Numerical higher-order calculations.

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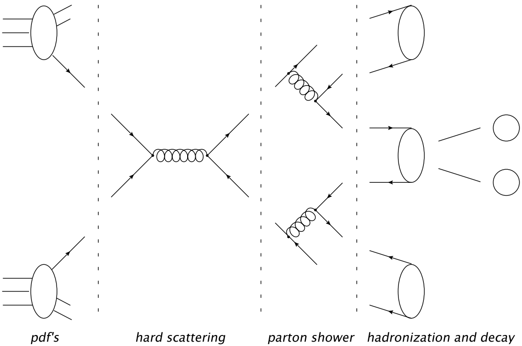
14. May 2013

Scattering processes at hadron-hadron colliders



- Cross-sections for hadron-hadron collisions:

$$\sigma = \sum_{a,b} \int_0^1 dx_a dx_b \int d\Phi_n f_a(x_a) f_b(x_b) \frac{1}{2\hat{s}} |\mathcal{A}_{n+2}|^2.$$

pQCD

- The hard scattering process is described by perturbation theory where the small parameter is the strong coupling constant.
- The perturbative expansion consists of virtual corrections, real emission processes and counter terms defined by your renormalisation scheme.
- Only the sum of the real and the virtual contributions leads to infrared finite results.
- We like to reduce theoretical uncertainties by calculating higher orders in the perturbative theory (more loops) and provide more realistic predictions by calculating processes with higher jet multiplicities (more legs).

pQCD

LO In 1990's significant progress in the multiplicity due Berends-Giele recursions.

NLO The NLO revolution starts with the establishment of the on-shell methods for the one-loop calculations. NLO Les Houches wishlist is fulfilled.
Processes with up to 6 particles in the final state are possible, $pp \to W + 5jets$[1].

NNLO Recent progress in $2 \to 2$ processes.
$pp \to t\bar{t}$ [2], $pp \to H + jet$ [3] and $gg \to dijet$[4];

[1] Z. Bern, L. J. Dixon, F. F. Cordero, S. Hoeche, H. Ita, D. A. Kosower, D. Maitre and K. J. Ozeren, arXiv:1304.1253 [hep-ph].

[2] M. Czakon, P. Fiedler and A. Mitov, arXiv:1303.6254 [hep-ph].

[3] R. Boughezal, F. Caola, K. Melnikov, F. Petriello and M. Schulze, arXiv:1302.6216 [hep-ph].

[4] A. G. -D. Ridder, T. Gehrmann, E. W. N. Glover and J. Pires, arXiv:1301.7310 [hep-ph].

pQCD

- Going to higher jet multiplicities in NNLO calculations are limited by the availability of the necessary two-loop amplitudes.
- One idea under investigation: Reduce the two-loop amplitudes to master integrals at amplitude level with on-shell methods.
- The master integrals, if not known analytically, are calculable with numerical methods, for example with SecDec[5].
- We investigate an alternative approach for calculating two-loop amplitudes based on local subtraction terms for the divergent two-loop amplitudes and contour deformation.

[5] S. Borowka and G. Heinrich, arXiv:1303.1157 [hep-ph].

Subtraction method for the virtual amplitude

- Instead of reducing the loop amplitudes to master integrals one performs the integration directly in the loop momentum space.
- The loop integrals are made finite by subtraction terms and a suitable contour deformation.
- The subtraction terms are known for a generic QCD one-loop amplitudes.
- The contour deformation is known to all loop levels (tested up to three loops).
- At NLO the method was used to calculate cross-sections in e^+e^- annihilation up to 7 jets[6] in the leading color approximation.

[6] S. Becker, D. Goetz, C. Reuschle, C. Schwan and S. Weinzierl, Phys. Rev. Lett. **108**, 032005 (2012) [arXiv:1111.1733 [hep-ph]].

