Perturbative QCD and Jets

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Lecture 1

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Outline

• Basics of QCD
  • Lagrangian and Feynman rules
  • Colour
  • QCD beta-function and asymptotic freedom
  • Factorisation

• QCD concepts in Phenomenology
  • e+e- to hadrons and infrared singularities
  • Scale variations
  • Hadronic collisions and PDFs
  • Jets
  • Selected topics: NLO automation, prompt photons, ...
    (time permitting)
Literature

• R. K. Ellis, W. J. Stirling and B. R. Webber, 
  *QCD and collider physics*,

• G. Dissertori, I. Knowles, M. Schmelling, 
  *Quantum Chromodynamics: High energy experiments and theory*
  International Series of Monographs on Physics No. 115, 

• J. M. Campbell, J. W. Huston and W. J. Stirling, 
  *Hard Interactions of Quarks and Gluons: A Primer for LHC Physics*, 


• G. P. Salam, *Towards Jetography*,

• M. Dasgupta, A. Fregoso, S. Marzani and G. P. Salam, 
  *Towards an understanding of jet substructure*, 

• J. Shelton, *TASI Lectures on Jet Substructure*,
Motivation

Why do we care about QCD?

• we have to: it dominates hadronic collisions
• can hide New Physics effects
• can fake New Physics effects
• is interesting by itself

the precision we can achieve on important measurements (e.g. Higgs properties) is directly linked to the control of QCD effects!

e.g. Higgs production in gluon fusion at NNLO:

\[
\sigma(m_H = 125 \text{ GeV}) = 19.27^{+7.2\%}_{-7.8\%}^{+7.5\%}_{-6.9\%} \text{ pb}
\]

D. de Florian, EPS '13
magnitudes of cross sections:

QCD dominates
**Basics of QCD**

Strong interactions are described by SU(3) gauge theory.

### Evidence for 3 Colours

\[
R_{\text{had}} = \frac{\sigma(ee \rightarrow \text{hadrons})}{\sigma(ee \rightarrow \mu\mu)} = 3 \cdot \sum_{i} Q_{i}^2
\]

#### Evolution of $R_{\text{had}}$ with rising CMS-energy

<table>
<thead>
<tr>
<th>$\sqrt{s}$</th>
<th>Quarks</th>
<th>$R_{\text{had}} = 3 \cdot \sum_{i} Q_{i}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; ~3 GeV</td>
<td>uds</td>
<td>3.6/9=2.00</td>
</tr>
<tr>
<td>&lt; ~10 GeV</td>
<td>udsc</td>
<td>3.10/9=3.33</td>
</tr>
<tr>
<td>&lt; ~350 GeV</td>
<td>udscb</td>
<td>3.11/9=3.67</td>
</tr>
<tr>
<td>&gt; ~350 GeV</td>
<td>udscbt</td>
<td>3.15/9=5.00</td>
</tr>
</tbody>
</table>

Data lies systematically higher than the prediction from the Quark Parton Model (QPM) and indicates gluon bremsstrahlung.
Evidence for 3 Colours

5.1 Elektron-Positron-Annihilation in Hadronen

\[ R = R_{\text{QED}}(1 + \alpha_s / \pi + ...) \]

[includes QCD corrections]

Resonances at beginning of step

\[ \sqrt{s} / \text{GeV} \]

\[ R \]

\[ J/\psi \psi' \gamma \]

\[ uds \]

\[ udsc \]

\[ udscb \]

\[ udscbt \]
QCD Lagrangian

\[ \mathcal{L}_{QCD} = \mathcal{L}_{\text{Yang Mills}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{ghost}} \]

\[ \mathcal{L}_{\text{Yang Mills}} = -\frac{1}{4} \mathcal{F}^A_{\mu\nu} \mathcal{F}^{\mu\nu}_A \]

\[ \mathcal{F}^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g f^{ABC} A^A_{\mu,B} A^B_{\nu,C} \]

Gluon self interactions

\[ A = 1, \ldots, 8 \]  
Gluons in adjoint representation of SU(3)

Non-Abelian gauge theory  
Different from QED!  
Important consequences
\[ \mathcal{L}_{\text{fermion}} = \sum_{\text{flavours}} \bar{q}_a \left( i \slashed{D}^{ab} - m \delta^{ab} \right) q_b \]

\[ \slashed{D}_{ab} = \gamma_\mu D^\mu_{ab} ; \quad D^\mu_{ab} = \partial^\mu \delta_{ab} + i g (t^A A^\mu_A)^{ab} \]

\( a, b \in \{1, 2, 3\} \) quarks in fundamental representation of SU(3)

\[ t^A = \lambda^A / 2 \]

