False vacuum as an unstable state: possible cosmological implications

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Contents

1. Introduction

The problem of false vacuum decay became famous after the publication of pioneer papers by Coleman and his colleagues,

[1] S. Coleman, Phys. Rev. D 15, 2929 (1977),
[2] C.G. Callan and S. Coleman, Phys. Rev. D 16, 1762 (1977),
[3] S. Coleman and F. de Lucia, Phys. Rev. D 21, 3305 (1980).

The instability of a physical system in a state which is not an absolute minimum of its energy density, and which is separated from the minimum by an effective potential barrier was discussed there. It was shown, in those papers, that even if the state of the early Universe is too cold to activate a "*thermal*" transition (via thermal fluctuations) to the lowest energy (i.e. "*true vacuum*") state, a quantum decay from the false vacuum to the true vacuum may still be possible through a barrier penetration via macroscopic quantum tunneling.

Not long ago, the decay of the false vacuum state in a cosmological context has attracted interest, especially in view of its possible relevance in the process of tunneling among the many vacuum states of the string landscape (a set of vacua in the low energy approximation of string theory). In many models the scalar field potential driving inflation has a multiple, low–energy minima or *"false vacuua"*. Then the absolute minimum of the energy density is the *"true vacuum"*.

Recently the problem of the instability the false vacuum state triggered much discussion in the context of the discovery of the Higgs–like resonance at 125 — 126 GeV (see, eg.,[4] — [7]).

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[4] A. Kobakhidze, A. Spencer–Smith, Phys. Lett. **B 722**, 130, (2013).

[5] G. Degrassi, *et al.*,JHEP 1208 (2012) 098.

[6] J. Elias–Miro, *et al.*, Phys. Lett. **B 709**, 222, (2012).

[7] Wei Chao, *et al.*, Phys. Rev. **D 86**, 113017, (2012).

In the recent analysis [5] assuming the validity of the Standard Model up to Planckian energies it was shown that a Higgs mass $m_h < 126$ GeV implies that the electroweak vacuum is a metastable state. This means that a discussion of Higgs vacuum stability must be considered in a cosmological framework, especially when analyzing inflationary processes or the process of tunneling among the many vacuum states of the string landscape.

Krauss nad Dent analyzing a false vacuum decay

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[8] L. M. Krauss, J. Dent, Phys. Rev. Lett., **100**, 171301 (2008); see also: S. Winitzki, Phys. Rev. **D 77**, 063508 (2008),

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pointed out that in eternal inflation, even though regions of false vacua by assumption should decay exponentially, gravitational effects force space in a region that has not decayed yet to grow exponentially fast.

This effect causes that many false vacuum regions can survive up to the times much later than times when the exponential decay law holds. In the mentioned paper by Krauss and Dent the attention was focused on the possible behavior of the unstable false vacuum at very late times, where deviations from the exponential decay law become to be dominat.

The aim of this talk is to analyze properties of the false vacuum state as an unstable state, the form of the decay law from the canonical decay times t up to asymptotically late times and to discuss the late time behavior of the energy of the false vacuum states.

2. Unstable states in short

If $|M\rangle$ is an initial unstable state then the survival probability, $\mathcal{P}(t)$, equals

$$\mathcal{P}(t) = |a(t)|^2,$$

where $a(t)$ is the survival amplitude,

$$a(t) = \langle M|M;t\rangle, \quad \text{and} \quad a(0) = 1,$$

and

$$|M;t\rangle = e^{-itH}\,|M\rangle,$$

H is the total Hamiltonian of the system under considerations.

The spectrum, $\sigma(H)$, of H is assumed to be bounded from below, $\sigma(H) = [E_{min}, \infty)$ and $E_{min} > -\infty$.

