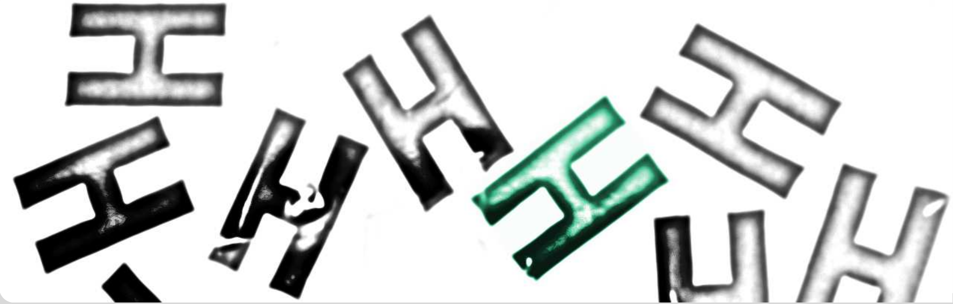
First steps to hadronic Higgs production at N^3LO:

NNLO partonic cross sections through order ϵ and convolutions with splitting functions

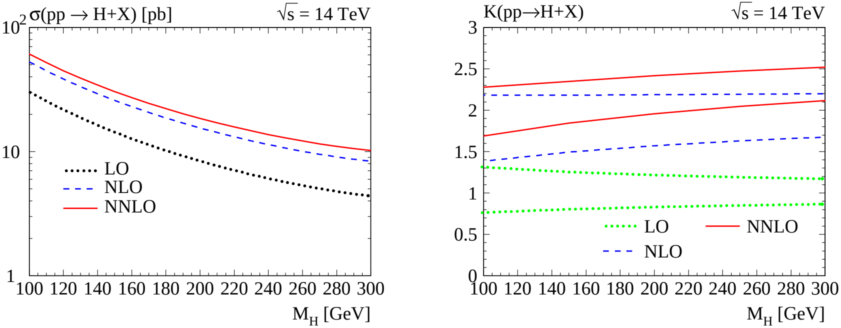
Maik Höschele, Jens Hoff, Alexey Pak, Matthias Steinhauser, and Takahiro Ueda | Tallahassee, 2013

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS, KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)



1. Motivation

- Higgs production via gluon fusion dominant process
- total cross section $\sigma(pp \to H + X)$, resp. K-factor



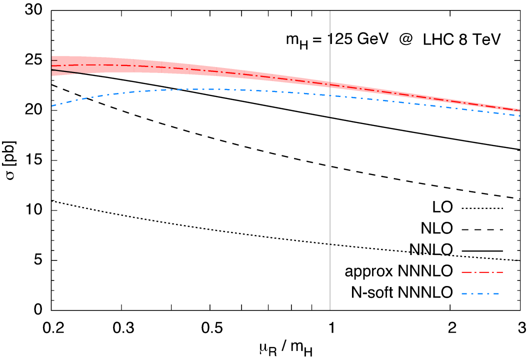
[Harlander, Kilgore; '02], [Anastasiou, Melnikov; '02], [Ravindran, Smith, van Neerven; '03]

- NNLO: corrections $\sim$100%, uncertainties $\sim$15% (PDFs + higher perturbative orders)

1. Motivation

- attempts to construct approximate N^{3}LO, resp. "beyond NNLO" results using resummations, high- and low-energy behaviour, ...