The subtraction method at NLO

- At NLO we have to calculate the real emission, containing soft and collinear emission of one additional particle and the virtual correction containing one closed loop.
- Both pieces are divergent and can not simple combined in a single phase space integration.
- One successful approach is to introduce subtraction terms for the real piece and combine the integrated subtraction terms with the integrated one-loop amplitude.
- We go one step further and introduce additional subtraction terms for the virtual piece such that the one-loop integral can performed numerically together with the phase space integration.
- By doing this we introduce a third piece where we combine the integrated subtraction terms from the real and the virtual piece.

Full NLO calculation

- We can write the NLO contribution as a sum of three finite pieces.

$$\langle O\rangle^{NLO} = \langle O\rangle^{NLO}_{real} + \langle O\rangle^{NLO}_{virtual} + \langle O\rangle^{NLO}_{insertion}$$

- For the *real* part we have

$$\langle O\rangle^{NLO}_{real} = \int_{n+1} \left(O_{n+1} d\sigma^R - O_n d\sigma^A \right).$$

- For the *virtual* part we have

$$\langle O\rangle^{NLO}_{virtual} = 2 \int d\phi_n \, \mathrm{Re} \int \frac{d^4k}{(2\pi)^4} \left[\mathcal{A}^{(0)^*} \left(\mathcal{G}^{(1)}_{bare} - \mathcal{G}^{(1)}_{IR} - \mathcal{G}^{(1)}_{UV} \right) \right] O_n,$$

where $\mathcal{G}^{(1)}_{IR}$ and $\mathcal{G}^{(1)}_{UV}$ the local subtraction terms for the infrared and ultraviolet divergences of the *bare* one-loop amplitude.

- For the *insertion* part we have

$$\langle O\rangle^{NLO}_{insertion} = \int_n O_n \left[\int d\sigma^A + \mathcal{A}^{(1)}_{CT} + \int \mathcal{G}^{(1)}_{IR} + \int \mathcal{G}^{(1)}_{UV} \right]$$

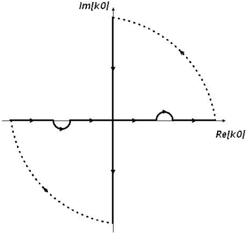
The deformation

- Even after subtraction of all physical divergencies, the virtual piece can not integrated in 4 real dimensions.
- The virtual amplitude has poles (on-shell propagators) at the real axis of the loop momentum.
- To avoid these poles, we are free to shift the integration contour into the complex plane, i.e. integrating in a four dimensional subspace of $\mathbb{C}^4$.
- The direction of the contour deformation is indicated by Feynman's $+\imath\epsilon$ rule.

Example: Tadpole

- The finite tadpole integral is not free of poles.

$$I_{Tad} = \int d^4k \frac{1}{[k^2 - m^2 + \imath\epsilon]^3}$$
$$k^0 = \pm\sqrt{\vec{k}^2 + m^2} \mp \imath\epsilon$$



- Feynman's $+\imath\epsilon$ tells us in which direction we have to avoid the poles.
- Wick rotation; replace the integration over the real axis by an integration over the imaginary axis for the time component of the loop momentum and make use of Cauchy's integral formula.
- By this transformation the poles in the denominator are vanish.

$$\{k^0, \vec{k}\} \rightarrow \{\imath k^0, \vec{k}\}; \qquad k^2 - m^2 \rightarrow -(k \circ k + m^2).$$
$$" \circ " = \text{euclidean scalar product.}$$

- In a numerical approach we deform all 4 components:

$$\{k^0, \vec{k}\} \rightarrow \{k^0, \vec{k}\} + \imath\{k^0, -\vec{k}\}, \qquad k^2 - m^2 \rightarrow \imath k \circ k - m^2.$$

Contour deformation: One-loop n-point function

- The pole structure of a primitive amplitude is given by a scalar n-point function.
- The different poles can overlap such that the contour is pinched.
- Such a pinch singularity is either an integratebale singularity or an infrared singularity, in this case we have our subtraction terms.
- By shifting the loop momentum into the complex plane **all** propagators ($D_i(k)$) become complex but only the imaginary part of the on-shell propagators have to be positive.

$$\begin{aligned} k &\rightarrow k + \imath\kappa(k) \\ D_i(k) &\rightarrow D_i(k + \imath\kappa) \\ D_i(k) = 0 &\rightarrow \mathrm{Im}\{D_i(k + \imath\kappa)\} > 0. \end{aligned}$$

- The numeric depends strongly on the choice of your deformation vector $\kappa(k)$.
- For the massless case we can use a known algorithm[7] for the contour deformation.
- This algorithm depends strongly on the structure of the kinematic of the external legs in the loop momentum space and is therefore not suitable for massive QCD and multi-loop calculations.