From basic principles of quantum theory it is known that the amplitude $a(t)$, and thus the decay law $\mathcal{P}(t)$ of the unstable state $|M\rangle$, are completely determined by the density of the energy distribution function $\omega(\mathcal{E})$ for the system in this state

$$a(t) = \int_{Spec.(H)} \omega(E)\, e^{-\,i\,E\,t}\, dE. \tag{1}$$

where

$$\omega(E) \geq 0 \;\; \text{for} \;\; E \geq E_{min} \;\; \text{and} \quad \omega(E) = 0 \quad \text{for} \quad E < E_{min}.$$

From this last condition and from the Paley–Wiener Theorem it follows that there must be (see [9])

[9] L. A. Khalfin, Zh. Eksp. Teor. Fiz. **33**, 1371 (1957)[Sov. Phys. JETP **6**, 1053 (1958)].

$$|a(t)| \;\geq\; A\, e^{-b\, t^q},$$

for $|t| \rightarrow \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$.

This means that the decay law $\mathcal{P}(t)$ of unstable states decaying in the vacuum can not be described by an exponential function of time t if time t is suitably long, $t \to \infty$, and that for these lengths of time $\mathcal{P}(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of t.

The analysis of the models of the decay processes shows that

$$\mathcal{P}(t) \simeq e^{-\varGamma_M t},$$

(where Γ_M is the decay rate of the state $|M\rangle$), to an very high accuracy at the canonical decay times t: From t suitably later than the initial instant t_0 up to

$$t \gg \tau_M = \frac{1}{\varGamma_M}$$

(τ_M is a lifetime) and smaller than $t = T$, where T is the crossover time and denotes the time t for which the non–exponential deviations of $a(t)$ begin to dominate.

In general, in the case of quasi–stationary (metastable) states it is convenient to express $a(t)$ in the following form

$$a(t) = a_{exp}(t) + a_{non}(t), \tag{2}$$

where $a_{exp}(t)$ is the exponential part of $a(t)$, that is

$$a_{exp}(t) = N\, e^{-it(E_M - \frac{i}{2}\,\Gamma_M)}, \tag{3}$$

(E_M is the energy of the system in the state $|M\rangle$ measured at the canonical decay times, N is the normalization constant), and $a_{non}(t)$ is the non–exponential part of $a(t)$.

For times $t \sim \tau_M$:

$$|a_{exp}(t)| \gg |a_{non}(t)|,$$

The crossover time T can be found by solving the following equation,

$$|a_{exp}(t)|^2 = |a_{non}(t)|^2. \tag{4}$$

The amplitude $a_{non}(t)$ exhibits inverse power–law behavior at the late time region: $t \gg T$. Indeed, the integral representation (1) of $a(t)$ means that $a(t)$ is the Fourier transform of the energy distribution function $\omega(E)$. Using this fact we can find asymptotic form of $a(t)$ for $t \to \infty$. Results are rigorous (see [10]).

[10] K. Urbanowski, *Eur. Phys. J. D*, **54**, (2009).

So, let us assume that $\lim_{E \to E_{min}+} \omega(E) \stackrel{\text{def}}{=} \omega_0 > 0$. Let derivatives $\omega^{(k)}(E)$, $(k = 0, 1, 2, \ldots, n)$, be continuous in $[E_{min}, \infty)$, (that is let for $E > E_{min}$ all $\omega^{(k)}(E)$ be continuous and all the limits $\lim_{E \to E_{min}+} \omega^{(k)}(E)$ exist) and let all these $\omega^{(k)}(E)$ be absolutely integrable functions then [10]

$$a(t) \underset{t \to \infty}{\sim} -\frac{i}{t}\, e^{-\,i\, E_{min} t} \sum_{k=0}^{n-1} (-1)^k \left(\frac{i}{t}\right)^k \omega_0^{(k)} = a_{non}(t), \tag{5}$$

where $\omega_0^{(k)} \stackrel{\text{def}}{=} \lim_{E \to E_{min}+} \omega^{(k)}(E)$.