Higgs hadron-level cross section



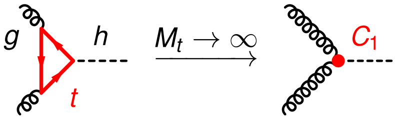
- [Moch, Vogt; '05] ("N-soft"), [Ball, Bonvini, Forte, Marzani, Ridolfi; '13] ("approx")

  $\Rightarrow$ show discrepancy

2. Introduction Effective Theory

- top-quark loop-induced process $\Rightarrow$ heavy-top effective theory

$$\mathcal{L}_{\mathrm{Y,eff.}} = -\frac{H^0}{v^0} C_1^0 \mathcal{O}_1^0 + \mathcal{L}_{\mathrm{QCD}}^{(5)}, \quad \text{with} \quad \mathcal{O}_1^0 = \frac{1}{4} G_{\mu\nu}^0 G^{0,\mu\nu}$$



- finite matching coefficient C_1

  e.g. [Chetyrkin, Kniehl, Steinhauser; '98], [Krämer, Laenen, Spira; '98] (3-loop QCD), or cf. next talk by Nikolai Zerf (3-loop SQCD)

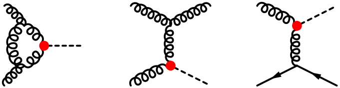
- small power corrections in $\frac{M_h}{M_t}$ (although $M_h \approx 125\,\mathrm{GeV}$, $M_t \approx 173\,\mathrm{GeV}$)

  e.g. [Marzani, Ball, Del Duca, Forte, Vicini; '08], [Pak, Rogal, Steinhauser; '09-'11], [Harlander, Ozeren; '09-'10]

$\Rightarrow$ **proceed to N^{3}LO**

2. Introduction Method

- higher order corrections: loops & final state particles

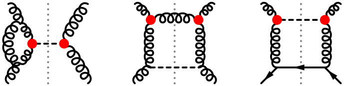
e.g. 

- different final states: cumbersome phase-space integration

Optical Theorem

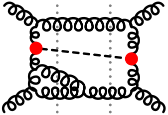
$$\int dPS\, |\mathcal{M}(gg \to h + X)|^2 \quad \longleftrightarrow \quad \mathrm{Im}\, \mathcal{M}(gg \to gg)$$

- $2 \to 2$ forward scattering (simplified kinematics)

e.g. 

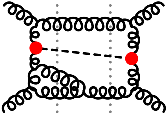
- common application of reduction techniques to real-virtual mixed parts to master integrals (MIs)
- approach due to [Anastasiou, Melnikov; '02]

3. Calculation Ingredients at N^3LO

e.g.  (3-, 4-particle cuts)

Ingredients at N^3LO (gg $\to$ gg to 3 loops):

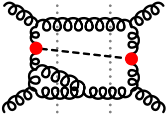
- ✓ UV-renormalization (only α_s^0 and $\mathcal{O}_1^0$) and C_1
- $\to$ finite lower order cross sections including higher orders in ϵ:
  - $\to$ reduction to MIs (contractions/traces, IBPs/Laporta)
  - ✓ MIs (NNLO: [Pak, Rogal, Steinhauser; '11], [Anastasiou, Bühler, Duhr, Herzog; '12])
  - $\to$ LO including $\mathcal{O}(\epsilon^3)$, NLO including $\mathcal{O}(\epsilon^2)$, NNLO including $\mathcal{O}(\epsilon)$
- $\to$ **mass factorization (initial-state radiation $\Rightarrow$ IR divergences)**
- ☐ 3 loop reduction and MIs
  - ...

e.g.  (3-, 4-particle cuts)

Ingredients at N^{3}LO (gg $\to$ gg to 3 loops):

- 3 loop reduction and MIs
  - channels: qq, qq', $q\bar{q}$, qg, gg
  - real, virtual, mixed contributions: v^3, v^2r, vr^2, r^3
    (v^3: [Baikov, Chetyrkin, Smirnov^2, Steinhauser; '09],
    [Gehrmann, Glover, Huber, Ikizlerli, Studerus; '10])
  - **soft expansion** (r^3) [Anastasiou, Duhr, Dulat, Mistlberger; '13]
    - expansion at diagram level in $y = 1 - M_h^2/s$ to $\mathcal{O}(y^2)$
    - compact reduction to 10 MIs (1 scale)
  - our approach
    - ...