\( \lambda^A : \) Gell-Mann matrices

\( \lambda^A : \) generators of SU(3)

\[ [t_A, t_B] = i f_{ABC} t^C \]

\( f_{ABC} : \) structure constants

NB conventions: doubly occurring indices are summed over
\[ \mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\lambda} \left( \partial_{\mu} A_{\mu}^{A} \right)^{2} \] (covariant gauges)

\[ \lambda = 1 : \text{Feynman gauge} \]
\[ \lambda \to 0 : \text{Landau gauge} \]

\[ \mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\lambda} \left( n^{\mu} A_{\mu}^{A} \right)^{2} \] (axial gauges: \( n \cdot A = 0 \))

\[ n^{2} = 0 : \text{light-cone gauge} \]

reminder: classical equation of motion
\[ K_{\mu\nu}^{AB} A_{B}^{\nu} = \delta^{AB} \left( -\Box g_{\mu\nu} + \partial_{\mu} \partial_{\nu} \right) A_{B}^{\nu} = J_{\mu}^{A} \]

cannot be solved because \( K_{\mu\nu}^{AB} \) is not invertible \( \Rightarrow \) need gauge fixing
Ghost fields

\[ \mathcal{L}_{\text{ghost}} = \partial_\mu (\eta^A)^\dagger (D_{A\mu}^B \eta^B) \] (covariant gauges)

\[ \mathcal{L}_{\text{ghost}} = -(\eta^A)^\dagger n_\mu (D_{A\mu}^B \eta^B) = -(\eta^A)^\dagger n_\mu (\partial^\mu \eta_A) \] (axial gauges)

\( \eta \) complex scalar field obeying Fermi statistics

(related to Jacobian of gauge transformations in path integral formulation)

- Covariant gauges introduce *unphysical* gluon polarisations at quantum level which are cancelled by ghost-gluon interactions.

- In axial gauges ghosts do not couple to gluons, only *physical* gluon polarisations propagate.

Therefore axial gauges are also called *physical* gauges.
Feynman Rules

\[ \Delta^{AB}_{\mu\nu}(p) = \frac{i \delta^{AB}}{p^2 + i \varepsilon} d_{\mu\nu} \]

\[ d_{\mu\nu} = \sum_{\text{polarisations } \alpha} \epsilon^*_\mu(p, \alpha) \epsilon_\nu(p, \alpha) \]

\[ = \begin{cases} 
- g_{\mu\nu} + (1 - \lambda) \frac{p_{\mu} p_{\nu}}{p^2} & \text{covariant gauge} \\
- g_{\mu\nu} + \frac{p_{\mu} n_{\nu} + p_{\nu} n_{\mu}}{p \cdot n} & \text{light-cone gauge} 
\end{cases} \]
Feynman Rules

\[ \delta^{AB}_{ij} \frac{i}{(p^2 + i\epsilon)} \quad \text{ghost propagator} \]

\[ \delta^{ab}_{ji} \frac{i}{(p' - m + i\epsilon)} \quad \text{fermion propagator} \]

\[ -g f^{ABC} [(p-q)^{\gamma} g^{a\beta} + (q-r)^{\alpha} g^{b\gamma} + (r-p)^{\beta} g^{\gamma a}] \]

(all momenta incoming, \( p + q + r = 0 \))

\[ -ig f^{XAC} f^{XBD} [g^{a\beta} g^{\gamma \delta} - g^{a\delta} g^{\beta \gamma}] \]

\[ -ig f^{XAD} f^{XBC} [g^{a\beta} g^{\gamma \delta} - g^{a\gamma} g^{\beta \delta}] \]

\[ -ig f^{XAB} f^{XCD} [g^{a\gamma} g^{\beta \delta} - g^{a\delta} g^{\beta \gamma}] \]

\[ g f^{ABC} q^{a} \quad \text{gluon-ghost vertex} \]

\[ -ig \left( t^{A}_{cb} \right)_{ij} (\gamma^{a})_{ji} \quad \text{gluon-quark vertex} \]
generators of $SU(N_c)$:

\[ N_c^2 - 1 \text{ hermitean traceless matrices } (t^A)_{ab} \quad (\text{fundamental representation}) \]

\[ [t_A, t_B] = i f_{ABC} t^C \]

\[ \begin{align*}
  a \rightarrow b & \quad \text{colour} \quad \delta_{ab} \\
  a \rightarrow \begin{array}{c}
    A \\
    a
  \end{array} & \quad \text{colour} \quad (t^A)_{ab} \\
  \begin{array}{c}
    A \\
    a
  \end{array} \rightarrow b & \quad \text{colour} \quad \delta_{AB} \\
  \begin{array}{c}
    A \\
    C
  \end{array} \rightarrow \begin{array}{c}
    B \\
    B
  \end{array} & \quad \text{colour} \quad i f^{ABC}
\end{align*} \]