Let us consider now a more complicated form of the density $\omega(E)$. Namely let $\omega(E)$ be of the form

$$\omega(E) = (E - E_{min})^{\lambda}\,\eta(E) \;\in\; L_1(-\infty, \infty), \tag{6}$$

where $0 < \lambda < 1$ and it is assumed that $\eta(E_{min}) > 0$ and $\eta^{(k)}(E)$, $(k = 0, 1, \ldots, n)$, exist and they are continuous in $[E_{min}, \infty)$, and limits $\lim_{E \to E_{min}+}\ \eta^{(k)}(E)$ exist, $\lim_{E\to\infty}\,(E - E_{min})^{\lambda}\,\eta^{(k)}(E) = 0$ for all above mentioned k, then

$$a(t) \underset{t\to\infty}{\sim} (-1)\,e^{-iE_{min}t}\,\Big[\Big(-\frac{i}{t}\Big)^{\lambda+1}\,\Gamma(\lambda+1)\,\eta_0 \tag{7}$$
$$+\,\lambda\,\Big(-\frac{i}{t}\Big)^{\lambda+2}\,\Gamma(\lambda+2)\,\eta_0^{(1)} \;+\; \ldots\Big] = a_{non}(t)$$

From (5), (7) it is seen that asymptotically late time behavior of the survival amplitude $a(t)$ depends rather weakly on a specific form of the energy density $\omega(E)$. The same concerns a decay curves $\mathcal{P}(t) = |a(t)|^2$. A typical form of a decay curve, that is the dependence on time t of $\mathcal{P}(t)$ when t varies from $t = t_0 = 0$ up to $t > 20\tau_M$ is presented in Fig. (1).

The decay curve, which one can observe in the case of the so–called broad resonances (when $(E_M^0 - E_{min})/\mathit{\Gamma}_M^0 \sim 1$), is presented in Fig (2).

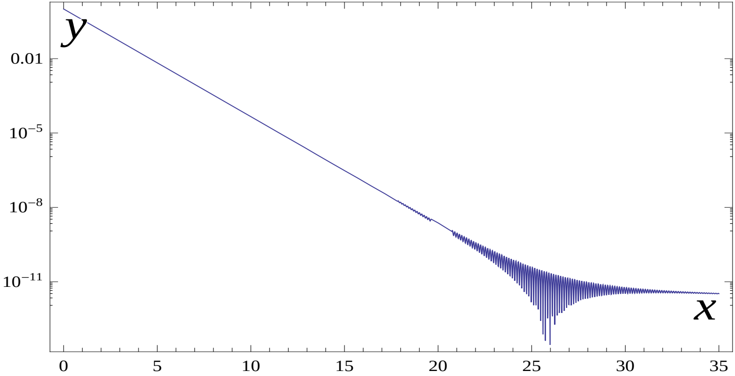

Figure: 1. Axes: $y = \mathcal{P}(t)$ — the logarithmic scale, $x = t/\tau_M$. $\mathcal{P}(t)$ is the survival probability. The time t is measured as a multiple of the lifetime τ_M. The case $(E_M^0 - E_{min})/\Gamma_M^0 = 50$

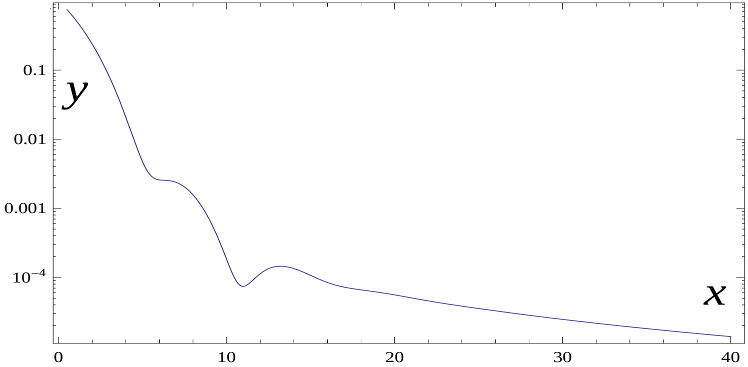

Figure: 2. Axes: $y = \mathcal{P}(t)$ — the logarithmic scale, $x = t/\tau_M$. $\mathcal{P}(t)$ is the survival probability. The time t is measured as a multiple of the lifetime τ_M. The case $(E_M^0 - E_{min})/\Gamma_M^0 = 1$.