3. Calculation Ingredients at N^{3}LO

e.g.  (3-, 4-particle cuts)

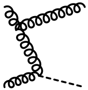
Ingredients at N^{3}LO (gg $\rightarrow$ gg to 3 loops):

- ☐ 3 loop reduction and MIs
  - our approach
    - **full** reduction (keep dependence $x = M_h^2/s$)
      $\Rightarrow$ complete set of MIs for full x-dependence
    - method of **differential equations**:
      $\rightarrow$ diff. in kinematic invariants (here: only x)
      $\rightarrow$ apply reduction tables $\rightarrow$ system of coupled DEQs
      $\rightarrow$ solve with **soft limit** as boundary condition

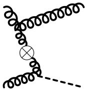
IR subtraction terms

convolution of UV-finite cross sections $\sigma_{ij}(x = M_h^2/s)$ with parton splitting functions $P_{ij}(x)$, each convolution coming along with an ϵ-pole

e.g. at NLO

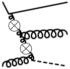


$\longrightarrow$

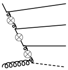


$\sim \frac{1}{\epsilon} P_{gg}^{(1)} \otimes \sigma_{gg}^{(0)}$

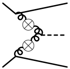
or at N^{3}LO



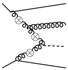
$\frac{1}{\epsilon^2} P_{gq}^{(1)} \otimes P_{gg}^{(1)} \otimes \sigma_{gg}^{(1)}$



$\frac{1}{\epsilon^3} P_{gq}^{(1)} \otimes P_{qg}^{(1)} \otimes P_{gq}^{(1)} \otimes \sigma_{gg}^{(0)}$



$\frac{1}{\epsilon^2} P_{gq}^{(2)} \otimes \sigma_{gg}^{(0)} \otimes P_{gq}^{(1)}$



$\frac{1}{\epsilon^3} P_{gq}^{(1)} \otimes P_{gg}^{(1)} \otimes \sigma_{gg}^{(0)} \otimes P_{gq}^{(1)}$

- ✓ ...
- ✓ 3-loop splitting functions in $\overline{\text{MS}}$, c.f. [Moch, Vermaseren, Vogt; '04]
- → **systematic** solution of integrals occurring in $\mathcal{O}(100)$ convolutions

4. Convolutions Definition

Convolution Integral

$$[f \otimes g](x) = \int_0^1 dx_1\, dx_2\, \delta(x - x_1 x_2)\, f(x_1)\, g(x_2)$$

- structure of $\sigma_{ij}^{(k)}(x)$, $P_{ij}^{(k)}(x)$:

$$\left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\right\} \times \left\{\delta(1-x), \left[\frac{\ln^k(1-x)}{1-x}\right]_+, H_{\vec{w}(4)}(\ldots)\right\}$$

- Mellin space: $M_n[f(x)] = \int_0^1 dx\, x^{n-1} f(x)$
- convolutions $\rightarrow$ **products**: $M_n[[f \otimes g](x)] = M_n[f(x)]\, M_n[g(x)]$
- index shift: $M_n\left[x^k f(x)\right] = M_{n+k}[f(x)]$
- integration-by-parts: $M_n\left[\frac{d}{dx} f(x)\right] = x^{n-1} f(x)\Big|_0^1 - (n-1) M_{n-1}[f(x)]$
- **problematic**: $\left[\frac{\ln^k(1-x)}{1-x}\right]_+$, $\frac{H_{\vec{w}}(\ldots)}{1+x}$, $\frac{H_{\vec{w}}(\ldots)}{1-x}$ (for $x \rightarrow 1$)

4. Convolutions (Harmonic Polylogarithms & Sums)

Harmonic PolyLogarithms (HPLs)

$$H_{w_1,w_2,\dots,w_n}(x) = \int_0^x dx' f_{w_1}(x') H_{w_2,\dots,w_n}(x') \quad \text{of weight } n$$

$$\text{with} \quad f_0(x) = \frac{1}{x},\ f_1(x) = \frac{1}{1-x},\ f_{-1}(x) = \frac{1}{1+x}$$

