Perturbative QCD and Jets

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Lecture 1

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MAX-PLANCK-GESELLSCHAFT

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Outline

- Basics of QCD
  - Lagrangian and Feynman rules
  - Colour
  - QCD beta-function and asymptotic freedom
  - Factorisation

- QCD concepts in Phenomenology
  - e+e- to hadrons and infrared singularities
  - Scale variations
  - Hadronic collisions and PDFs
  - Jets
  - Selected topics: NLO automation, prompt photons, ...

    (time permitting)

Literature

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  Oxford University Press, Feb. 2003, 2005.

- J. M. Campbell, J. W. Huston and W. J. Stirling,
  Hard Interactions of Quarks and Gluons: A Primer for LHC Physics,
  Rept. Prog. Phys. **70** (2007) 89 [hep-ph/0611148].

- J. Alcaraz Maestre et al., *The SM and NLO Multileg and SM MC Working Groups: Summary Report of the Les Houches 2011 workshop on Physics at TeV Colliders*, arXiv:1203.6803 [hep-ph].

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- M. Dasgupta, A. Fregoso, S. Marzani and G. P. Salam,
  Towards an understanding of jet substructure,
  arXiv:1307.0007 [hep-ph].

- J. Shelton, *TASI Lectures on Jet Substructure*,
  arXiv:1302.0260 [hep-ph].

Motivation

Why do we care about QCD ?

- we have to : it dominates hadronic collisions
- can <u>hide</u> New Physics effects
- can <u>fake</u> New Physics effects
- is interesting by itself

the precision we can achieve on important measurements (e.g. Higgs properties) is directly linked to the control of QCD effects!

e.g. Higgs production in gluon fusion at NNLO:

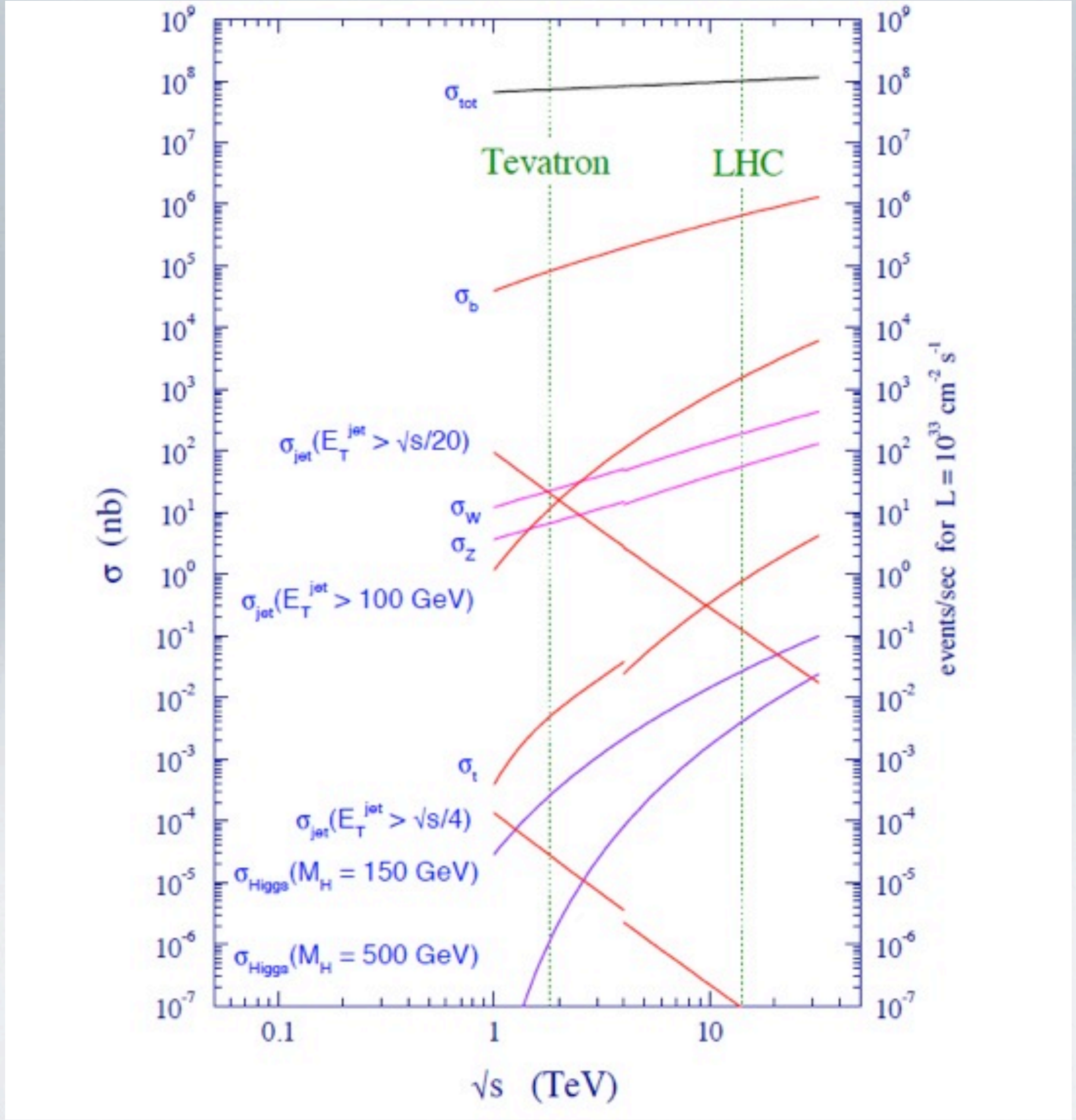
$$\sigma(m_H = 125\,\mathrm{GeV}) = 19.27^{+7.2\%}_{-7.8\%}\,{}^{+7.5\%}_{-6.9\%}\ \mathrm{pb}$$

(scale, pdf + α_S)

D. de Florian, EPS '13

magnitudes of cross sections:

QCD dominates



Basics of QCD

strong interactions are described by SU(3) gauge theory

Evidence for 3 Colours

N_C

$$R_{had} = \frac{\sigma(ee \to hadrons\,)}{\sigma(ee \to \mu\mu)} = 3 \cdot \sum_i Q_i^2$$

fractional quark charges

Evolution of R_{had}
with rising CMS-energy

$\sqrt{s}$	Quarks	$R_{had} = 3 \cdot \sum_i Q_i^2$
< ~3 GeV	uds	3·6/9=2.00
< ~10 GeV	udsc	3·10/9=3.33
< ~350 GeV	udscb	3·11/9=3.67
> ~350 GeV	udscbt	3·15/9=5.00

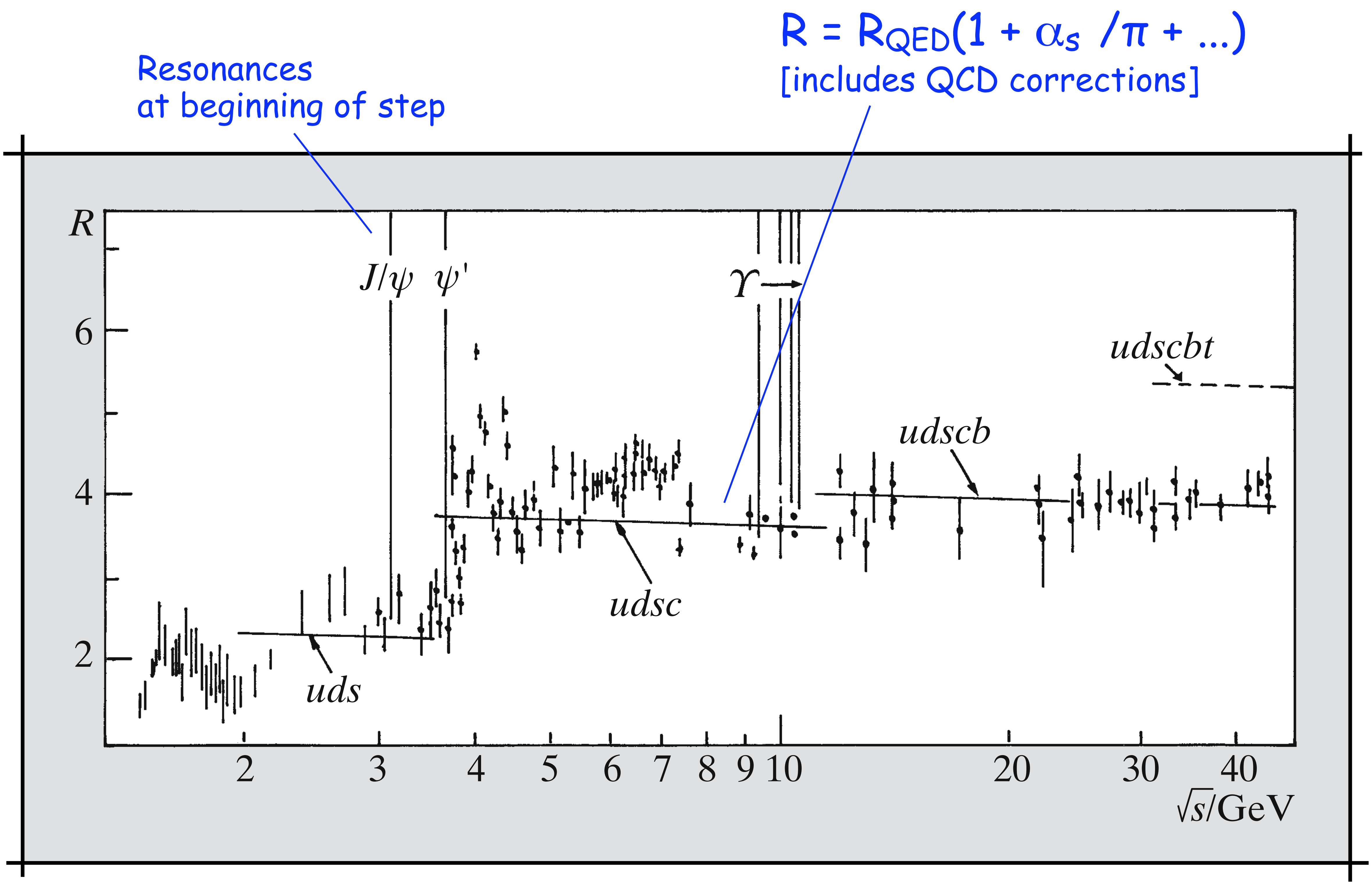
d,s   u

$3 \cdot [2 \cdot (1/3)^2 + (2/3)^2]$

$3 \cdot [2 \cdot (1/3)^2 + 2\,(2/3)^2]$

$3 \cdot [3 \cdot (1/3)^2 + 2\,(2/3)^2]$

d,s,b   u,b



Resonances
at beginning of step
R = R_QED(1 + αs /π + ...)
[includes QCD corrections]
R
J/ψ
ψ'
Υ
6
4
2
udscbt
udscb
udsc
uds
2
3
4
5
6
7
8
9
10
20
30
40
√s/GeV

QCD Lagrangian

$$\mathcal{L}_{QCD} = \mathcal{L}_{\text{Yang Mills}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{ghost}}$$

$$\mathcal{L}_{\text{Yang Mills}} = -\frac{1}{4}\mathcal{F}^{A}_{\mu\nu}\mathcal{F}^{\mu\nu}_{A}$$

$$\mathcal{F}^{A}_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu - g\, f^{ABC} \mathcal{A}_{\mu,B}\mathcal{A}_{\nu,C}$$

gluon self interactions

$A = 1, \ldots, 8$ gluons in **adjoint** representation of SU(3)

non-Abelian gauge theory $\rightarrow$ different from QED ! important consequences

$$\mathcal{L}_{\rm fermion} = \sum_{\rm flavours} \bar{q}_a \left(i \not{D}^{ab} - m\,\delta^{ab} \right) q_b$$

$$\not{D}_{ab} = \gamma_\mu D^\mu_{ab} \; ; \; D^\mu_{ab} = \partial^\mu \delta_{ab} + i\,g\,(t^A \mathcal{A}^\mu_A)_{ab}$$