[7] W. Gong, Z. Nagy and D. E. Soper, Phys. Rev. D **79**, 033005 (2009) [arXiv:0812.3686 [hep-ph]].

Contour deformation: Massive n-point function

- To make use of the subtraction method in a broader range of processes, we have to deal with heavy particles in the loop.
- The new algorithm[8] is much more general, we can construct the deformation vector out of an arbitrary string of propagators.
- The propagators can massive and can depend on more then one variable momenta;

$$\begin{aligned} \{D_i\,(q(k_1,\ldots,k_n);m)\,|\,i\in\mathcal{I}\} &\quad\rightarrow\quad \kappa(k_1,\ldots,k_n) \\ q(\tilde{k}_1,\ldots,\tilde{k}_n) &\quad\rightarrow\quad q+\imath\kappa \\ D_i(q;m) \;=\; 0 &\quad\rightarrow\quad \mathrm{Im}\,\{D_i(q+\imath\kappa;m)\} \;>\; 0. \end{aligned}$$

- This opens the door to a contour deformation for multi-loop processes[9].

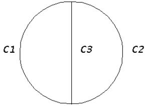
[8] S. Becker and S. Weinzierl, Phys. Rev. D **86**, 074009 (2012) [arXiv:1208.4088 [hep-ph]].
[9] S. Becker and S. Weinzierl, Eur. Phys. J. C **73**, 2321 (2013) [arXiv:1211.0509 [hep-ph]].

Motivation:
Find local subtraction terms for the divergencies of the two-loop amplitude and perform the loop integrals and the phase space integral in a single Monte Carlo integration.

Contour deformation: Two-loop n-point function

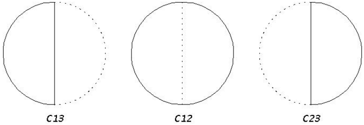
- Shift all loop momenta into the complex plane, but how does these shifts are constructed?
- We group our propagators into chains corresponding to the different linear combinations of the loop momenta in the loop diagram.

$$\begin{aligned} C_1 &= \{D_i(k_1)\}, \quad C_2 = \{D_i(k_2)\}, \\ C_3 &= \{D_i(k_3),\ k_3 = k_1 - k_2\} \end{aligned}$$



- We combine the different chains to closed cycles and apply the one-loop algorithm to these cycles.

$$\begin{aligned} C_1 \cup C_2 &\to \kappa_{12}, \\ C_1 \cup C_3 &\to \kappa_{13}, \\ C_2 \cup C_3 &\to \kappa_{23} \end{aligned}$$



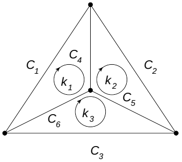
- The momentum shift for a loop momentum is the sum of all deformation vectors which depends on the corresponding chain:

$$\begin{aligned} \kappa_1 &= \kappa_{12} + \kappa_{13}, \quad \kappa_2 = \kappa_{12} + \kappa_{23}, \\ \kappa_3 &= \kappa_1 - \kappa_2 = \kappa_{13} - \kappa_{23}. \end{aligned}$$

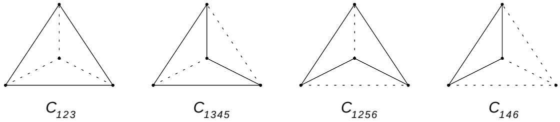
- A pinch singularity in one chain still allows to deform in the other chains.