\[ \text{Tr}(t^A) = 0 \]
 Colour Algebra

 some pictorial identities:

$$T_r(t^A t^B) = T_R \delta^{AB}, \quad T_R = \frac{1}{2}$$

$$\sum_A t^A t^A = C_F \delta_{ab}, \quad C_F = \frac{N_c^2 - 1}{2N_c}$$

$$\sum_{C,D} f^{CDA} f^{CDB} = C_A \delta^{AB}, \quad C_A = N_c$$

$$(t^A)_{ab} (t^A)_{cd} = \frac{1}{2} \delta_{ad} \delta_{bc} - \frac{1}{2N_c} \delta_{ab} \delta_{cd}$$

(Fierz identity)
\[ f^{ABC} = -2i \text{Tr}([t^A, t^B] t^C) \]

we can write every n-gluon tree graph colour factor as a sum of traces of matrices: 
\[ Tr(t^{A_1} t^{A_2} \ldots t^{A_n}) + \text{all non-cyclic permutations} \]

similarly 
\[ q\bar{q}gggg \Rightarrow Tr(t^{A_1} t^{A_2} \ldots t^{A_n})_{ab} + \text{permutations} \]

\[ M_{n}^{\text{tree}}(\{p_i, a_i, h_i\}) = g^{n-2}Tr(t^{A_1} t^{A_2} \ldots t^{A_n}) M_{n}^{\text{tree}}(1^{h_1}, 2^{h_2} \ldots n^{h_n}) + \text{all non-cyclic permutations} \]

**important:** as 
\[ M_{n}^{\text{tree}}(1^{h_1}, 2^{h_2} \ldots n^{h_n}) \] comes from diagrams with cyclic ordering of external legs, it only has singularities in adjacent invariants 
\[ s_{i,i+1} = (p_i + p_{i+1})^2 \]
(see later)
Colour expansion

\[ d\sigma_{\text{tree}}^{\text{tree}}(\{p_i, a_i, h_i\}) \sim \sum_{a_i} \sum_{h_i} |M_{\text{tree}}^{\text{tree}}(\{p_i, a_i, h_i\})|^2 \]

insert colour ordered amplitude and perform the colour sum:

\[ d\sigma_{\text{tree}}^{\text{tree}}(\{p_i, a_i, h_i\}) \sim N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |M_{\text{tree}}^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}) \ldots \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{n-2}) \]

Non-planar topologies are subleading in colour

Note: parton showers usually do not take subleading colour into account
... contains one of the most important minus signs in physics!

QED:

Roughly speaking, the gluon self couplings reverse the sign of the beta-function.

In more detail ...
QCD beta-function

- consider a dimensionless observable $R$ which can be expanded in $\alpha_s = \frac{g^2}{4\pi}$ and which depends on a single large energy scale $Q$

- dimensional analysis $\rightarrow$ $R$ should be independent of $Q$

- however, $R$ needs UV renormalisation!

- this introduces another mass scale $\mu$, the point at which the subtractions of the UV divergences are performed

- therefore $R$ will depend on the ratio $Q/\mu$

- the renormalized coupling $\alpha_s$ will also depend on $\mu$

- as $\mu$ is arbitrary, $R$ cannot depend on it $\Rightarrow$

$$\mu^2 \frac{d}{d\mu^2} R \left( \frac{Q^2}{\mu^2}, \alpha_s \right) = \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right] R = 0$$
We shall see QCD is asymptotically free: and hence we can safely use perturbation theory. Then knowledge of

\[ \tau = \ln \left( \frac{Q^2}{\mu^2} \right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2}, \]

then

\[ \left[ -\frac{\partial}{\partial \tau} + \beta(\alpha_S) \frac{\partial}{\partial \alpha_S} \right] R = 0 \]

renormalisation group equation

solved by running coupling \( \alpha_S(Q) : \)

\[ \tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)}, \quad \alpha_S(\mu) \equiv \alpha_S \]

\[ \frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) \]

\[ \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} \]

the beta function has the expansion

\[ \beta(\alpha_s) = -b_0 \alpha_s^2 (1 + b_1 \alpha_s) + O(\alpha_s^4) \]

\[ b_0 = \frac{1}{12\pi} (11N_c - 2N_f), \quad b_1 = \frac{17N_c^2 - 5N_cN_f - 3C_FN_f}{2\pi (11N_c - 2N_f)} \]

where \( N_f \) is the number of active flavours. Terms up to \( O(\alpha_s^4) \) are known.
Asymptotic freedom

Roughly speaking, quark loop diagram (a) contributes negative $N_f$ term in $b$, while gluon loop (b) gives positive $C_A$ contribution, which makes $\beta$ function negative overall.