$a, b \in \{1, 2, 3\}$ quarks in **fundamental** representation of SU(3)

$t^A = \lambda^A / 2$ $\quad$ λ^A : Gell-Mann matrices

generators of SU(3)

$[t_A, t_B] = i\, f_{ABC}\, t^C$ $\quad$ f_{ABC} : structure constants

NB conventions: doubly occurring indices are summed over

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\lambda}\left(\partial^\mu \mathcal{A}_\mu^A\right)^2 \quad \text{(covariant gauges)}$$

$$\lambda = 1 : \text{Feynman gauge}$$

$$\lambda \to 0 : \text{Landau gauge}$$

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\lambda}\left(n^\mu \mathcal{A}_\mu^A\right)^2 \quad \text{(axial gauges: } n \cdot A = 0)$$

$$n^2 = 0 : \quad \text{light-cone gauge}$$

reminder: classical equation of motion $K_{\mu\nu}^{AB} \mathcal{A}_B^\nu = \delta^{AB}(-\Box g_{\mu\nu} + \partial_\mu \partial_\nu)\, \mathcal{A}_B^\nu = J_\mu^A$

cannot be solved because $K_{\mu\nu}^{AB}$ is not invertible $\Longrightarrow$ need gauge fixing

Ghost fields

$$\mathcal{L}_{\rm ghost} = \partial_\mu(\eta^A)^\dagger(D^\mu_{AB}\eta^B) \quad \text{(covariant gauges)}$$

$$\mathcal{L}_{\rm ghost} = -(\eta^A)^\dagger n_\mu(D^\mu_{AB}\eta^B) = -(\eta^A)^\dagger n_\mu(\partial^\mu \eta_A) \quad \text{(axial gauges)}$$

η complex scalar field obeying Fermi statistics

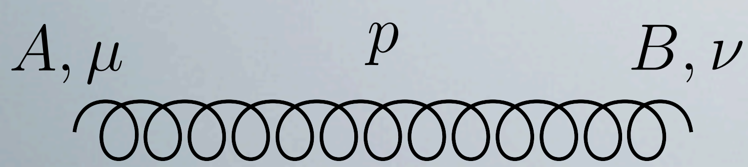
(related to Jacobian of gauge transformations in path integral formulation)

- Covariant gauges introduce *unphysical* gluon polarisations at quantum level which are cancelled by ghost-gluon interactions.

- In axial gauges ghosts do not couple to gluons, only *physical* gluon polarisations propagate.

Therefore axial gauges are also called *physical* gauges.

Feynman Rules

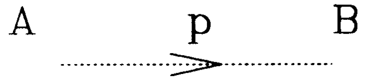


gluon propagator

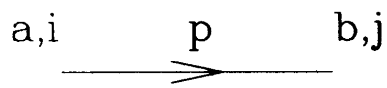
$$\Delta^{AB}_{\mu\nu}(p) = \frac{i\,\delta^{AB}}{p^2 + i\,\varepsilon}\, d_{\mu\nu}$$

$$\begin{aligned} d_{\mu\nu} &= \sum_{\text{polarisations}\,\alpha} \epsilon^*_\mu(p,\alpha)\epsilon_\nu(p,\alpha) \\ &= \begin{cases} -g_{\mu\nu} + (1-\lambda)\,\frac{p_\mu p_\nu}{p^2} & \text{covariant gauge} \\ -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} & \text{light-cone gauge} \end{cases} \end{aligned}$$

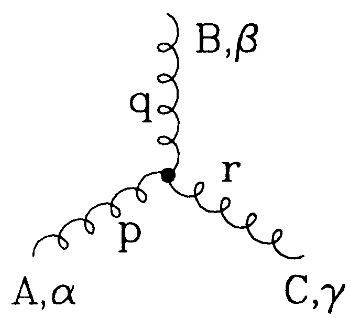
Feynman Rules



$$\delta^{AB}\,\frac{i}{(p^2+i\epsilon)}$$ ghost propagator

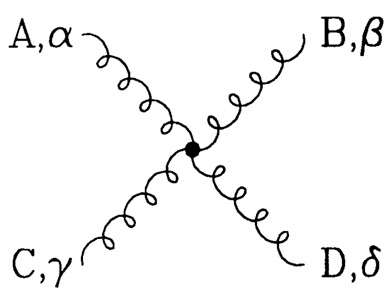


$$\delta^{ab}\,\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$ fermion propagator

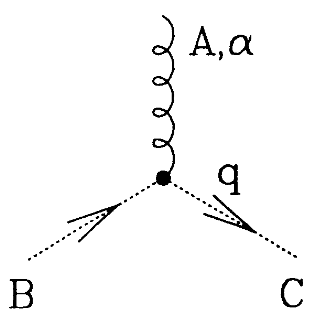


$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

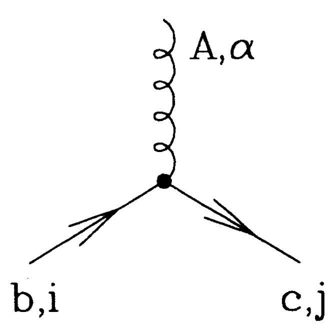
(all momenta incoming, $p+q+r=0$)



$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$g\ f^{ABC}q^{\alpha}$$ gluon-ghost vertex



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$ gluon-quark vertex

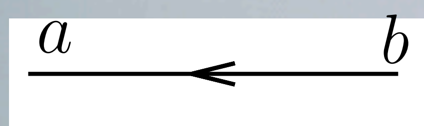
conventions from
Ellis, Stirling, Webber
QCD and Collider Physics

Colour Algebra

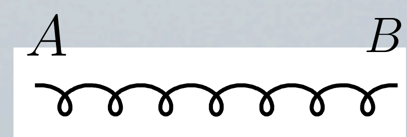
generators of $SU(N_c)$:

$N_c^2 - 1$ hermitean traceless matrices $(t^A)_{ab}$ (fundamental representation)

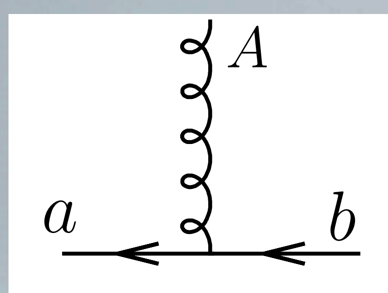
$$[t_A, t_B] = i\, f_{ABC}\, t^C$$



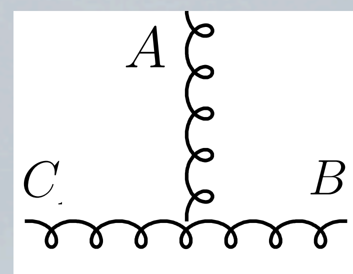
$\stackrel{\text{colour}}{=} \delta_{ab}$



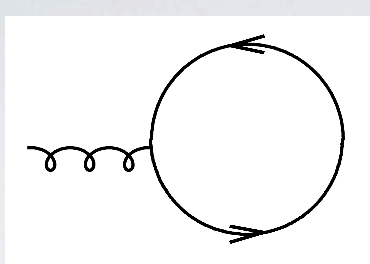
$\stackrel{\text{colour}}{=} \delta^{AB}$



$\stackrel{\text{colour}}{=} (t^A)_{ab}$



$\stackrel{\text{colour}}{=} i\, f^{ABC}$

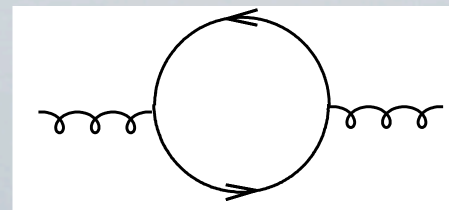
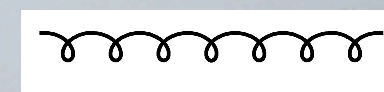


$$Tr(t^A) = 0$$

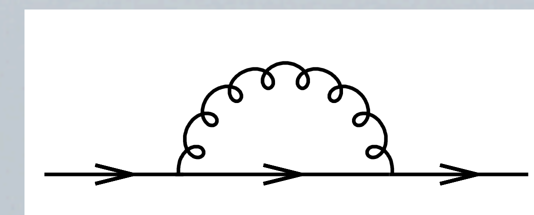
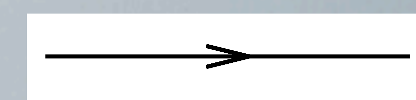
Colour Algebra

some pictorial identities:

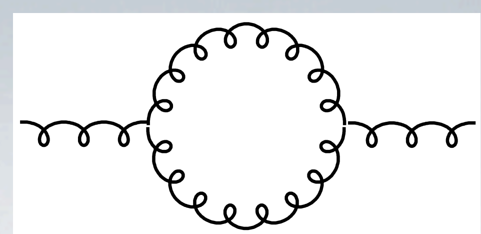
$$Tr(t^A t^B) = T_R \delta^{AB} \;,\; T_R = \frac{1}{2}$$

 $\overset{\text{colour}}{=}$ T_R 

$$\sum_A t^A_{ac} t^A_{cb} = C_F \, \delta_{ab} \;,\; C_F = \frac{N_c^2 - 1}{2N_c}$$

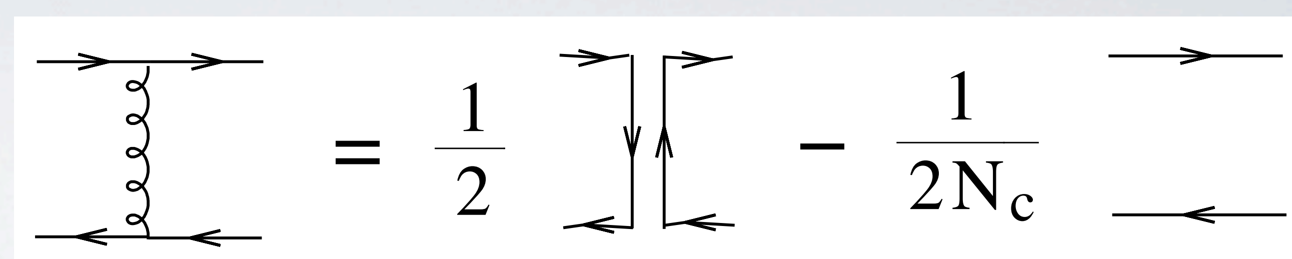
 $\overset{\text{colour}}{=}$ C_F 

$$\sum_{C,D} f^{CDA} f^{CDB} = C_A \, \delta^{AB} \;,\; C_A = N_c$$

 $\overset{\text{colour}}{=}$ C_A 

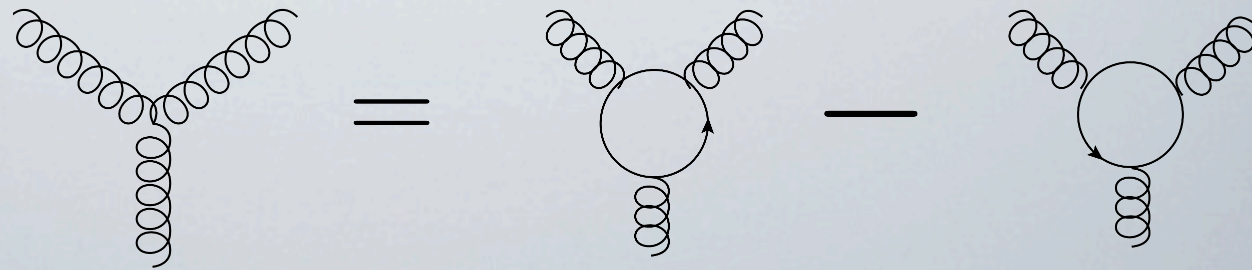
$$(t^A)_{ab}(t^A)_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{2N_c}\delta_{ab}\delta_{cd}$$

