 3\zeta_3 \right), \quad \mathscr{G}_{A_N}^{(110)} = 2\,\beta_{210}^N,\; \mathscr{G}_{A_M}^{(110)} = 2\,\beta_{120}^M,$$

$$\mathscr{G}_{f^i_{\{b,n\}}}^{(101)} = C_{F_N}\,Q_{f^i_{\{b,n\}}}^2 \left(\frac{3}{16} - \frac{3}{2}\zeta_2 + 3\zeta_3 \right), \; \mathscr{G}_{A_N}^{(101)} = 2\,\beta_{201}^N,\; \mathscr{G}_{A_U}^{(101)} = 2\,\beta_{102}^U,$$

$$\mathscr{G}_{f^i_{\{b,m\}}}^{(011)} = C_{F_M}\,Q_{f^i_{\{b,m\}}}^2 \left(\frac{3}{16} - \frac{3}{2}\zeta_2 + 3\zeta_3 \right), \; \mathscr{G}_{A_M}^{(011)} = 2\,\beta_{021}^M,\; \mathscr{G}_{A_U}^{(011)} = 2\,\beta_{012}^U.$$

Application to QCD$\times$QED

In the Standard Model, there are only two unbroken gauge symmetries at low energy: QCD and QED. The results presented here can be applied directly to this case by identifying the $SU(N)$ symmetry with QCD, the $U(1)$ symmetry with QED, the F_n multiplets with the quarks and the F_l multiplets with the leptons and dropping everything involving the $SU(M)$ symmetry and the F_m and F_b multiplets.

My results for the quark and lepton anomalous dimensions agree with those found previously by WK and Sturm. While the mixed photon and gluon anomalous dimensions have now been honestly calculated for the first time, they too fully agree with prior expectations.

Summary

- I have computed the anomalous dimensions that govern the two-loop infrared structure of mixed gauge interactions.
- I find no mixed corrections to the cusp and soft anomalous dimensions at this order.
- The corrections to the $\mathscr{G}$ anomalous dimensions are in complete agreement with expectations.
- These results are directly applicable to calculations of two-loop $QCD \times QED$ amplitudes that will be useful for precision studies at LHC.