- weight 1: $H_0(x) = \ln(x)$, $H_1(x) = -\ln(1-x)$, $H_{-1}(x) = \ln(1+x)$
- weight 2: e.g. $H_{0,0}(x) = \frac{1}{2}\ln^2(x)$, $H_{0,1}(x) = \mathrm{Li}_2(x), \dots$
- from weight 4: larger class than Nielsen polylogarithms
  $\rightarrow$ cover many problems in multi-loop
- transformations between related arguments
- shuffle algrebra: express products as sums $\rightarrow$ minimal basis
  $H_{\vec{w}_1}(x) H_{\vec{w}_2}(x) = \sum_{\vec{w} \in \vec{w}_1 \uplus \vec{w}_2} H_{\vec{w}}(x)$

4. Convolutions (Harmonic Polylogarithms & Sums)

- transcendental numbers for $H_{\vec{w}}(1)$ (Riemann $\zeta_i, \ldots$)
- convergent series expansion $\rightarrow$ numeric evaluation
- Mellin transforms of HPLs: combinations of harmonic sums $S_{\vec{w}}(n)$

Harmonic Sums

$$S(n) = 1, \; S_{w_1,w_2,\ldots,w_n}(n) = \sum_{i=1}^{n} f_{w_1}(i) S_{w_2,\ldots,w_n}(i)$$

$$\text{with} \quad f_w(i) = \begin{cases} i^{-w}, & w \geq 0, \\ (-1)^i i^w, & w < 0 \end{cases}$$

- `FORM` package `harmpol.h` [Remiddi, Vermaseren; '00], `Mathematica` package `HPL.m` [Maitre; '06, '12]

Convolution Integral

$$[f \otimes g](x) = \int_0^1 dx_1 \, dx_2 \, \delta(x - x_1 x_2) \, f(x_1) \, g(x_2)$$

- structure of $\sigma_{ij}^{(k)}(x)$, $P_{ij}^{(k)}(x)$:

$$\left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right\} \times \left\{ \delta(1-x), \left[\frac{\ln^k(1-x)}{1-x} \right]_+, H_{\vec{w}(4)}(\ldots) \right\}$$

- Mellin space: $M_n[f(x)] = \int_0^1 dx \, x^{n-1} f(x)$
- convolutions $\to$ **products**: $M_n[[f \otimes g](x)] = M_n[f(x)] \, M_n[g(x)]$
- index shift: $M_n\left[x^k f(x)\right] = M_{n+k}[f(x)]$
- integration-by-parts: $M_n\left[\frac{d}{dx} f(x)\right] = x^{n-1} f(x)\Big|_0^1 - (n-1) M_{n-1}[f(x)]$
- **problematic**: $\left[\frac{\ln^k(1-x)}{1-x}\right]_+$, $\frac{H_{\vec{w}}(\ldots)}{1+x}$, $\frac{H_{\vec{w}}(\ldots)}{1-x}$ (for $x \to 1$)

Regularized Derivative

Karlsruhe Institute of Technology

$$\frac{d}{dx} H_1(x) = -\frac{d}{dx} \ln(1-x) = \frac{1}{1-x} \rightarrow \left[\frac{1}{1-x}\right]_+ : M_n \left[\left[\frac{1}{1-x}\right]_+\right] = -S_1(n-1)$$

Regularized Derivative (RD)

$$M_n \left[\hat{\partial}_x f(x)\right] = R[f(x)] - (n-1) M_{n-1} [f(x)] \quad \text{with}$$

$$R\left[g_k(x) \ln^k(1-x) + g_{k-1}(x) \ln^{k-1}(1-x) + \ldots + g_0(x)\right] = g_0(1)$$

and $g_j(1) \neq 0 \quad \forall j > 0$

- regularize singular behavior of HPLs $\sim \ln^k(1-x)$ in terms of generalized functions $(\delta, +)$
- $f(x)$ non-divergent $\Rightarrow \hat{\partial}_x = \frac{d}{dx}$