Results presented in Figs (1), (2) were obtained for the Breit–Wigner energy distribution function,

$$\omega(E) \equiv \frac{N}{2\pi}\, \Theta(E - E_{min})\, \frac{\Gamma_M^0}{(E - E_M^0)^2 + (\Gamma_M^0/2)^2}, \tag{8}$$

where $\Theta(E)$ is the unit step function.

The crossover time T for this model:

$$\begin{aligned} \Gamma_M^0\, T \;\; &\simeq \;\; 8,28 + 4\,\ln\big(\frac{E_M^0 - E_{min}}{\Gamma_M^0}\big) \\ &\qquad + 2\,\ln\big[8,28 + 4\,\ln\big(\frac{E_M^0 - E_{min}}{\Gamma_M^0}\big)\big] + \ldots \end{aligned} \tag{9}$$

where $(E_M^0 - E_{min}/\Gamma_M^0) > 10$.

3. Instantaneous energy and instantaneous decay rate

The amplitude $a(t)$ contains information about the decay law $\mathcal{P}(t)$ of the state $|M\rangle$, that is about the decay rate $\mathit{\Gamma}_M^0$ of this state, as well as the energy E_M^0 of the system in this state. This information can be extracted from $a(t)$. Indeed if $|M\rangle$ is an unstable (a quasi–stationary) state then

$$a(t) \cong e^{-i(E_M^0 - \frac{i}{2}\mathit{\Gamma}_M^0)\,t}, \quad (t \sim \tau_M). \tag{10}$$

So, there is

$$E_M^0 - \frac{i}{2}\mathit{\Gamma}_M^0 \equiv i\,\frac{\partial a(t)}{\partial t}\,\frac{1}{a(t)}, \tag{11}$$

in the case of quasi–stationary states.

The standard interpretation and understanding of the quantum theory and the related construction of our measuring devices are such that detecting the energy E_M^0 and decay rate $\mathit{\Gamma}_M^0$ one is sure that the amplitude $a(t)$ has the form (10) and thus that the relation (11) occurs.

Taking the above into account one can define the "effective Hamiltonian", h_M, for the one–dimensional sub-space of states $\mathcal{H}_{||}$ spanned by the normalized vector $|M\rangle$ as follows

$$h_M \stackrel{\text{def}}{=} i\,\frac{\partial a(t)}{\partial t}\,\frac{1}{a(t)}. \tag{12}$$

In general, h_M can depend on time t, $h_M \equiv h_M(t)$. One meets this effective Hamiltonian when one starts with the Schrödinger Equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for the distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$. The equivalent expression for $h_M \equiv h_M(t)$ has the following form [10]

$$h_M(t) \equiv \frac{\langle M|H|M;t\rangle}{\langle M|M;t\rangle} \stackrel{\text{def}}{=} \mathcal{E}_M(t) - \frac{i}{2}\,\gamma_M(t). \tag{13}$$

Details can be found in [10] and in

[11] K. Urbanowski, Cent. Eur. J. Phys. **7**, (2009), (see also references one can find therein).

Thus, one finds the following expressions for the energy and the decay rate of the system in the state $|M\rangle$ under considerations, to be more precise for the instantaneous energy $\mathcal{E}_M(t)$ and the instantaneous decay rate, $\gamma_M(t)$,

$$\mathcal{E}_M \equiv \mathcal{E}_M(t) = \Re\,(h_M(t)), \tag{14}$$

$$\gamma_M \equiv \gamma_M(t) = -\,2\,\Im\,(h_M(t)), \tag{15}$$

where $\Re\,(z)$ and $\Im\,(z)$ denote the real and imaginary parts of z respectively.

Using (12) and (21), (22) one can find that

$$\mathcal{E}_M(0) = \langle M|H|M\rangle, \tag{16}$$

$$\mathcal{E}_M(t \sim \tau_M) \simeq E_M^0 \neq \mathcal{E}_M(0), \tag{17}$$

$$\gamma_M(0) = 0, \tag{18}$$

$$\gamma_M(t \sim \tau_M) \simeq \Gamma_M^0. \tag{19}$$

So, there is $\mathcal{E}_M(t) = E_M^0$ at the canonical decay time.