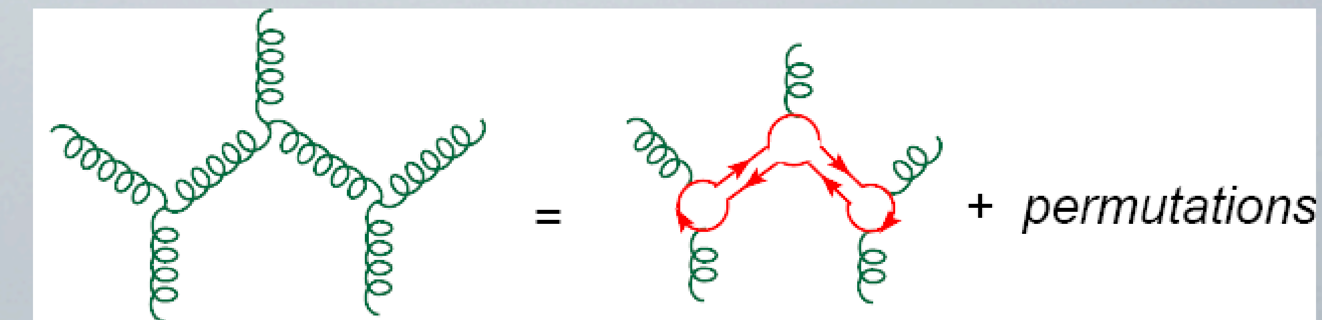
(Fierz identity)



Colour decomposition

$$f^{ABC} = -2\,i\,\mathrm{Tr}([t^A, t^B]\,t^C)$$





we can write every n-gluon tree graph colour factor as a sum of traces of matrices: $Tr(t^{A_1}t^{A_2}\cdots t^{A_n})\; + \;\text{all non-cyclic permutations}$

similarly $\quad q\bar{q}gggg\ldots \Rightarrow Tr(t^{A_1}t^{A_2}\cdots t^{A_n})_{ab}\; + \;\text{permutations}$

$$\mathcal{M}_n^{\text{tree}}(\{p_i, a_i, h_i\}) = g^{n-2}Tr(t^{A_1}t^{A_2}\cdots t^{A_n})\, M_n^{\text{tree}}(1^{h_1}, 2^{h_2}\ldots n^{h_n}) + \text{all non-cyclic permutations}$$

momenta colour helicities

colour ordered subamplitude, colour factors stripped off

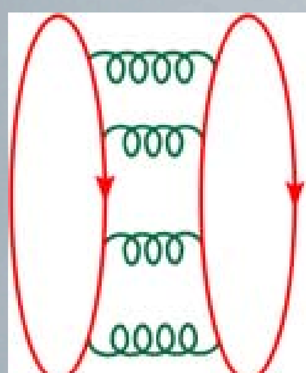
important: as $M_n^{\text{tree}}(1^{h_1}, 2^{h_2}\ldots n^{h_n})$ comes from diagrams with cyclic ordering of external legs, it only has singularities in adjacent invariants $s_{i,i+1} = (p_i + p_{i+1})^2$

(see later)

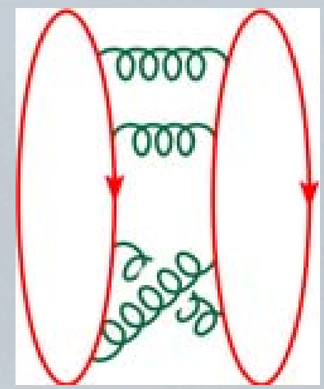
Colour expansion

$$d\sigma^{\text{tree}}(\{p_i, a_i, h_i\}) \sim \sum_{a_i} \sum_{h_i} \left| M_n^{\text{tree}}(\{p_i, a_i, h_i\}) \right|^2$$

insert colour ordered amplitude and perform the colour sum:



$$= N_c^n$$



$$= N_c^n \times \frac{1}{N_c^2}$$

$$d\sigma^{\text{tree}}(\{p_i, a_i, h_i\}) \sim N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} \left| M_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}) \ldots \sigma(n^{h_n})) \right|^2 + \mathcal{O}(N_c^{n-2})$$

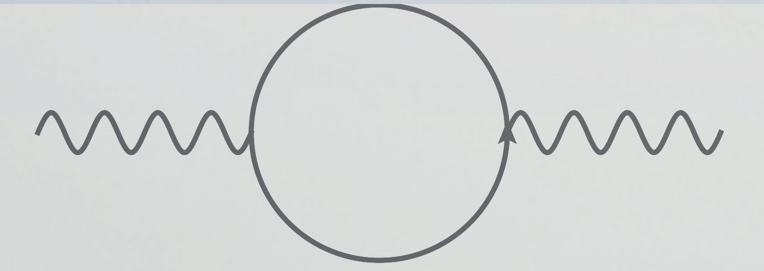
Non-planar topologies are subleading in colour

Note: parton showers usually do not take subleading colour into account

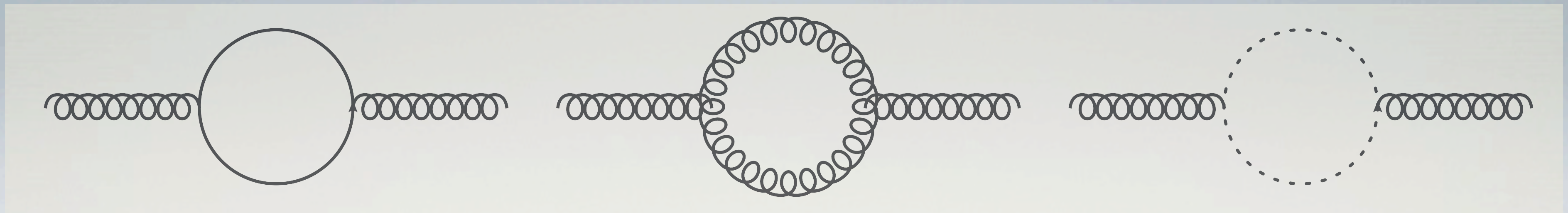
QCD beta-function

... contains one of the most important minus signs in physics !

QED:



QCD:



Roughly speaking, the gluon self couplings reverse the sign of the beta-function.

In more detail ...

QCD beta-function

- consider a dimensionless observable R which can be expanded in $\alpha_s = \frac{g^2}{4\pi}$ and which depends on a single large energy scale Q
- dimensional analysis $\longrightarrow$ R should be independent of Q
- however, R needs UV renormalisation !
- this introduces another mass scale

  μ : the point at which the subtractions of the UV divergences are performed
- therefore R will depend on the ratio Q/μ
- the renormalized coupling α_s will also depend on μ
- as μ is arbitrary, R can not depend on it $\Longrightarrow$

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0$$

QCD beta function

define $\tau = \ln\left(\frac{Q^2}{\mu^2}\right)$, $\beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2}$,

then $\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S} \right] R = 0$ **renormalisation group equation**

solved by running coupling $\alpha_s(Q)$: $\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)}$, $\alpha_S(\mu) \equiv \alpha_S$

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) \, , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)}$$

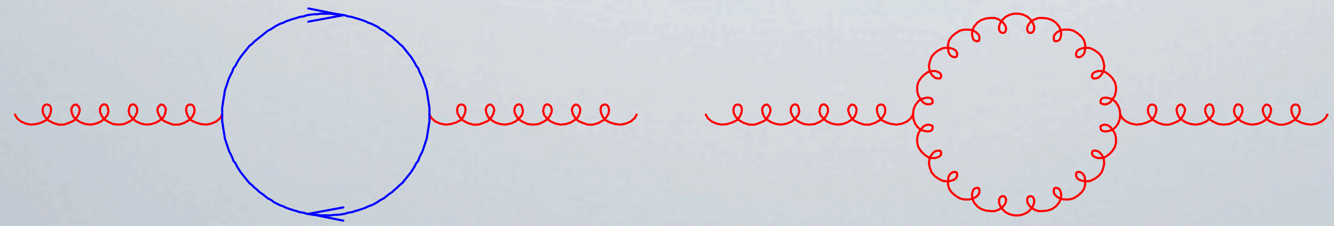
the beta function has the expansion

$$\beta(\alpha_s) = -b_0 \, \alpha_s^2 \, (1 + b_1 \, \alpha_s) + \mathcal{O}(\alpha_s^4)$$

$$b_0 = \frac{1}{12\pi}(11N_c - 2N_f) \, , \; b_1 = \frac{17N_c^2 - 5N_cN_f - 3C_FN_f}{2\pi\,(11N_c - 2N_f)}$$

where N_f is the number of active flavours. Terms up to $\mathcal{O}(\alpha_s^4)$ are known.

asymptotic freedom



(a) (b) (includes ghost loop)

QCD: $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0 \, \ln\left(\frac{Q^2}{\mu^2}\right)} \qquad b_0 = \frac{1}{12\pi}(11N_c - 2N_f)$$