$\Rightarrow$ **common** treatment of functions related to derivatives of HPLs

e.g. $\frac{H_{\vec{w}}(x)}{1+x}, \left[\frac{H_{\vec{w}}(x)}{1-x}\right]_+, \ldots$

Karlsruhe Institute of Technology

RDs and their Mellin Images

$$\hat{\partial}_x 1 = \delta(1-x) \qquad M_n\left[\hat{\partial}_x 1\right] = 1$$

$$\hat{\partial}_x H_1(x) = \left[\frac{1}{1-x}\right]_+ \qquad M_n\left[\hat{\partial}_x H_1(x)\right] = -S_1(n-1)$$

$$\hat{\partial}_x H_0(x) = \frac{d}{dx} H_0(x) = \frac{1}{x} \qquad \longrightarrow \qquad M_n\left[\hat{\partial}_x H_0(x)\right] = \frac{1}{n-1}$$

$$\hat{\partial}_x H_{-1}(x) = \frac{d}{dx} H_{-1}(x) = \frac{1}{1+x} \qquad M_n\left[\hat{\partial}_x H_{-1}(x)\right] = (-1)^{n-1}\left(S_{-1}(n-1) + \ln(2)\right)$$

$$\hat{\partial}_x H_{\vec{w}}(x) = \ldots \qquad M_n\left[\hat{\partial}_x H_{\vec{w}}(x)\right] = \ldots$$

KIT Karlsruhe Institute of Technology

Algorithm

1. transform expressions to mellin space
2. tabulate Mellin transforms of HPLs to certain weight; here: 5
   $\Rightarrow$ harmonic sums and transcendental numbers
3. tabulate Mellin transforms of RDs of HPLs
4. solve system for

$$\left\{ \frac{1}{n^k}, \frac{S_{\ldots}(n)}{n^k}, (-1)^n \frac{S_{\ldots}(n)}{n^k} \right\}$$

$\Rightarrow$ inverse Mellin transforms (and relations)

- implementation in `Mathematica` package `MT.m` **(to be published soon)**
- using the `HPL.m` package [Maitre; '06, '12]

KIT Karlsruhe Institute of Technology

Mellin Transforms of HPLs

$$M_n[1] = \frac{1}{n},$$
$$M_n[H_0(x)] = -\frac{1}{n^2},$$
$$M_n[H_1(x)] = \frac{S_1(n)}{n},$$
$$M_n[H_{-1}(x)] = -\frac{(-1)^n}{n}\left(S_{-1}(n) + \ln(2)\right) + \frac{\ln(2)}{n},$$
$$M_n[H_{\vec{w}}(x)] = \ldots$$

Inverse Mellin Transforms

$$M_x^{-1}\left[\frac{1}{n}\right] = 1,$$
$$M_x^{-1}[1] = \hat{\partial}_x 1,$$
$$M_x^{-1}\left[\frac{1}{n^2}\right] = -H_0(x),$$
$$M_x^{-1}[S_1(n)] = -x\hat{\partial}_x H_1(x),$$
$$M_x^{-1}\left[\frac{S_1(n)}{n}\right] = H_1(x),$$
$$M_x^{-1}[\ldots] = \ldots$$

- tables generated using `harmpol.h` [Remiddi, Vermaseren; '00]
- Mellin images of structures at NNLO also studied in [Blümlein, Kurth; '99], [Ablinger, Blümlein, Schneider; '11]

5. Conclusion & Outlook

Karlsruhe Institute of Technology

Conclusion

- NNLO partonic cross sections to $\mathcal{O}(\epsilon)$
- algorithm for the computation of convolution integrals involving $+$-distributions and HPLs
- systematic evaluation of all convolutions with splitting functions entering a N^{3}LO calculation

$\Rightarrow$ [arXiv:1211.6559 (hep-ph)], [M. Höschele et al., Physics Letters B (2013)]

Outlook

- techniques to be applied to the Drell-Yan process
- publication of `MT.m` [in preparation]