QED $\beta$ function is $\beta_{\text{QED}}(\alpha) = \frac{\alpha}{3\pi} \ln \left( \frac{Q^2}{m_e^2} \right)$

Thus $b$ coefficients in QED and QCD have opposite signs.

From earlier slides, $\partial \alpha S(Q) / \partial \tau = -b \alpha^2 S(Q) + O(\alpha^4 S(Q))$.

Neglecting $b'$ and higher coefficients gives $\alpha S(Q) = \alpha S(\mu) \left( 1 + \alpha S(\mu) b \tau \right)$, $\tau = \ln \left( \frac{Q^2}{\mu^2} \right)$.

QCD:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2) b_0 \ln \left( \frac{Q^2}{\mu^2} \right)}$$

$b_0 = \frac{1}{12\pi} (11N_c - 2N_f)$

Coupling decreases with energy $\Rightarrow$ asymptotic freedom

QED:

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \ln \left( \frac{Q^2}{m_e^2} \right)}$$

$1/137$ coupling grows with energy

Coupling grows with energy

QCD and Monte Carlo Methods

Lecture I: QCD, asymptotic freedom and infrared safety – p.27/38
Asymptotic freedom

\[ \alpha_s(Q) \]

July 2009

- Deep Inelastic Scattering
- e^+e^- Annihilation
- Heavy Quarkonia

1. Deep Inelastic Scattering
2. e^+e^- Annihilation
3. Heavy Quarkonia

QCD \( \alpha_s(M_Z) = 0.1184 \pm 0.0007 \)

\( Q \) [GeV]

S. Bethke

\( \alpha_s(M_Z) = 0.1184 \pm 0.0007 \)

\( \alpha_{QED} \)

10 \( \alpha_{QED} \)

10 \( \alpha_{QED} \)

10 \( \alpha_{QCD} \)

\( Q \) (GeV)

\( 1 \times 10^1 \)

\( 1 \times 10^2 \)

\( 1 \times 10^3 \)

\( 1 \times 10^4 \)

\( 1 \times 10^5 \)

\( 1 \times 10^6 \)

\( 1 \times 10^7 \)

\( 1 \times 10^8 \)

\( 1 \times 10^9 \)

\( 1 \times 10^{10} \)

\( 1 \times 10^{11} \)

\( 1 \times 10^{12} \)

\( 1 \times 10^{13} \)

\( 1 \times 10^{14} \)

\( 1 \times 10^{15} \)

\( 1 \times 10^{16} \)

\( 1 \times 10^{17} \)

\( 1 \times 10^{18} \)

\( 1 \times 10^{19} \)

\( 1 \times 10^{20} \)

\( \alpha_s(M_Z) \)

\( 0.11 \)

\( 0.12 \)

\( 0.13 \)

\( \alpha_s(M_Z) \)
Confinement

at small scales: running coupling diverges, so perturbation theory cannot be applied

⇒ domain of lattice QCD

\[ \alpha_s(\mu) = \alpha_s(\mu_0) + \alpha_s(\mu_0) \beta_0 \ln \frac{\mu^2}{\mu_0^2} \]

\( \Lambda_{QCD} \) is a scale that is introduced dynamically in the theory, it is the price to pay for renormalisability.

The running of the coupling tells us that we cannot believe PT theory for scales less than 500 MeV.

The growth of the PT coupling at small scales is consistent with the fact that quarks cannot be observed as free objects, but always confined to form hadrons.

**confinement:** partons (quarks and gluons) are only found in colour singlet bound states (hadrons)

**hadronisation:** partons produced in hard scattering processes reorganize themselves to form hadrons.
Lambda Parameter

It is useful to define a dimensionful parameter $\Lambda$ (integration constant) setting the scale at which the coupling becomes large.

$$\ln \left( \frac{Q^2}{\Lambda^2} \right) = - \int_{\alpha_s(Q)}^\infty \frac{dx}{\beta(x)} = \int_{\alpha_s(Q)}^\infty \frac{dx}{b_0 x^2 (1 + b_1 x + \ldots)}$$

Keeping only $b_0$(LO), $b_1$(NLO)

$$\alpha_s(Q) = \frac{1}{b_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} \quad \text{(LO)} \quad \alpha_s(Q) = \frac{1}{b_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} \left[ 1 - \frac{b_1 \ln \ln \left( \frac{Q^2}{\Lambda^2} \right)}{b_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} \right] \quad \text{(NLO)}$$

Note that $\Lambda$ depends on the number of active flavours $N_f$.