Contour deformation: Three loop n-point function

- The construction of the integration contour is analogues to the two loop case.
- We have now six chains and three independent loop momenta.



- One again construct all possible cycles from the chains and apply our one-loop methods to them.
- Example: The decomposition into cycles which contains chain C_1.



$$
\begin{aligned}
\kappa_1 &= \kappa_{123} + \kappa_{1345} + \kappa_{1256} + \kappa_{146} \\
\kappa_2 &= \dots \\
\kappa_3 &= \dots
\end{aligned}
$$

Contour deformation: Summary

- A multi-loop process defines a set S of cycles.
- For each cycle we can calculate the corresponding deformation vector using one-loop methods.
- To get the right deformation for a single loop-momentum/chain, we take the subset S_r of S which contains all cycles which contain the chain C_r.
- The deformation of chain C_r is then given by a sum of deformation vectors κ_α, where each κ_α corresponds to one cycle in S_r.

$$\kappa_r = \sum_{\alpha \in S_r} \kappa_\alpha$$

- By doing this we always deform correctly even if the contour is pinched in some of the chains.

How far to deform?

- Assume that the κ_i deforms into the right directions, we have to ensure that the deformations are not to large such we cross other poles.
- Introduce a global scaling factor for the deformation vectors:

$$\kappa_i \;\to\; \lambda\kappa_i$$

- Choose λ as large as possible but small enough to not cross any poles in any propagator.
- We put the scaled deformation vector into our propagators, set the propagators to zero and solve the resulting quadratic equations in λ_j;

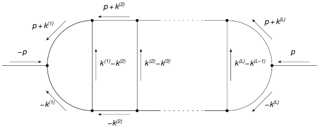
$$D_j(k_i + \imath\lambda_j\kappa_i) \;=\; 0 \quad\to\quad \lambda_j \;=\; \ldots$$

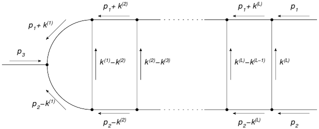
- If the solution is closed to the real axis, the λ_j is one half of the absolute value of this solution. Otherwise λ_j only have to be a smooth function in the loop momenta.
- The global λ is the minimum of all λ_j.

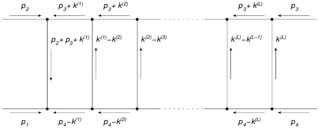
$$\lambda \quad=\quad \min[1, \lambda_j]$$

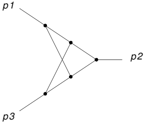
Tests of the algorithm

- The contour deformation for multi-loop processes was tested with two- and three-loop ladder diagrams where all external momenta are off-shell.



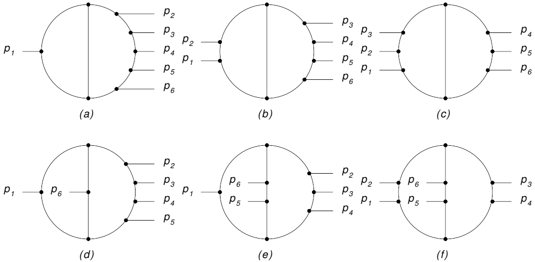






integral	numerical result	analytical result
$B^{(2)}[\mathrm{GeV}^{-2}]$	$(-3.59 \pm 0.05) \cdot 10^{-8}$	$-3.571 \cdot 10^{-8}$
$C^{(2)}[\mathrm{GeV}^{-4}]$	$(-1.80 \pm 0.05) \cdot 10^{-11}$	$-1.832 \cdot 10^{-11}$
$C^{(2)}_{\mathrm{np}}[\mathrm{GeV}^{-4}]$	$(-2.93 \pm 0.04) \cdot 10^{-11}$	$-2.904 \cdot 10^{-11}$
$D^{(2)}[\mathrm{GeV}^{-6}]$	$(-5.88 \pm 0.07) \cdot 10^{-14}$	$-5.897 \cdot 10^{-14}$
$B^{(3)}[\mathrm{GeV}^{-4}]$	$(-7.9 \pm 0.5) \cdot 10^{-14} i$	$-8.027 \cdot 10^{-14} i$
$C^{(3)}[\mathrm{GeV}^{-6}]$	$(-5.3 \pm 0.6) \cdot 10^{-17} i$	$-5.389 \cdot 10^{-17} i$
$D^{(3)}[\mathrm{GeV}^{-8}]$	$(-7.1 \pm 0.7) \cdot 10^{-19} i$	$-6.744 \cdot 10^{-19} i$