coupling decreases with energy $\Rightarrow$

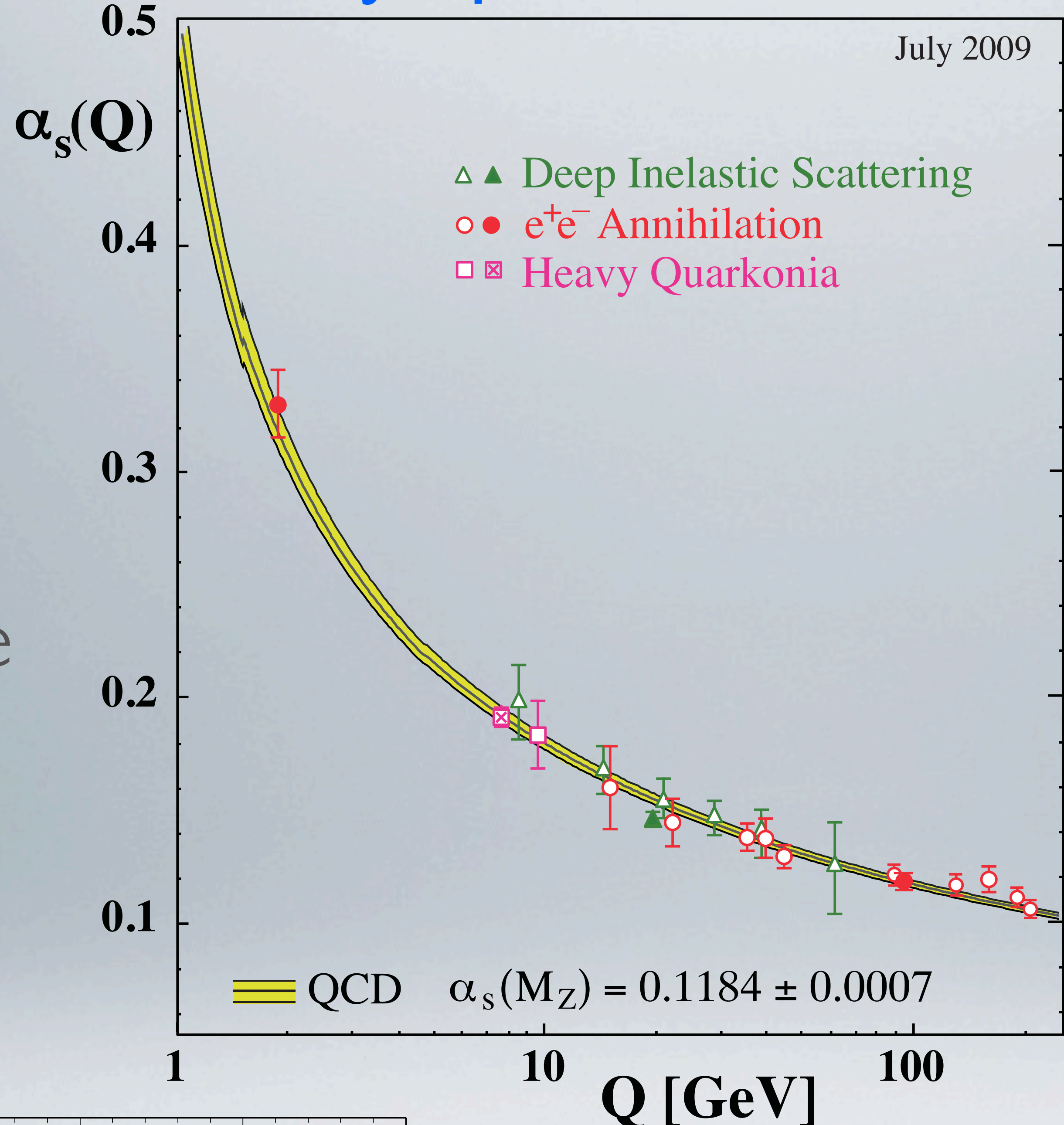
asymptotic freedom

1/137

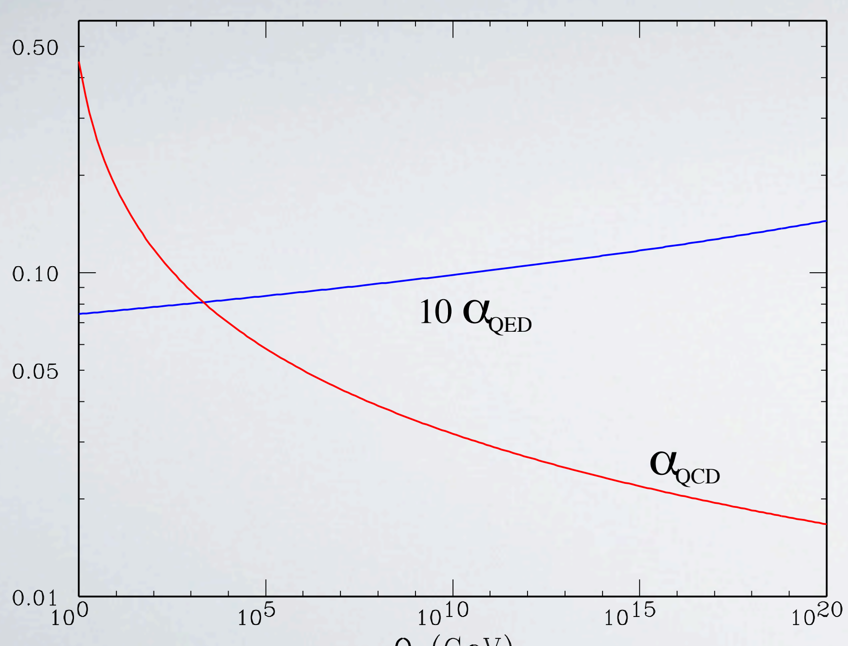
QED: $$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \, \ln\left(\frac{Q^2}{m_e^2}\right)}$$

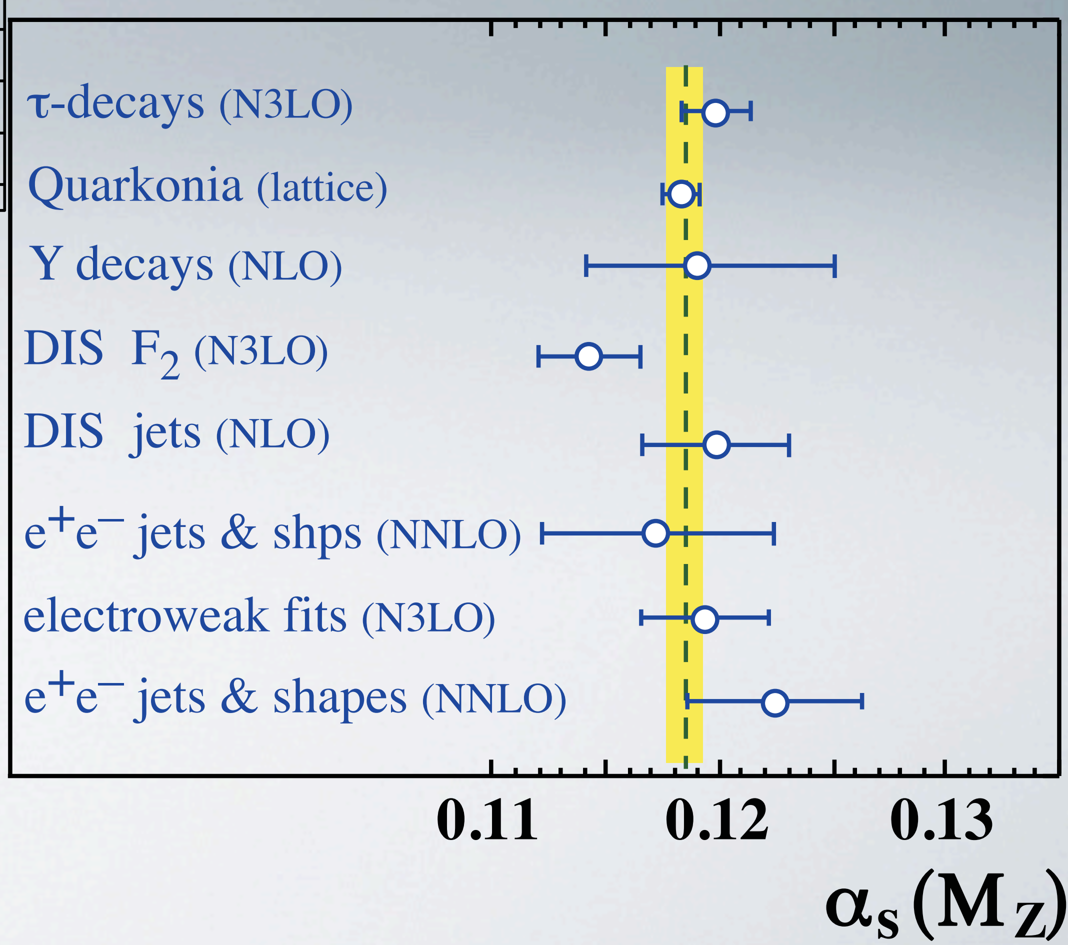
coupling grows with energy

Asymptotic freedom



S. Bethke

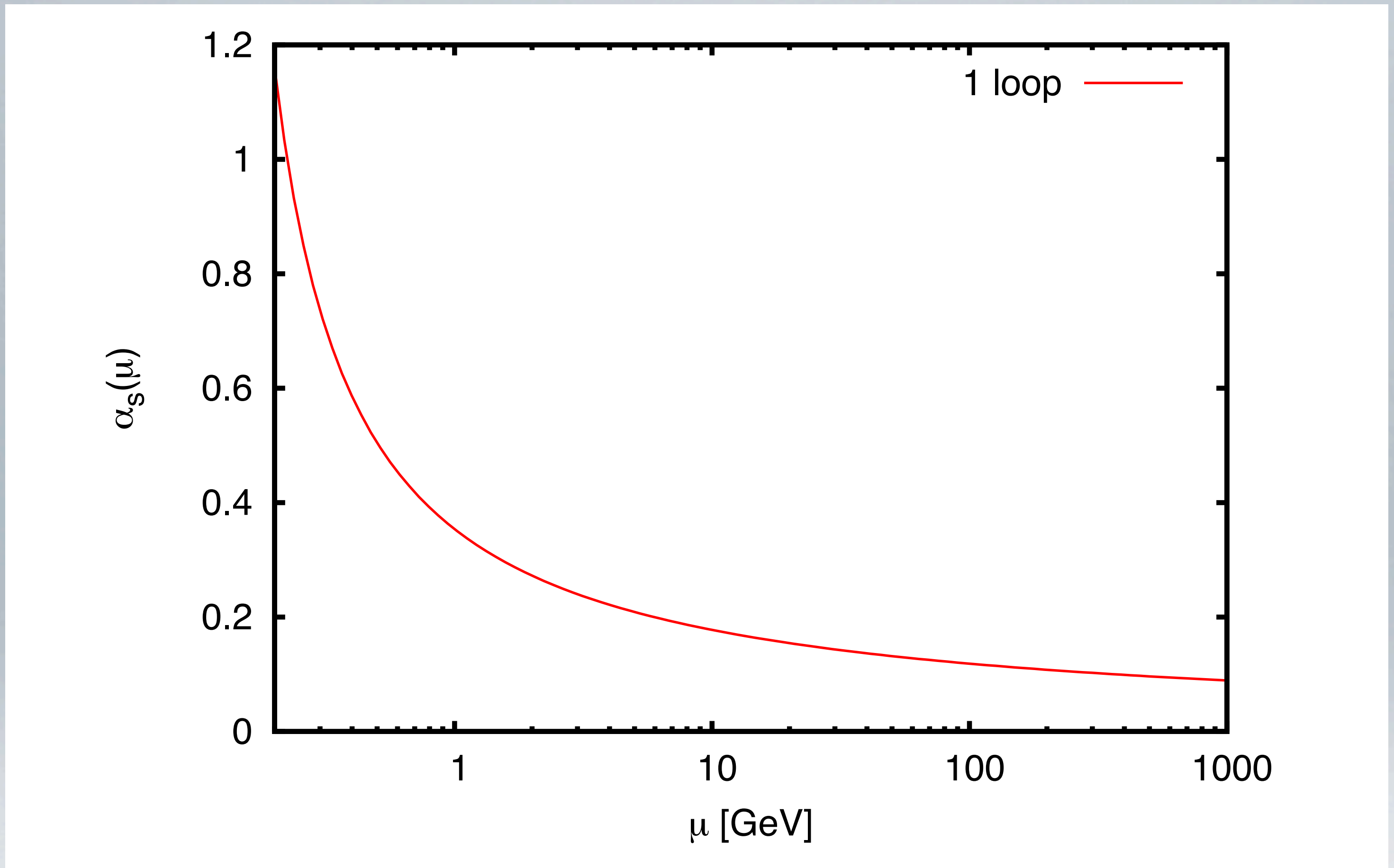




Confinement

at small scales: running coupling diverges, so perturbation theory cannot be applied

$\Rightarrow$ domain of **lattice QCD**



confinement: partons (quarks and gluons) are only found in colour singlet bound states (hadrons)

hadronisation: partons produced in hard scattering processes reorganize themselves to form hadrons

Lambda Parameter

It is useful to define a dimensionful parameter Λ (integration constant)

setting the scale at which the coupling becomes large.

$$\ln\left(\frac{Q^2}{\Lambda^2}\right) = -\int_{\alpha_s(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_s(Q)}^{\infty} \frac{dx}{b_0\, x^2\,(1 + b_1\, x + \ldots)}$$