Starting from the asymptotic expressions (5) and (7) for $a(t)$ and using (12) after some algebra one finds for times $t \gg T$ that

$$h_M(t)\|_{t\to\infty} \simeq E_{min} + (-\frac{i}{t})\, c_1 \, + \, (-\frac{i}{t})^2\, c_2 \, + \, \ldots, \tag{20}$$

where $c_i = c_i^*, \;\; i = 1, 2, \ldots;$ (coefficients c_i depend on $\omega(E)$).
This last relation means that

$$\mathcal{E}_M(t) \;\; \simeq \;\; E_{min} - \frac{c_2}{t^2} \ldots, \quad (\text{for} \;\; t \gg T), \tag{21}$$

$$\gamma_M(t) \;\; \simeq \;\; 2\,\frac{c_1}{t} + \ldots, \quad (\text{for} \;\; t \gg T), \tag{22}$$

These properties take place for all unstable states which survived up to times $t \gg T$.

From (21) it follows that $\lim_{t\to\infty} \mathcal{E}_M(t) = E_{min}$.

For the most general form (6) of the density $\omega(E)$ (i. e. for $a(t)$ having the asymptotic form given by (7)) we have

$$c_1 = \lambda + 1, \quad c_2 = (\lambda + 1)\,\frac{\eta^{(1)}(E_{min})}{\eta(E_{min})}. \tag{23}$$

The energy densities $\omega(E)$ considered in quantum mechanics and in quantum field theory can be described by $\omega(E)$ of the form (6), eg. quantum field theory models correspond with $\lambda = \frac{1}{2}$.
The average energy measured at some time interval (t_1, t_2) (with $t_1, t_2 \gg T$) equals

$$\overline{\mathcal{E}_M(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \mathcal{E}_M(t)\, dt \simeq E_{min} - \frac{c_2}{t_1\, t_2} + \ldots, \tag{24}$$

A general form of $(\mathcal{E}_M(t) - E_{min})/(E_M^0 - E_{min})$ as a function of time t varying from $t = t_0 = 0$ up to $t > T$ is presented in Figs (3), (4). These results were obtained for the model considered in the previous Section and correspond with Figs (1), (2).

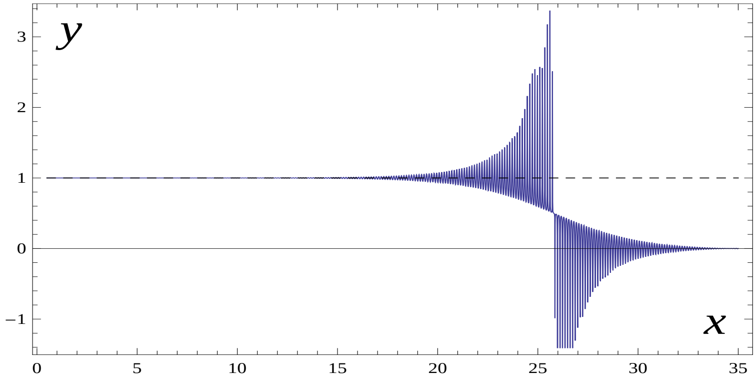

Figure: 3. Axes: $y = (\mathcal{E}_M(t) - E_{min})/(E_M^0 - E_{min})$, $x = t/\tau_M$. The difference of energies $(\mathcal{E}_M(t) - E_{min})$ is measured as a multiple of the difference $(E_M^0 - E_{min})$. The case $(E_M^0 - E_{min})/\Gamma_M^0 = 50$.