- We checked the algorithm at higher multiplicities by calculating the real part of various finite two-loop six-point functions.



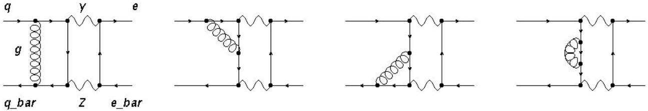
topology	our result
(a) $[\mathrm{GeV}^{-10}]$	$(-8.66 \pm 0.08) \cdot 10^{-19}$
(b) $[\mathrm{GeV}^{-10}]$	$(-1.17 \pm 0.02) \cdot 10^{-18}$
(c) $[\mathrm{GeV}^{-10}]$	$(-7.75 \pm 0.13) \cdot 10^{-19}$
(d) $[\mathrm{GeV}^{-10}]$	$(-1.91 \pm 0.02) \cdot 10^{-19}$
(e) $[\mathrm{GeV}^{-10}]$	$(-4.64 \pm 0.08) \cdot 10^{-19}$
(f) $[\mathrm{GeV}^{-10}]$	$(-1.03 \pm 0.02) \cdot 10^{-18}$

Summary

- The direct numerical integration of loop integrals is motivated by the subtraction method for the virtual piece.
- At NLO the subtraction method works well, $e^+e^- \to 7jets$.
- The two-loop amplitudes limit NNLO calculations.
- The subtraction method for the virtual part + contour deformation is probably a solution for high multiplicity NNLO calculations.
- A suitable contour deformation is available.
- The subtraction terms are work in progress.

Subtraction terms: First two-loop example

- We construct the subtraction terms for the mixed EW initial-final state corrections with QCD corrections.



- This amplitude is divergent for:
  - Soft, collinear and ultraviolet gluon momenta.
  - Soft and (collinear) photon.
- It is known that initial-final state corrections are free of collinear divergencies.
- But locally we have to deal with these pinch singularities with subtraction terms.
- After an analytic integration of the subtraction terms, these unphysical divergencies are vanish.

Testing the subtraction terms

- Compare the scaling behavior of the bare amplitude with the bare amplitude minus subtraction terms in the divergent limits.
- Soft limit:

$$k \rightarrow k/\lambda$$

- Collinear limit:

$$k \rightarrow \alpha p_i + p_\perp/\lambda$$

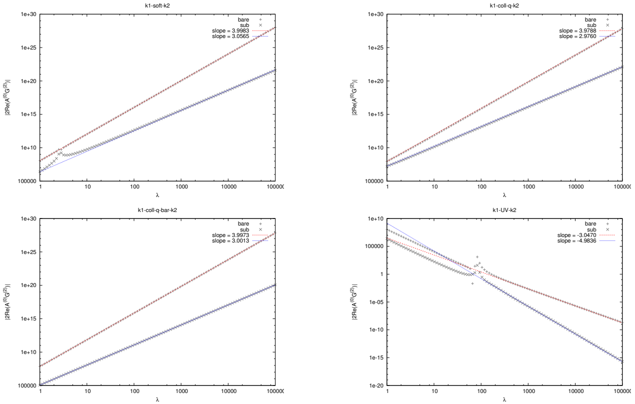
- Ultraviolet limit:

$$k \rightarrow \lambda k$$

- Expected results:

	$G^{(2)}_{bare}$	$G^{(2)}_{bare} - G^{(2)}_{sub}$
IR	λ^4	λ^3
UV	$1/\lambda^3$	$1/\lambda^5$
IRIR	λ^8	λ^7
IRUV	λ	$1/\lambda^2$