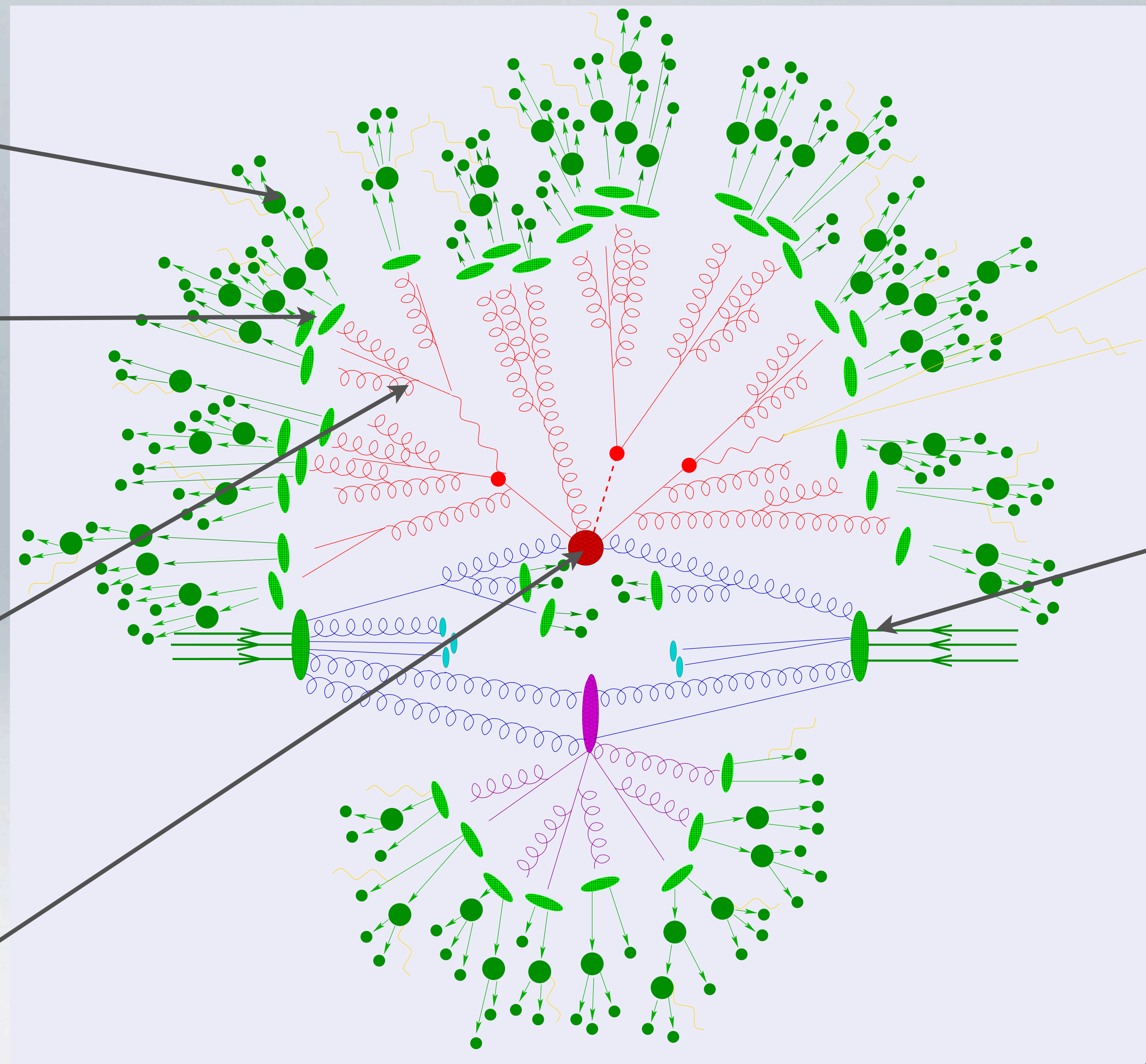
Keeping only $b_0(\mathrm{LO}), b_1(\mathrm{NLO})$

$$\alpha_s(Q) = \frac{1}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)} \quad \text{(LO)} \qquad \alpha_s(Q) = \frac{1}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}\left[1 - \frac{b_1\ \ln\ln\left(\frac{Q^2}{\Lambda^2}\right)}{b_0\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}\right] \quad \text{(NLO)}$$

Note that Λ depends on the number of active flavours N_f .

Comment: as it sets the scale of hadron masses, it is quite an important parameter in particle physics!
Not as famous as the Higgs though . . .

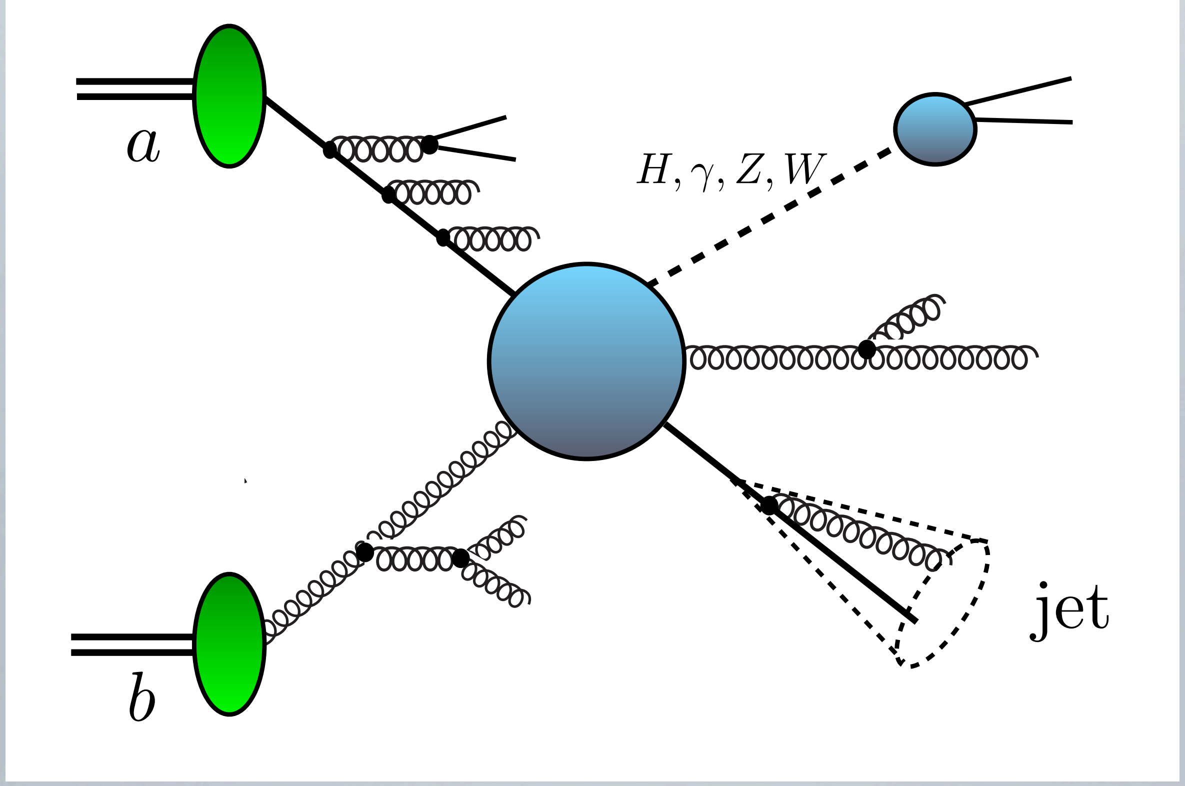
Hadron collider event



artist: Frank Krauss

how can we describe this?

Factorisation: separate hard and soft scales



$$\sigma_{pp\to X} = \sum_{a,b,c} f_a(x_1,\mu_f^2) f_b(x_2,\mu_f^2) \otimes \hat{\sigma}_{ab}\left(p_1,p_2,\frac{Q^2}{\mu_f^2},\frac{Q^2}{\mu_r^2},\alpha_s(\mu_r^2)\right)$$
$$\otimes D_{c\to X}(z,\mu_f^2) + \mathcal{O}(\Lambda/Q)$$

f_a, f_b: parton distribution functions (from fits to data)

$\hat{\sigma}_{ab}$: partonic hard scattering cross section

calculable order by order in perturbation theory

$D_{c\to X}(z,\mu_f^2)$: describing the final state e.g. fragmentation function, jet observable, etc.

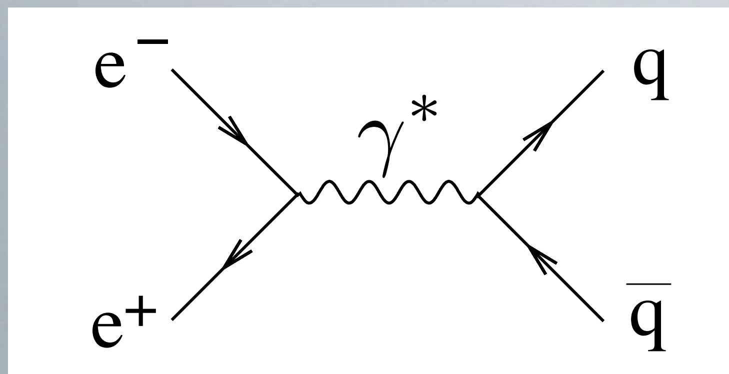
Without factorisation we would be quite lost, but there are still a number of open (QCD) questions, e.g.

- hard scattering cross section:
  - which order in the perturbative expansion is precise enough? (LO, NLO, NNLO ...)
  - is fixed order adequate, or do we need to resum large logarithms?
- how to combine the partonic hard scattering result with a parton shower?
- do we know the parton distribution functions (PDFs) well enough?
- how to model hadronisation?

we will concentrate mostly on the hard scattering cross section in the following

e+e- annihilation

start with simple example:



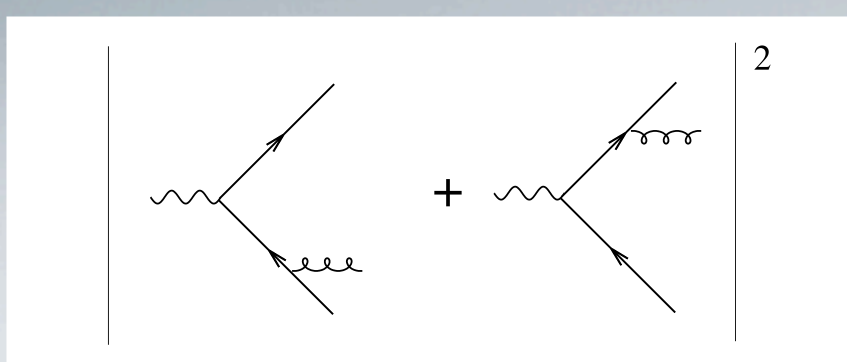
$$R = \frac{\sigma\left(e^+e^- \rightarrow \text{hadrons}\right)}{\sigma\left(e^+e^- \rightarrow \mu^+\mu^-\right)}$$

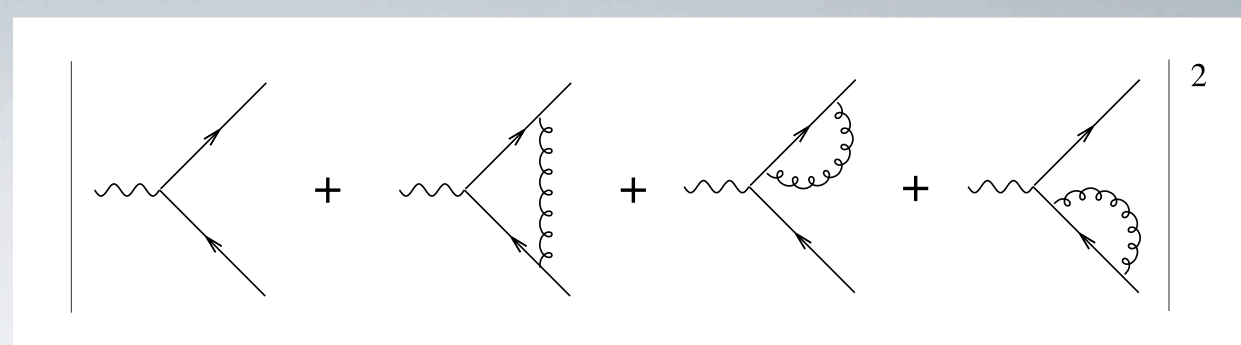
at leading order:

$$R_0 = N_c \sum_q Q_q^2$$

(we will not consider Z exchange here)

what happens if one of the quarks emits a gluon?