Comment: as it sets the scale of hadron masses, it is quite an important parameter in particle physics!
Not as famous as the Higgs though . . . 🤔
Introduction to Event Generators

- Hadron collider event
- Hard scattering
- Parton shower
- Hadronisation
- Hadron decays

- Initial state (proton)

artist: Frank Krauss
how can we describe this?

**Factorisation:** separate hard and soft scales

\[ \sigma_{pp \to X} = \sum_{a,b,c} f_a(x_1, \mu_f^2) f_b(x_2, \mu_f^2) \otimes \hat{\sigma}_{ab}(p_1, p_2, \frac{Q^2}{\mu_f^2}, \frac{Q^2}{\mu_r^2}, \alpha_s(\mu_r^2)) \]

\[ \otimes D_{c \to X}(z, \mu_f^2) + O(\Lambda/Q) \]

- \( f_a, f_b \): parton distribution functions (from fits to data)
- \( \hat{\sigma}_{ab} \): partonic hard scattering cross section
- **calculable order by order in perturbation theory**
- \( D_{c \to X}(z, \mu_f^2) \): describing the final state e.g. fragmentation function, jet observable, etc.
Without factorisation we would be quite lost, but there are still a number of open (QCD) questions, e.g.

- **hard scattering** cross section:
  - which **order** in the perturbative expansion is precise enough? (LO, NLO, NNLO ...)
  - is fixed order adequate, or do we need to **resum large logarithms**?

- how to combine the partonic hard scattering result with a **parton shower**?

- do we know the **parton distribution functions** (PDFs) well enough?

- how to **model hadronisation**?
e+e- annihilation

Start with a simple example:

\[ R = \frac{\sigma (e^+e^- \rightarrow \text{hadrons})}{\sigma (e^+e^- \rightarrow \mu^+\mu^-)} \]

At Born level, for \( n_f \) massless quarks, we have

\[ R_0 = N_c \sum_q Q_q^2 \]

(we will not consider Z exchange here)

What happens if one of the quarks emits a gluon?

Real radiation

Virtual corrections

To work consistently at order \( \alpha_s \) we need both real and virtual corrections
Infrared singularities

In a gauge theory with massless particles both soft and collinear divergences can occur.

Consider the emission of a gluon from a hard quark:

\[
p = E (1, 0, 0, 1)
\]
\[
k = \omega (1, 0, \sin \theta, \cos \theta)
\]

\[
(p + k)^2 = 2E \omega (1 - \cos \theta)
\]

will go to zero if the gluon becomes soft \((\omega \to 0)\)
or if quark and gluon become collinear \((\theta \to 0)\)

Soft and collinear divergences also occur in virtual corrections.

They cancel in an inclusive quantity where “degenerate energy states” are summed over (KLN theorem, see later).

Note: are UV divergent. These UV singularities cancel with vertex diagram due to Ward Identity
Soft singularities

Consider real emission diagrams in more detail:

\[
\mathcal{M}_{q\bar{q}g}^\mu = \bar{u}(p_1) (-ie\gamma^\mu) \frac{i(p_1 + k)}{(p_1 + k)^2} (-ie\gamma^\mu) v(p_2)
\]

\[
+ \bar{u}(p_1) (-ie\gamma^\mu) \frac{-i(p_2 + k)}{(p_2 + k)^2} (-ie\gamma^\mu) v(p_2)
\]

If gluon becomes soft: neglect \( k \) except if it is in denominator:

\[
\mathcal{M}_{q\bar{q}g}^\mu \overset{soft}{=} -iegt^A \bar{u}(p_1) \gamma^\mu \left( \frac{\phi_1}{2p_1k} - \frac{\phi_2}{2p_2k} \right) v(p_2)
\]

\[
|\mathcal{M}_{q\bar{q}g}|^2 \overset{soft}{\rightarrow} |\mathcal{M}_{q\bar{q}}|^2 g^2 C_F \frac{p_1 p_2}{(p_1 k)(p_2 k)}
\]

Note: colour will in general \textbf{not} factorize in the soft limit

Factorisation into Born matrix element and Eikonal factor
Soft singularities

Consider real emission diagrams in more detail:

\[
\mathcal{M}^\mu_{q\bar{q}g} = \bar{u}(p_1) (-igt^A \phi) \frac{i(p_1 + k)}{(p_1 + k)^2} (-ie\gamma^\mu) u(p_2)
\]

\[
\quad + \quad \bar{u}(p_1) (-ie\gamma^\mu) \frac{-i(p_2 + k)}{(p_2 + k)^2} (-igt^A \phi) u(p_2)
\]