Divergent gluon



IR photon

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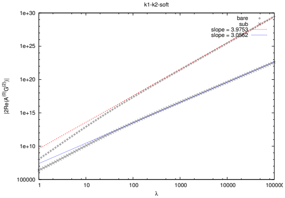
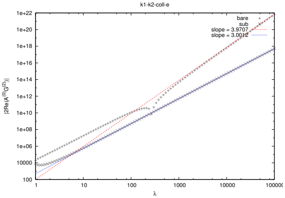
Introduction

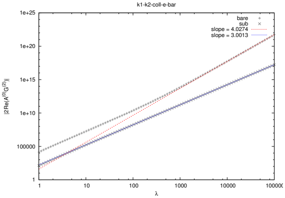
Subtraction method

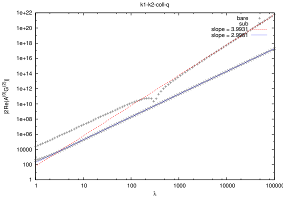
Contour deformation

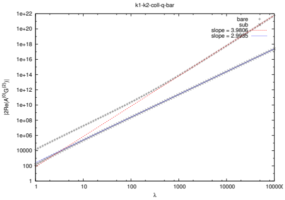
Summary



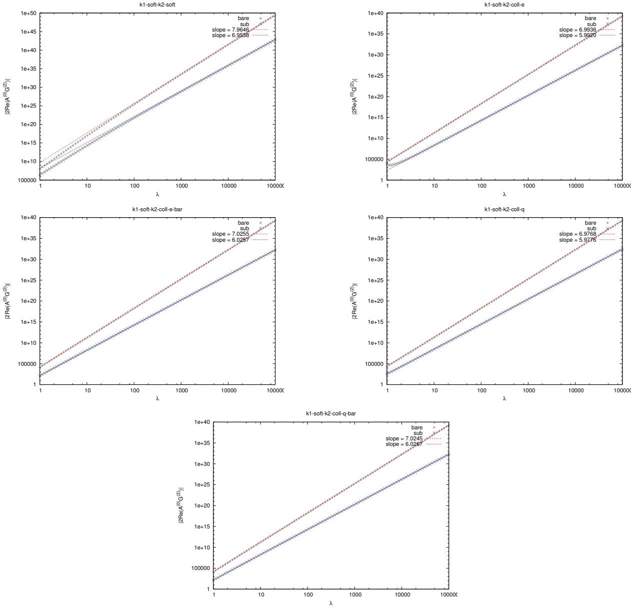




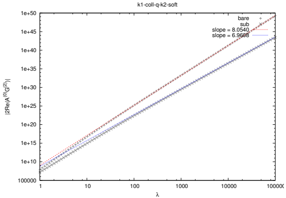


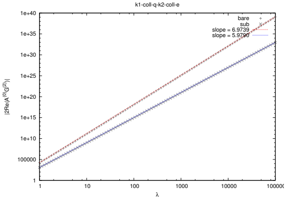


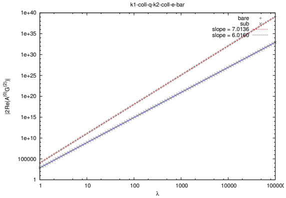
Soft gluon, IR photon

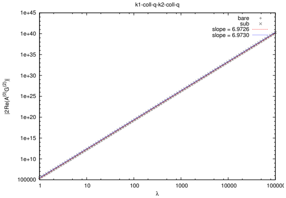


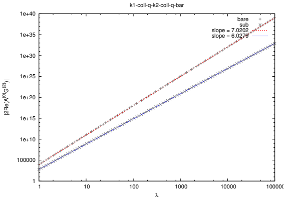
Collinear gluon (q), IR photon



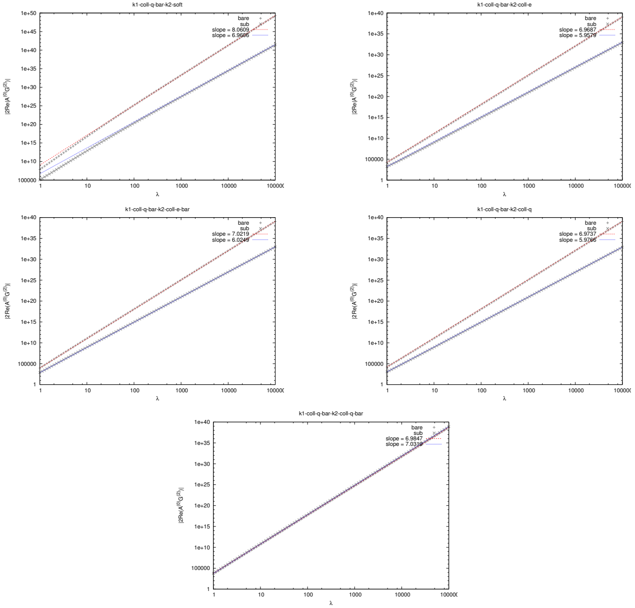




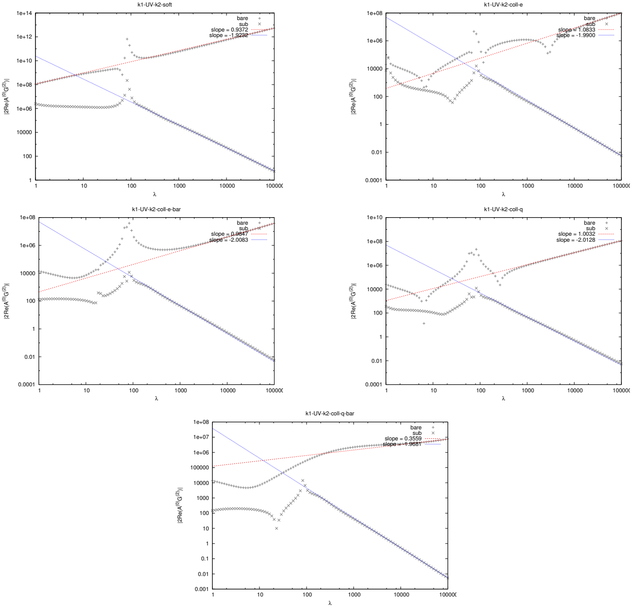




Collinear gluon ($\bar{q}$), IR photon