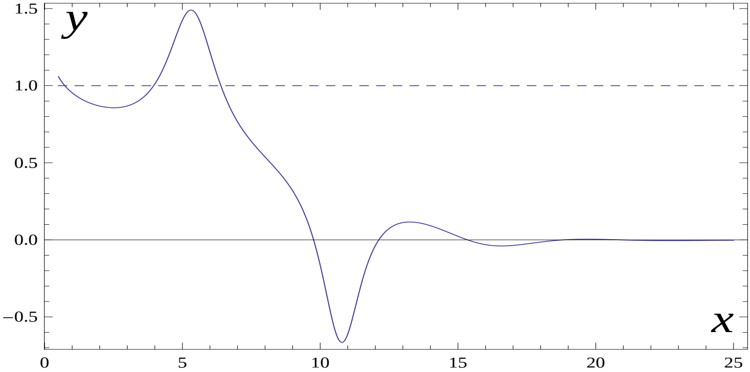

Figure: 4. Axes: $y = (\mathcal{E}_M(t) - E_{min})/(E_M^0 - E_{min})$, $x = t/\tau_M$. The difference of energies $(\mathcal{E}_M(t) - E_{min})$ is measured as a multiple of the difference $(E_M^0 - E_{min})$. The case $(E_M^0 - E_{min})/\Gamma_M^0 = 1$.

4. Cosmological applications

Krauss and Dent in their paper [8] mentioned earlier made a hypothesis that some false vacuum regions do survive well up to the time T or later. Let $|M\rangle = |0\rangle^{false}$, be a false, $|0\rangle^{true}$ – a true, vacuum states and E_0^{false} be the energy of a state corresponding to the false vacuum measured at the canonical decay time and E_0^{true} be the energy of true vacuum (i.e. the true ground state of the system). As it is seen from the results presented in previous Section, the problem is that the energy of those false vacuum regions which survived up to T and much later differs from E_0^{false},

[12] K. Urbanowski, *Phys. Rev. Lett.*, **107**, 209001 (2011), (see also references one can find therein).

Now, if one assumes that $E_0^{true} \equiv E_{min}$ and $E_0^{false} = E_M^0$ and takes into account results of the previous Section (including those in Figs (3), (4)) then one can conclude that the energy of the system in of false vacuum state has the following general properties:

$$\mathcal{E}_0^{false}(t) = E_0^{true} + \Delta E \; \cdot \; \Phi(t), \tag{25}$$

where $\Delta E = E_0^{false} - E_0^{true}$ and $\Phi(t) = \frac{\mathcal{E}_0^{false}(t) - E_0^{true}}{\Delta E} \simeq 1$ for $t \sim \tau_0^{false} < T$. $\Phi(t)$ is a fluctuating function of t at $t \sim T$ (see Figs (3), (4)) and $\Phi(t) \propto \frac{1}{t^2}$ for $t \gg T$.
At asymptotically late times $t \gg T$ one finds that

$$\mathcal{E}_0^{false}(t) \simeq E_0^{true} - \frac{c_2}{t^2} \ldots \; \neq \; E_0^{false}. \tag{26}$$

Similarly

$$\gamma_0^{false}(t) \simeq +2\,\frac{c_1}{t} \ldots \quad (\text{for} \;\; t \gg T). \tag{27}$$

Two last properties of the false vacuum states mean that

$$\mathcal{E}_0^{false}(t) \to E_0^{true} \;\; \text{and} \;\; \gamma_0^{false}(t) \to 0 \;\; \text{as} \;\; t \to \infty. \qquad (28)$$

The basic physical factor forcing the wave function $|M; t\rangle$ and thus the amplitude $a(t)$ to exhibit inverse power law behavior at $t \gg T$ is a boundedness from below of $\sigma(H)$. This means that if this condition takes place and

$$\int_{-\infty}^{+\infty} \omega(E)\, dE \; < \; \infty, \qquad (29)$$

then all properties of $a(t)$, including a form of the time–dependence at $t \gg T$, are the mathematical consequence of them both. The same applies by (12) to properties of $h_M(t)$ and concerns the asymptotic form of $h_M(t)$ and thus of $\mathcal{E}_M(t)$ and $\gamma_M(t)$ at $t \gg T$.

Note that properties of $a(t)$ and $h_M(t)$ discussed above do not take place when $\sigma(H) = (-\infty, +\infty)$.