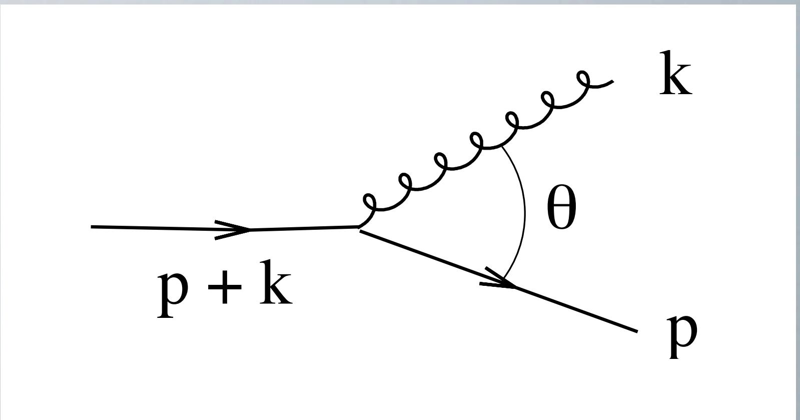
real radiation

virtual corrections

to work consistently at order α_s we need both real and virtual corrections

Infrared singularities

In a gauge theory with massless particles both soft and collinear divergences can occur.



Consider the emission of a gluon from a hard quark:

$$
\begin{aligned}
p &= E\,(1,0,0,1) \\
k &= \omega\,(1,0,\sin\theta,\cos\theta)
\end{aligned}
$$

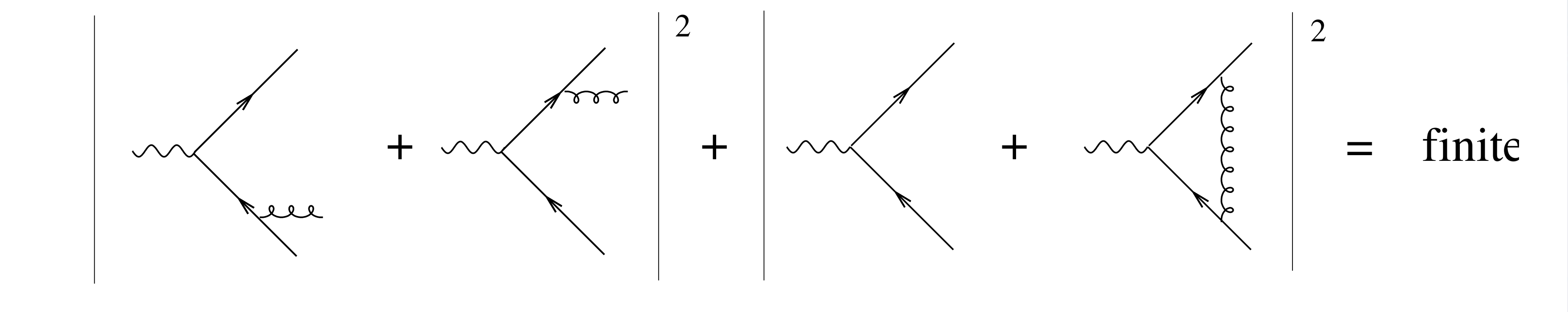
$$(p+k)^2 = 2E\,\omega\,(1-\cos\theta)$$

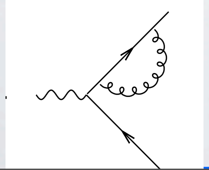
will go to zero if the gluon becomes soft $(\omega \to 0)$

or if quark and gluon become collinear $(\theta \to 0)$

Soft and collinear divergences also occur in virtual corrections.

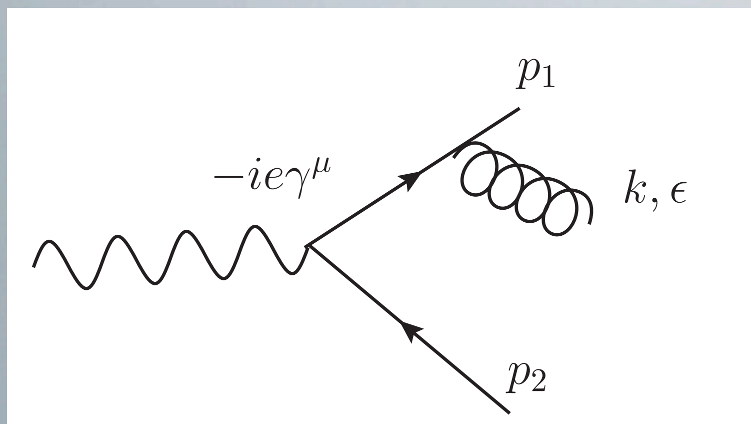
They cancel in an inclusive quantity where "degenerate energy states" are summed over (KLN theorem, see later).



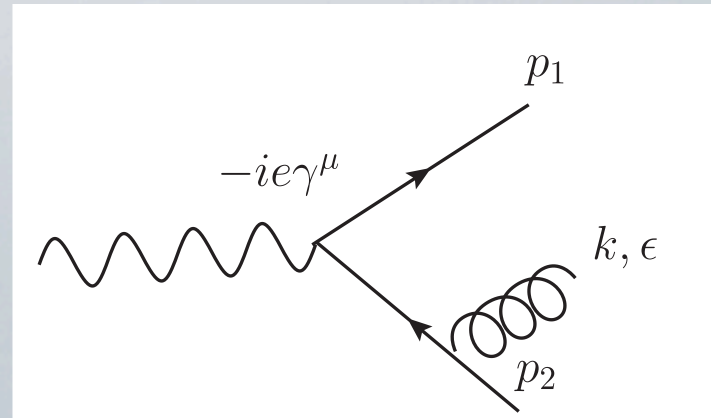
Note:  are UV divergent. These UV singularities cancel with vertex diagram due to Ward Identity

Soft singularities

Consider real emission diagrams in more detail:



$+$



$=$

$$\mathcal{M}^\mu_{q\bar{q}g} = \bar{u}(p_1)\,(-igt^A \not\epsilon)\,\frac{i(\not p_1 + \not k)}{(p_1+k)^2}\,(-ie\gamma^\mu)\,v(p_2)$$

$$+ \quad \bar{u}(p_1)\,(-ie\gamma^\mu)\,\frac{-i(\not p_2 + \not k)}{(p_2+k)^2}\,(-igt^A \not\epsilon)\,v(p_2)$$

If gluon becomes soft: neglect *k* except if it is in denominator:

$$\mathcal{M}^\mu_{q\bar{q}g} \overset{soft}{=} -iegt^A\,\bar{u}(p_1)\,\gamma^\mu \left(\frac{\not\epsilon \not p_1}{2p_1k} - \frac{\not p_2 \not\epsilon}{2p_2k}\right) v(p_2)$$

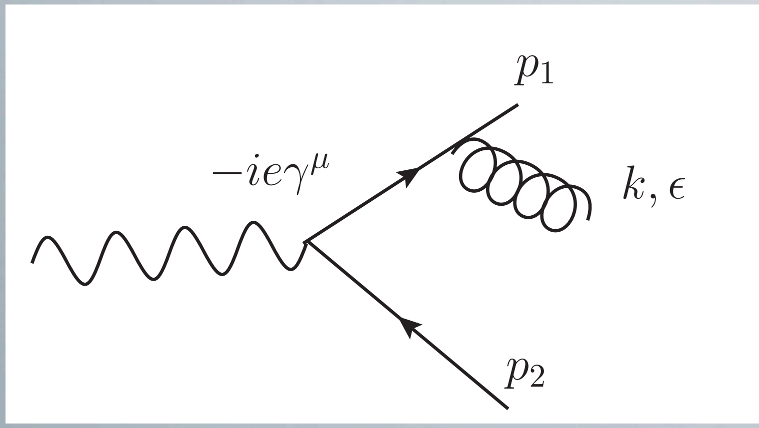
$$|\mathcal{M}_{q\bar{q}g}|^2 \overset{soft}{\rightarrow} |\mathcal{M}_{q\bar{q}}|^2\; g^2 C_F\, \frac{p_1p_2}{(p_1k)(p_2k)}$$

Note: colour will in general **not** factorize in the soft limit

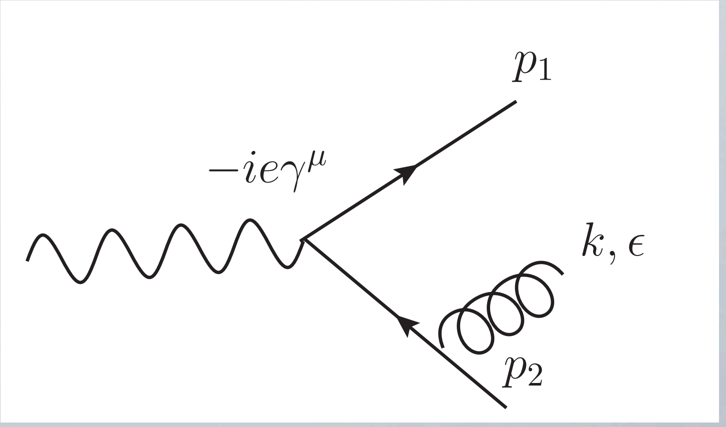
Factorisation into Born matrix element and Eikonal factor

Soft singularities

Consider real emission diagrams in more detail:



$+$



$=$

$$\mathcal{M}^\mu_{q\bar{q}g} = \bar{u}(p_1)\,(-igt^A \not{\epsilon})\,\frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}\,(-ie\gamma^\mu)\,v(p_2)$$
$$+ \quad \bar{u}(p_1)\,(-ie\gamma^\mu)\,\frac{-i(\not{p}_2 + \not{k})}{(p_2+k)^2}\,(-igt^A \not{\epsilon})\,v(p_2)$$

If gluon becomes soft: neglect *k* except if it is in denominator:

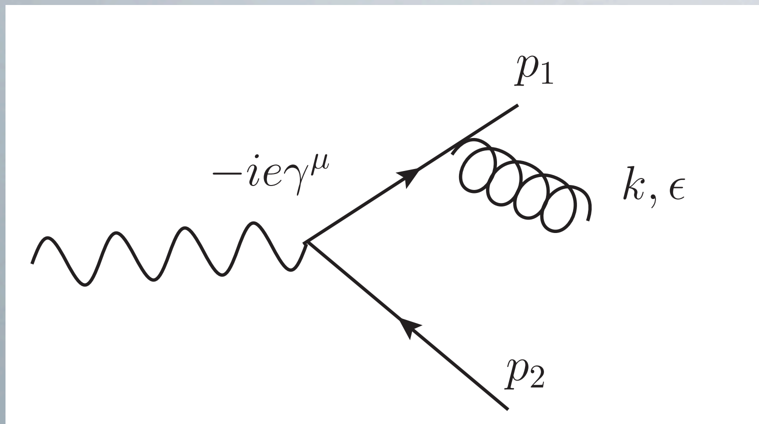
$$\mathcal{M}^\mu_{q\bar{q}g} \overset{soft}{=} -iegt^A\,\bar{u}(p_1)\,\gamma^\mu \left(\frac{\not{\epsilon}\not{p}_1}{2p_1k} - \frac{\not{p}_2\not{\epsilon}}{2p_2k}\right) v(p_2)$$

$$|\mathcal{M}_{q\bar{q}g}|^2 \overset{soft}{\rightarrow} |\mathcal{M}_{q\bar{q}}|^2\; g^2 C_F\, \frac{p_1p_2}{(p_1k)(p_2k)}$$