If gluon becomes soft: neglect \(k\) except if it is in denominator:

\[
\mathcal{M}^\mu_{q\bar{q}g} \overset{soft}{=} -iegt^A \bar{u}(p_1) \gamma^\mu \left( \frac{\phi_1}{2p_1 k} - \frac{\phi_2}{2p_2 k} \right) u(p_2)
\]

\[
|\mathcal{M}_{q\bar{q}g}|^2 \overset{soft}{\to} g^2 C_F \frac{p_1 p_2}{(p_1 k)(p_2 k)}
\]

Note: colour will in general not factorize in the soft limit

Factorisation into Born matrix element and Eikonal factor
Collinear singularities

\[(p_1 + k)^2 = 2E\omega (1 - \cos \theta) \rightarrow 0 \quad \text{for} \quad \theta \rightarrow 0\]

note: if \(p_1\) is a massive particle: \[p_1 = E (1, 0, 0, v), \quad v = \sqrt{1 - \frac{m_1^2}{E^2}}\]

\[(p_1 + k)^2 = 2E\omega (1 - v \cos \theta)\]

no singular denominator for \(\theta \rightarrow 0\)

\[\Rightarrow \quad \text{only massless particles can lead to a collinear singularity}\]

convenient parametrisation of momenta ("Sudakov parametrisation")

\[p_1 = z p^\mu + k^\mu_{\perp} - \frac{k_{\perp}^2}{z} \frac{n^\mu}{2p_1 \cdot n}\]

\[k = (1 - z) p^\mu - k^\mu_{\perp} - \frac{k_{\perp}^2}{1 - z} \frac{n^\mu}{2p_1 \cdot n}\]

\[\Rightarrow 2p_1 k = -\frac{k_{\perp}^2}{z(1 - z)}\]

\[|\mathcal{M}_1(p_1, k, p_2)|^2 \xrightarrow{\text{coll}} g^2 \frac{1}{p_1 \cdot k} P_{qq}(z) |\mathcal{M}_0(p_1 + k, p_2)|^2\]

\(P_{qq}(z):\) splitting functions
DGLAP splitting functions

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

\[ \hat{P}_{qq}(z) = C_F \left[ \frac{1 + z^2}{(1 - z)} \right], \]
\[ \hat{P}_{qg}(z) = T_R \left[ z^2 + (1 - z)^2 \right], \quad T_R = \frac{1}{2}, \]
\[ \hat{P}_{gg}(z) = C_A \left[ \frac{z}{(1 - z)} + \frac{1 - z}{z} + z (1 - z) \right], \]
\[ \hat{P}_{gq}(z) = C_F \left[ \frac{1 + (1 - z)^2}{z} \right]. \]

(details see later in PDF discussion)
Real radiation matrix element

in general we sum over final state polarizations and
colours and average over initial state pols., colours:

$$\left| \mathcal{M} \right|^2 \rightarrow \sum_{\lambda,c} \left| \mathcal{M}_{\lambda,c} \right|^2 = \frac{1}{\prod_{\text{initial}} N_{\text{pol}} N_{\text{col}}} \sum_{\text{final pol, col}} \left| \mathcal{M}_{\lambda,c} \right|^2$$

at LO, we obtain

$$\left| \mathcal{M}_0 \right|^2 = \left( \frac{1}{3} \right) 4e^2 Q^2 N_c s$$

with extra gluon radiation:

$$s_{ij} = (p_i + p_j)^2$$

$$p^\gamma = \sqrt{s} (1, 0, 0, 0)$$

$$p_1 = E_1 (1, 0, 0, 1)$$

$$p_2 = E_2 (1, 0, \sin \theta, \cos \theta)$$

$$k \equiv p_3 = p^\gamma - p_1 - p_2$$

$$\left| \mathcal{M}_1 \right|^2 = \left| \mathcal{M}_0 \right|^2 \frac{2g^2 C_F}{s} \left( \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s \frac{s_{12}}{s_{13}s_{23}} \right)$$

defining

$$x_1 = \frac{2E_1}{\sqrt{s}}, \quad x_2 = \frac{2E_1}{\sqrt{s}}$$

$$\left| \mathcal{M}_1 \right|^2 = \left| \mathcal{M}_0 \right|^2 \frac{2g^2 C_F}{s} \left( \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \right)$$

gluon energy:

$$E_g = \sqrt{s} (1 - x_1 - x_2)$$
Real radiation matrix element

\[ |\mathcal{M}_1|^2 = |\mathcal{M}_0|^2 \frac{2g^2 C_F}{s} \left( \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s \frac{s_{12}}{s_{13}s_{23}} \right) \]

\[ = |\mathcal{M}_0|^2 \frac{2g^2 C_F}{s} \left( \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \right) \]

\[ x_1 \to 1: \text{ collinear singularity} \quad p_1 \parallel p_3 , \quad x_2 \to 1: \text{ collinear singularity} \quad p_2 \parallel p_3 \]

\[ x_1 \to 1 - x_2: \text{ soft gluon} \quad E_g = \sqrt{s} (1 - x_1 - x_2) \]

in these limits the matrix element is singular!