UV gluon, IR photon



Divergence		Scaling λ^x	
k_g	k_γ	$G^{(2)}_{bare}$	$G^{(2)}_{bare} - G^{(2)}_{sub}$
soft	fix	4	3
coll p_q	fix	4	3
coll $p_{\bar{q}}$	fix	4	3
UV	fix	-3	-5
fix	soft	4	3
fix	coll p_e	4	3
fix	coll $p_{\bar{e}}$	4	3
fix	coll p_q	4	3
fix	coll $p_{\bar{q}}$	4	3
soft	soft	8	7
soft	coll p_e	7	6
soft	coll $p_{\bar{e}}$	7	6
soft	coll p_q	7	6
soft	coll $p_{\bar{q}}$	7	6
coll p_q	soft	8	7
coll p_q	coll p_e	7	6
coll p_q	coll $p_{\bar{e}}$	7	6
coll p_q	coll p_q	7	7
coll p_q	coll $p_{\bar{q}}$	7	6
coll $p_{\bar{q}}$	soft	8	7
coll $p_{\bar{q}}$	coll p_e	7	6
coll $p_{\bar{q}}$	coll $p_{\bar{e}}$	7	6
coll $p_{\bar{q}}$	coll p_q	7	6
coll $p_{\bar{q}}$	coll $p_{\bar{q}}$	7	7
UV	soft	1	-2
UV	coll p_e	1	-2
UV	coll $p_{\bar{e}}$	1	-2
UV	coll p_q	1	-2
UV	coll $p_{\bar{q}}$	1/4	-2