Note: colour will in general **not** factorize in the soft limit

Factorisation into Born matrix element and Eikonal factor

Collinear singularities



$$(p_1 + k)^2 = 2E\,\omega\,(1 - \cos\theta) \to 0 \quad \text{for} \quad \theta \to 0$$

note: if p1 is a massive particle: $p_1 = E\,(1, 0, 0, v)\,,\ v = \sqrt{1 - \frac{m_1^2}{E^2}}$

$$(p_1 + k)^2 = 2E\omega\,(1 - v\,\cos\theta)$$

no singular denominator for $\theta \to 0$

$\Rightarrow$ only massless particles can lead to a collinear singularity

convenient parametrisation of momenta ("Sudakov parametrisation")

$$\begin{aligned} p_1 &= z\,p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\,\frac{n^\mu}{2p_1 n} \\ k &= (1-z)\,p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\,\frac{n^\mu}{2p_1 n} \\ \Rightarrow 2p_1 k &= -\frac{k_\perp^2}{z(1-z)} \end{aligned}$$

p^μ collinear direction

n^μ light-like auxiliary vector

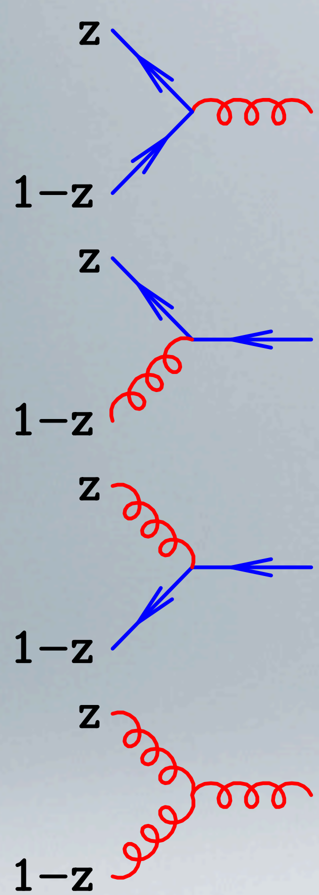
$k_\perp p = k_\perp n = 0 \qquad z = \frac{E_1}{E_1 + E_g}$

$$|\mathcal{M}_1(p_1, k, p_2)|^2 \overset{coll}{\rightarrow} g^2\,\frac{1}{p_1 \cdot k}\,P_{qq}(z)\,|\mathcal{M}_0(p_1 + k, p_2)|^2$$

$P_{qq}(z)$: splitting functions

DGLAP splitting functions

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)



$$
\begin{aligned}
\hat{P}_{qg}(z) &= T_R\left[z^2+(1-z)^2\right], \quad T_R=\frac{1}{2}\,, \\
\hat{P}_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)}\right]\,, \\
\hat{P}_{gq}(z) &= C_F\left[\frac{1+(1-z)^2}{z}\right]\,, \\
\hat{P}_{gg}(z) &= C_A\left[\frac{z}{(1-z)}+\frac{1-z}{z}+z\,(1-z)\right]
\end{aligned}
$$

(details see later in PDF discussion)

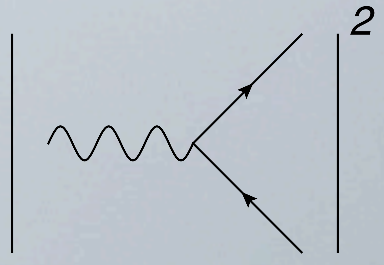
Real radiation matrix element

in general we sum over final state polarizations and colours and average over initial state pols., colours:

$$|\overline{\mathcal{M}}|^2 \;\to\; \overline{\sum}_{\lambda,c}|\mathcal{M}_{\lambda,c}|^2 = \frac{1}{\prod_{\text{initial}} N_{\text{pol}} N_{\text{col}}} \sum_{\text{final pol,col}} |\mathcal{M}_{\lambda,c}|^2$$

at LO, we obtain

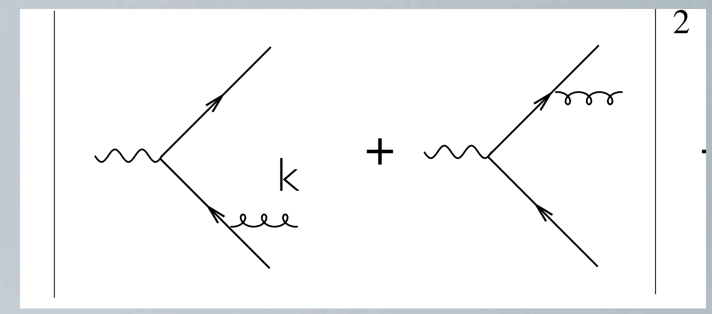
$$|\overline{\mathcal{M}}_0|^2 \;=\; \frac{1}{3}\, 4e^2 Q_q^2 N_c\, s$$



with extra gluon radiation:

$$s_{ij} = (p_i + p_j)^2$$

$$\begin{aligned} p^\gamma &= \sqrt{s}\,(1,0,0,0) \\ p_1 &= E_1\,(1,0,0,1) \\ p_2 &= E_2\,(1,0,\sin\theta,\cos\theta) \\ k &\equiv p_3 = p^\gamma - p_1 - p_2 \end{aligned}$$



$$|\overline{\mathcal{M}}_1|^2 \;=\; |\overline{\mathcal{M}}_0|^2 \frac{2g^2 C_F}{s}\left(\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s\,\frac{s_{12}}{s_{13}s_{23}}\right)$$

defining $\quad x_1 = 2E_1/\sqrt{s}\,,\; x_2 = 2E_1/\sqrt{s}$

$$|\overline{\mathcal{M}}_1|^2 \;=\; |\overline{\mathcal{M}}_0|^2 \frac{2g^2 C_F}{s}\left(\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\right)$$

gluon energy:

$$E_g = \sqrt{s}\,(1 - x_1 - x_2)$$

Real radiation matrix element

$$|\overline{\mathcal{M}}_1|^2 = |\overline{\mathcal{M}}_0|^2 \, \frac{2g^2\, C_F}{s} \left(\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s\, \frac{s_{12}}{s_{13}s_{23}} \right)$$

$$= |\overline{\mathcal{M}}_0|^2 \, \frac{2g^2\, C_F}{s} \left(\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \right)$$

$x_1 = 2E_1/\sqrt{s}\ ,\ x_2 = 2E_1/\sqrt{s}$

$x_1 \to 1$: collinear singularity $p_1 \parallel p_3$, $x_2 \to 1$: collinear singularity $p_2 \parallel p_3$

$x_1 \to 1 - x_2$: soft gluon $E_g = \sqrt{s}\,(1 - x_1 - x_2)$

in these limits the matrix element is singular !

- how can we interpret this ?
- how can we remedy this ?

Cancellation of IR divergences

- interpretation: A quark-antiquark pair with a soft an collinear gluon cannot be distinguished experimentally from just a q qbar pair, so this is not an observable final state.
  Physical final states are hadrons or jets.

$\boxed{\text{KLN Theorem}}$

Kinoshita, Lee, Nauenberg, 60's

Soft and collinear singularities cancel in the sum over degenerate states

- what are degenerate states ?

For example, a quark emitting a soft gluon cannot be distinguished from simply a quark.

Exchange of virtual gluons also leads to IR singularities (same oder in alpha_s).

Singularities cancel between real and virtual corrections.

Cancellation of IR divergences

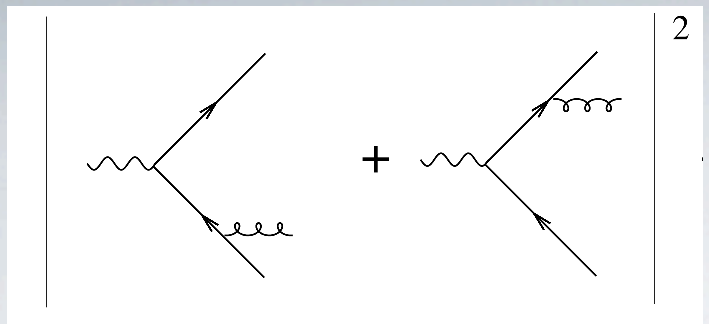
IR singularities cancel between real and virtual corrections.

Really?

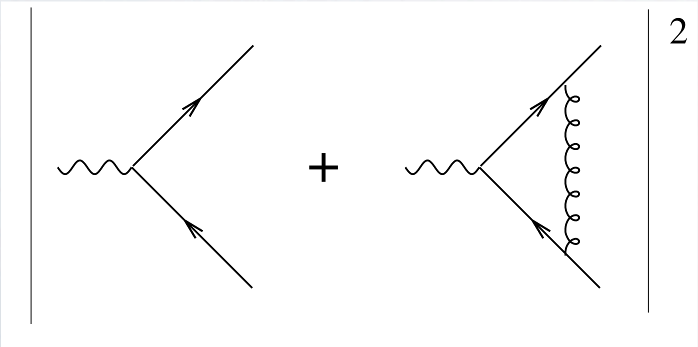
Well, not always: in hadronic collisions, initial state collinear singularities do not cancel, but need to be absorbed into the parton distribution functions, as we cannot sum over “degenerate states” in the proton (see later).

In practice (calculation):

We need to isolate the singularities before we can cancel them, as real and virtual corrections live on different phase spaces.



3-particle phase space



2-particle phase space

Dimensional Regularization

A convenient way to isolate singularities is dimensional regularisation:

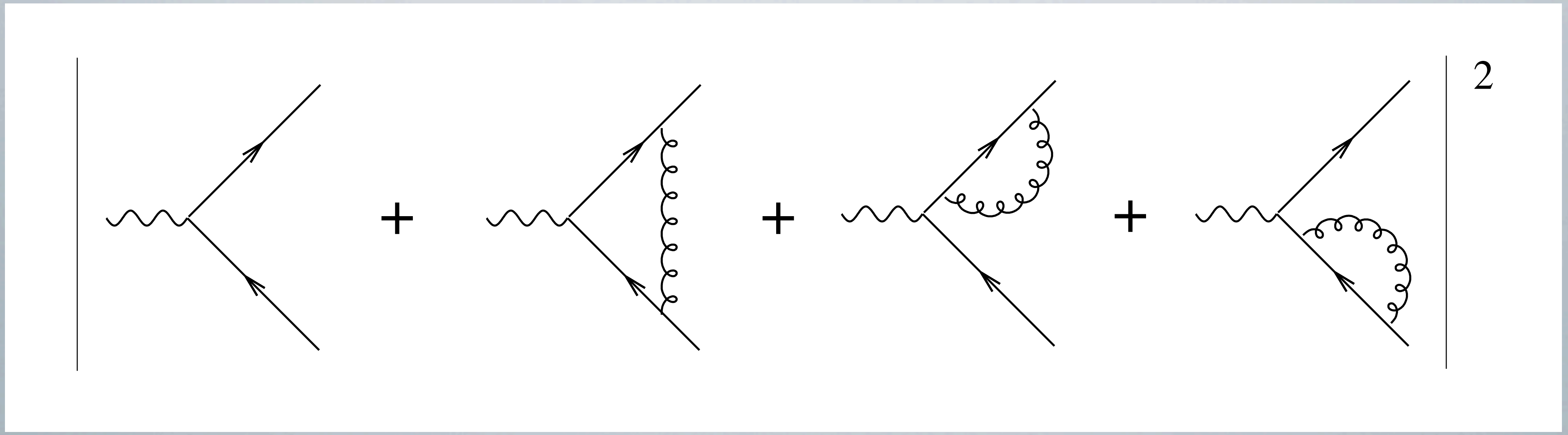
we work in $D = 4 - 2\epsilon$ dimensions.