- how can we interpret this?
- how can we remedy this?
Cancellation of IR divergences

• interpretation: A quark-antiquark pair with a soft and collinear gluon cannot be distinguished experimentally from just a q qbar pair, so this is not an observable final state. Physical final states are hadrons or jets.

KLN Theorem

Soft and collinear singularities cancel in the sum over degenerate states

• what are degenerate states?

For example, a quark emitting a soft gluon cannot be distinguished from simply a quark. Exchange of virtual gluons also leads to IR singularities (same order in alpha_s).

Singularities cancel between real and virtual corrections.
Cancellation of IR divergences

IR singularities cancel between real and virtual corrections.

Really?

Well, not always: in hadronic collisions, initial state collinear singularities do not cancel, but need to be absorbed into the parton distribution functions, as we cannot sum over “degenerate states” in the proton (see later).

In practice (calculation):

We need to isolate the singularities before we can cancel them, as real and virtual corrections live on different phase spaces.

In the partonic ratio $\hat{R}$, IRC singularities cancel in the inclusive sum of real and virtual contributions $+$ finite $2$ $\hat{R}$ is insensitive to soft and collinear emission. It is then called an infrared and collinear safe observable.

In case of $\hat{R}$, there is complete cancellation of soft and collinear radiation, $\hat{R}$ becomes completely insensitive to soft and collinear emissions up to the hard scale $Q^2 = s$.

The key features that guarantee cancellation of IRC divergences are:

1. In the IRC region, the matrix elements for real and virtual corrections are equal but with opposite sign.
2. The observable assigns the same weight to real emissions and virtual corrections.
Dimensional Regularization

A convenient way to isolate singularities is dimensional regularisation:

we work in \( D = 4 - 2\epsilon \) dimensions.

- regulates both UV and IR divergences
- does not violate gauge invariance
- poles can be isolated in terms of \( 1/\epsilon^b \)

- need phase space integrals in D dimensions
- need integration over virtual loop momenta in D dimensions

\[
g^2 \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \rightarrow g^2 \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D}
\]
Virtual corrections

\[
R_{\text{virt}} = R^{\text{LO}} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left( \frac{-s}{4\pi\mu^2} \right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\}
\]
Phase space in D dimensions

1 to N particle phase space:

\[ Q \rightarrow p_1 + \ldots + p_N \]

\[ \int d\Phi_D^N = (2\pi)^{N-D(N-1)} \int \prod_{j=1}^{N} d^D p_j \delta^+(p_j^2 - m_j^2) \delta^{(D)} \left( Q - \sum_{i=1}^{N} p_i \right) \]

In the following consider massless case \( p_j^2 = 0 \). Use for \( i = 1, \ldots, N - 1 \)

\[ \int d^D p_i \delta^+(p_i^2) \equiv \int d^D p_i \delta(p_i^2) \theta(E_i) = \int d^{D-1} \vec{p}_i \, dE_i \, \delta(E_i^2 - \vec{p}_i^2) \theta(E_i) \]

\[ = \frac{1}{2E_i} \int d^{D-1} \vec{p}_i \bigg|_{E_i = |\vec{p}_i|} \]

and eliminate \( p_N \) by momentum conservation

\[ \Rightarrow \int d\Phi_D^N = (2\pi)^{N-D(N-1)} \, 2^{1-N} \int \prod_{j=1}^{N-1} d^{D-1} \vec{p}_j \frac{\Theta(E_j)}{E_j} \delta^+([Q - \sum_{i=1}^{N-1} p_i]^2) \bigg|_{E_j = |\vec{p}_j|} \]

phase space volume of unit sphere in D dimensions

\[ \int d\Omega_{D-1} = V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \]

\[ V(D) = \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin \theta_2 \ldots \int_0^{\pi} d\theta_{D-1} (\sin \theta_{D-1})^{D-2} \]
Real radiation in D dimensions