Going from quantum mechanics to quantum field theory one should take into account among others a volume factors so that survival probabilities per unit volume per unit time should be considered. The standard false vacuum decay calculations shows that the same volume factors should appear in both early and late time decay rate estimations (see Krauss and Dent [8]). This means that the calculations of cross–over time T can be applied to survival probabilities per unit volume. For the same reasons within the quantum field theory the quantity $\mathcal{E}_M(t)$ can be replaced by the energy per unit volume $\rho_M(t) = \mathcal{E}_M(t)/V$ because these volume factors V appear in the numerator and denominator of the formula (10) for $h_M(t)$.

- The general properties of the energy of the system in the unstable false vacuum state as a function of time t,

$$\mathcal{E}_0^{false}(t) = E_0^{true} + \Delta E \ \cdot \ \Phi(t), \tag{30}$$

where $\Delta E = E_0^{false} - E_0^{true}$ and $\Phi(t) \simeq 1$ for $t \sim \tau_0^{false} < T$. $\Phi(t)$ is a fluctuating function of t at $t \sim T$ and $\Phi(t) \propto \frac{1}{t^2}$ for $t \gg T$.

- or, of the energy density $\rho_0^{false}(t)$:

$$\rho_0^{false}(t) = \rho_0^{true} + D \ \cdot \ F(t), \tag{31}$$

(where $D = D^*$, $\rho_0^{true} \equiv \rho_0^{bare}$, $F(t) \simeq 1$ for $t \sim \tau_0^{false}$, $F(t)$ is fluctuating at $t \sim T$ and $F(t) \sim 1/t^2$ at $t \gg T$).

Similarly:

- The late time behavior of the energy of the system in the false vacuum state,

$$\mathcal{E}_0^{false}(t) \simeq E_0^{true} - \frac{c_2}{t^2}\ldots, \quad \text{for} \quad t \gg T, \tag{32}$$

(where $c_2 = c_2^*$ and it can be positive or negative depending on the model considered),

- or, of the energy density $\rho_0^{false}(t)$:

$$\rho_0^{false}(t) \simeq \rho_0^{true} - \frac{d_2}{t^2}\ldots, \quad \text{for} \quad t \gg T, \tag{33}$$

(where $d_2 = d_2^*$, $\rho_0^{true} \equiv \rho_0^{bare}$), is the pure quantum effect following from the basic principles of the quantum theory.

The standard relation is

$$\rho_0 \equiv \rho_0^{true} = \frac{\Lambda_0}{8\pi G}, \tag{34}$$

where $\Lambda_0 \equiv \Lambda^{bare}$ is the bare cosmological constant.

5. Final Remarks

The late time properties of the energy of the unstable false vacuum state discussed in the previous Section give a strong support for cosmological models using:

- $$\rho_0(t) = \rho_0^{bare} + \frac{A_0}{t^2}, \quad \text{or equivalently,} \quad \Lambda(t) = \Lambda^{bare} + \frac{B_0}{t^2},$$

  where A_0, B_0 are real and can be positive or negative depending on the model considered;

- $$\rho_0(t) = \rho_0^{bare} + A_1\, H_u^2 \quad \text{or equivalently,} \quad \Lambda(t) = \Lambda^{bare} + B_1\, H_u^2,$$

  where A_1, B_1 are real and H_u is the Hubble parameter. (There is $H_u \equiv H_u(t) \propto \frac{1}{t}$).

Cosmologies using such parameters are consistent with the quantum theoretical treatment of unstable vacua.

Open problems:

- Properties of the energy $\mathcal{E}_0^{false}(t)$ are determined by the form of the energy density $\omega(E)$ (The sign of $\eta^{(1)}(E_{min})$ and thus the sign of c_2 in the formula (23) depends on the form of $\eta(E)$ and thus of $\omega(E)$): It is necessary to find at least an approximate form of $\omega(E)$ for false vacuum states.
- As it is seen from Figs (3), (4), the energy $\mathcal{E}_0^{false}(t)$ of the unstable false vacuum state should fluctuate at transition times $t \sim T$. This means that $\rho_0^{false}(t)$ and $\Lambda(t)$ should also fluctuate at these times.

  Question: What are possible consequences of this effect?
- If the vacuum is unstable (or even metastable) why our Universe still exists?

Thank you for your attention