1 to 3 particle phase space:

\[ p^\gamma = (\sqrt{s}, \vec{0}^{(D-1)}) \]
\[ p_1 = E_1 (1, \vec{0}^{(D-2)}, 1) \]
\[ p_2 = E_2 (1, \vec{0}^{(D-3)}, \sin \theta, \cos \theta) \]
\[ p_3 = p^\gamma - p_2 - p_1 \]

\[ x_i = \frac{2p_i \cdot p^\gamma}{s} \]

\[ d\Phi_{1\to3} = \frac{1}{4} (2\pi)^{3-2D} dE_1 dE_2 d\theta [E_1 E_2 \sin \theta]^{D-3} d\Omega_{D-2} d\Omega_{D-3} \]
\[ = (2\pi)^{3-2D} \frac{2^{4-D}}{32} s^{D-3} d\Omega_{D-2} d\Omega_{D-3} [(1 - x_1)(1 - x_2)(1 - x_3)]^{D/2-2} \]
\[ dx_1 dx_2 dx_2 \Theta(1 - x_1) \Theta(1 - x_2) \Theta(1 - x_3) \delta(2 - x_1 - x_2 - x_3) \]

\[ |\overline{M}_1|^2 = |\overline{M}_0^{(D)}|^2 \frac{2g^2 C_F}{s} \left( \frac{(x_1^2 + x_2^2)(1 - \epsilon) + 2\epsilon(1 - x_3)}{(1 - x_1)(1 - x_2)} - 2\epsilon \right) \]
Combine to final result

\[ R_{\text{real}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 3\epsilon)} \left( \frac{s}{4\pi \mu^2} \right)^{-\epsilon} \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right\} \]

 gluon both soft and collinear

remember virtual corrections:

\[ R_{\text{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left( \frac{-s}{4\pi \mu^2} \right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\} \]

KLN theorem at work!

\[ R = R^{LO} \times \left\{ 1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right\} \]

scale dependence
Scale dependence

\[ R = R_0 \times \Delta_{QCD} = 3 \sum_q Q_q^2 \times \Delta_{QCD} \]
\[ \Delta_{QCD} = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n=2}^{\infty} C_n \left( \frac{s}{\mu^2} \right)^n \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n \]

\[ \frac{dR}{d\mu} = 0 \Rightarrow \mu^2 \frac{\partial R}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial R}{\partial \alpha_s} = 0 \]

The more higher orders are included in \( R \) and \( \alpha_s \) the more the scale dependence is reduced.
More on scale dependence

dijet production at NNLO (leading colour):

Higgs + jet production (pure gluon only):

Gehrmann-De Ridder, Gehrmann, Glover, Pires 2013

Boughezal, Caola, Melnikov, Petriello, Schulze 2013
More on scale dependence

e+e- to 3 jets up to NNLO:

2jet to 3jet transition parameter $Y_3$

- reduction of scale uncertainty
- better description of the data
- NNLO, NLO not within LO uncertainty band!

- scale variations of LO result do not necessarily give a realistic error estimate
- choice of a convenient central scale is important (and not straightforward)
- NLO does a reasonable job, LO does not

bands from scale variations $M_Z/2 \leq \mu \leq 2 M_Z$

Gehrmann-De Ridder, Gehrmann, Glover, GH, 2008
More on scale dependence

\[ \overline{p}p \rightarrow (Z, \gamma^*) + X \]

\[ \frac{d^2 \sigma}{dM/dY} \text{ [pb/GeV]} \]

\[ \sqrt{s} = 1.96 \text{ TeV} \]

\[ M = M_Z \]

\[ M/2 \leq \mu \leq 2M \]

[Anastasiou et al. 04]
K-factors

\[ K = \frac{\sigma^{N(N)\text{LO}}}{\sigma^{\text{LO}}} \]

example Higgs production:

\[ K(gg \rightarrow H) \]
\[ \sqrt{s} = 14 \text{ TeV} \]

\[ \frac{1}{2} < \frac{\mu_F}{\mu_R} < 2 \]
\[ \frac{1}{2} < \frac{M_H}{\mu_F, \mu_R} < 2M_H \]

NNLO: Harlander, Kilgore ’02
Anastasiou Melnikov ’02
Ravindran, Smith van Neerven, ’03

K-factors very large!
K-factors

**Note:** K-factors for distributions are in general not constant!

**example:** \( pp \rightarrow W^+ W^- b \bar{b} \)

\( W^+ W^- b \bar{b} \): Transverse momentum of the \( b \bar{b} \) system

LHC 7 TeV

\( H_T/4 < \mu < H_T \)

MSTW2008(n)lo pdf

GH, J. Schlenk, J. Winter
What is a convenient scale choice?

Example from W+3 jets:

Possible scale choices:

\[ E_T^W = \sqrt{M_W^2 + p_T^2(W)} \]

\[ H_T = \sum_{\text{jets}} E_T^{\text{jet}} + E_T^{\text{lepton}} + E_T^{\text{miss}} \]

\[ M_{12} = (p_{\text{jet1}} + p_{\text{jet2}})^2 \]

C. Berger et al (Blackhat) ’09

\[ H_T \] much better reflects the scale of the hard interaction