- regulates both UV and IR divergences
- does not violate gauge invariance
- poles can be isolated in terms of $1/\epsilon^b$

- need phase space integrals in D dimensions
- need integration over virtual loop momenta in D dimensions

$$g^2 \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \longrightarrow g^2 \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D}$$

Virtual corrections



we will not go through the calculation but only quote the result:

$$R^{\text{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right\}$$

Phase space in D dimensions

1 to N particle phase space:

$$Q \to p_1 + \ldots + p_N$$

$$\int d\Phi_N^D = (2\pi)^{N-D(N-1)} \int \prod_{j=1}^{N} d^D p_j \, \delta^+(p_j^2 - m_j^2) \delta^{(D)}\Big(Q - \sum_{i=1}^{N} p_i\Big)$$

In the following consider massless case $p_j^2 = 0$. Use for $i = 1, \ldots, N-1$

$$\begin{aligned} \int d^D p_i \delta^+(p_i^2) &\equiv \int d^D p_i \delta(p_i^2)\theta(E_i) = \int d^{D-1}\vec{p}_i \, dE_i \, \delta(E_i^2 - \vec{p}_i^{\,2})\theta(E_i) \\ &= \frac{1}{2E_i} \int d^{D-1}\vec{p}_i \Big|_{E_i = |\vec{p}_i|} \end{aligned}$$

and eliminate p_N by momentum conservation

$$\Rightarrow \quad \int d\Phi_N^D = (2\pi)^{N-D(N-1)} \, 2^{1-N} \int \prod_{j=1}^{N-1} d^{D-1}\vec{p}_j \, \frac{\Theta(E_j)}{E_j} \, \delta^+([Q - \sum_{i=1}^{N-1} p_i]^2)\Big|_{E_j = |\vec{p}_j|}$$

phase space volume of unit sphere in D dimensions

$$\int d\Omega_{D-1} = V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$$

$$V(D) = \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \ldots \int_0^{\pi} d\theta_{D-1} (\sin\theta_{D-1})^{D-2}$$

Real radiation in D dimensions

1 to 3 particle phase space:

$$\begin{aligned} p^\gamma &= (\sqrt{s}, \vec{0}^{(D-1)}) \\ p_1 &= E_1\,(1, \vec{0}^{(D-2)}, 1) \\ p_2 &= E_2\,(1, \vec{0}^{(D-3)}, \sin\theta, \cos\theta) \\ p_3 &= p^\gamma - p_2 - p_1 \end{aligned}$$

$$x_i = \frac{2p_i \cdot p^\gamma}{s}$$

$$\begin{aligned} d\Phi_{1\to 3} = \frac{1}{4}(2\pi)^{3-2\,D}\; dE_1 dE_2 d\theta\,[E_1 E_2 \sin\theta]^{D-3} d\Omega_{D-2}\; d\Omega_{D-3} \\ = \;\; (2\pi)^{3-2\,D}\,\frac{2^{4-D}}{32}\; s^{D-3}\; d\Omega_{D-2}\; d\Omega_{D-3}\,[(1-x_1)\,(1-x_2)\,(1-x_3)]^{D/2-2} \\ dx_1\, dx_2\, dx_2\, \Theta(1-x_1)\,\Theta(1-x_2)\,\Theta(1-x_3)\delta(2-x_1-x_2-x_3) \end{aligned}$$

$$|\overline{\mathcal{M}}_1|^2 \;\; = \;\; |\overline{\mathcal{M}}_0^{(D)}|^2\,\frac{2g^2\,C_F}{s}\left(\frac{(x_1^2+x_2^2)(1-\epsilon)+2\epsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\epsilon\right)$$

Combine to final result

$$R^{\text{real}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right\}$$

gluon both soft and collinear

remember virtual corrections:

$$R^{\text{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\}$$

KLN theorem at work!

$$R = R^{LO} \times \left\{ 1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

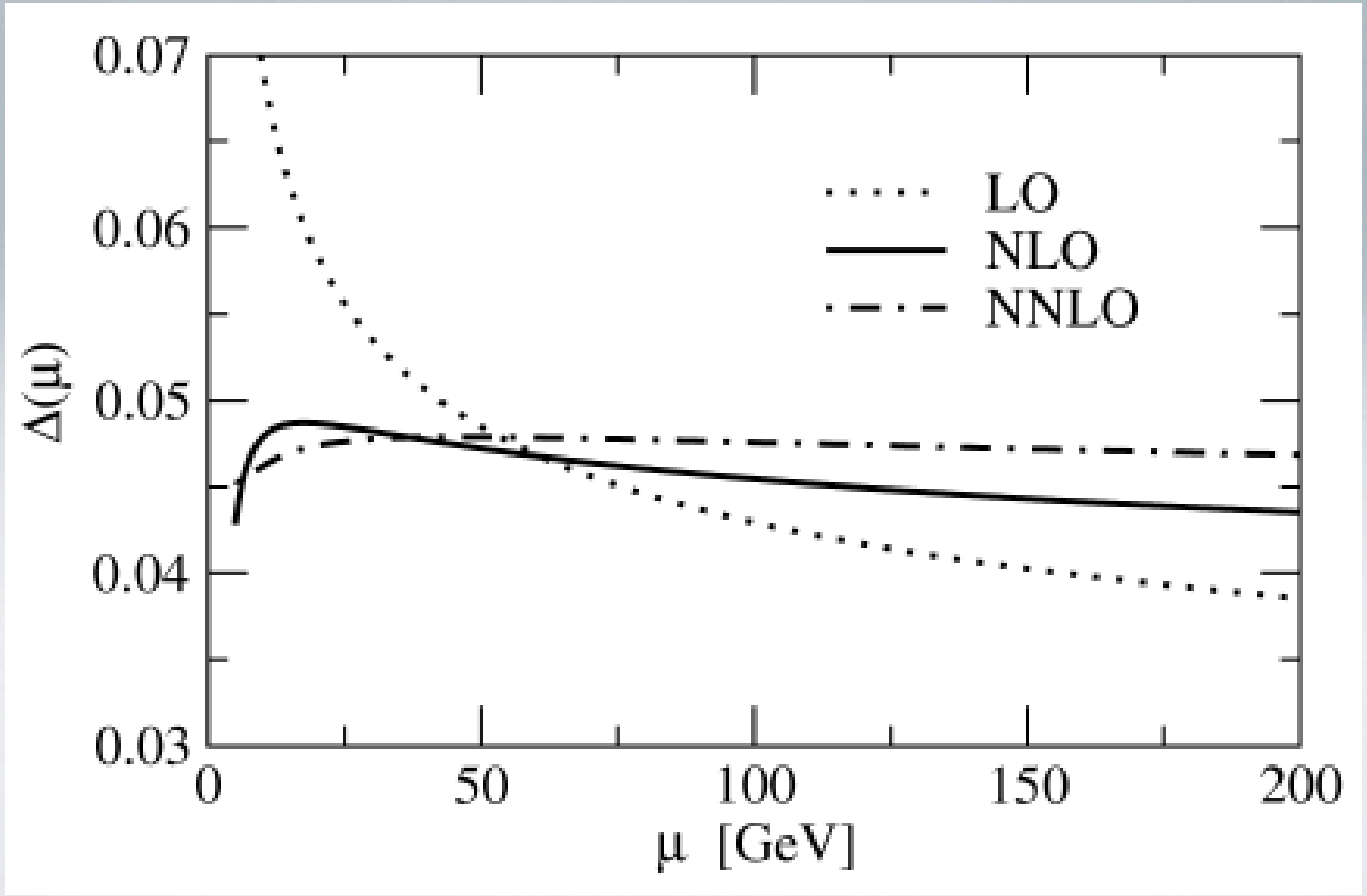
scale dependence

Scale dependence

$$R = R_0 \times \Delta_{QCD} = 3\sum_q Q_q^2 \times \Delta_{QCD} \qquad \Delta_{QCD} = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n=2}^{\infty} C_n \left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

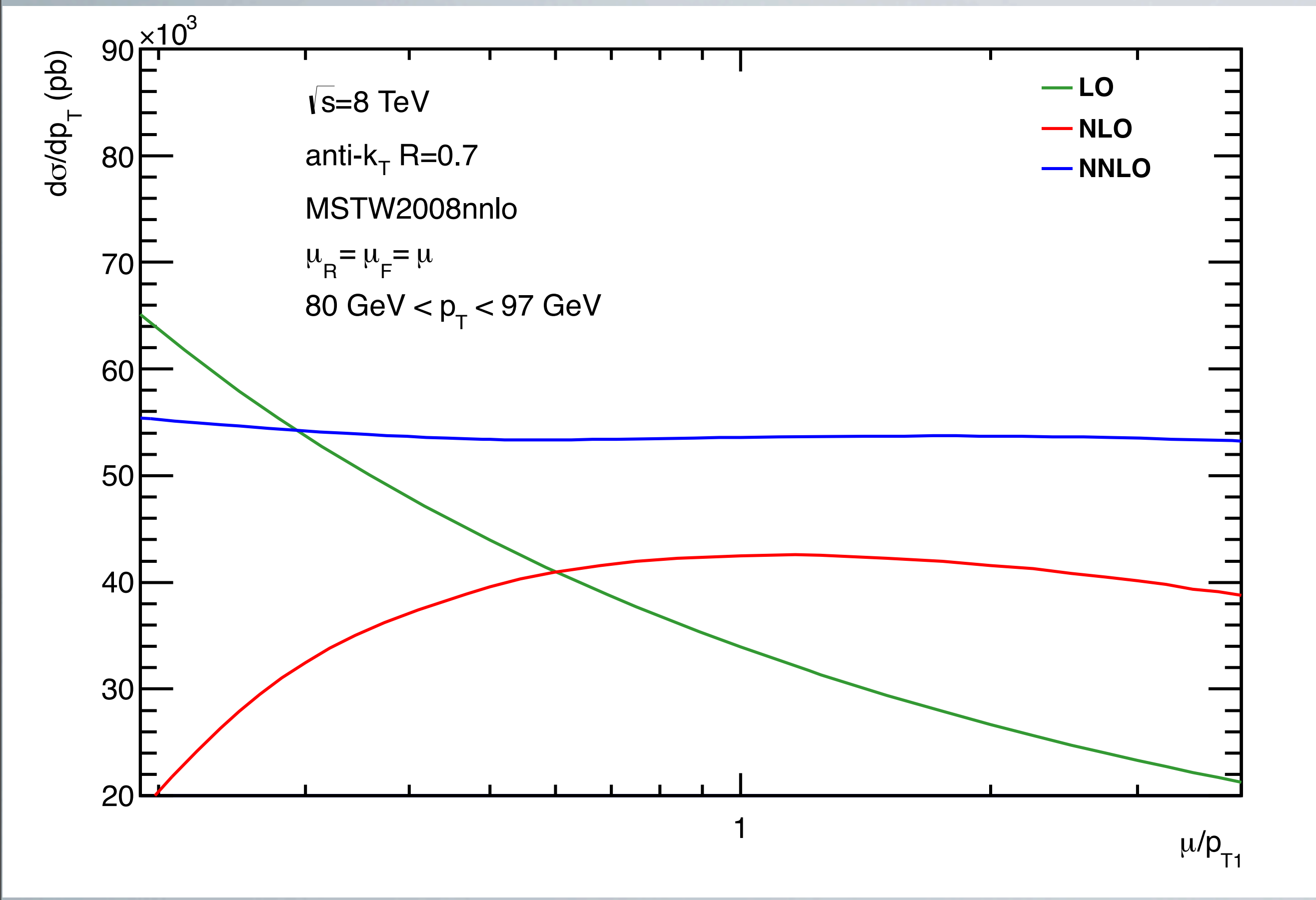
$$\frac{dR}{d\mu} = 0 \Rightarrow \mu^2 \frac{\partial R}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial R}{\partial \alpha_s} = 0$$

The more higher orders are included in R and α_s the more the scale dependence is reduced



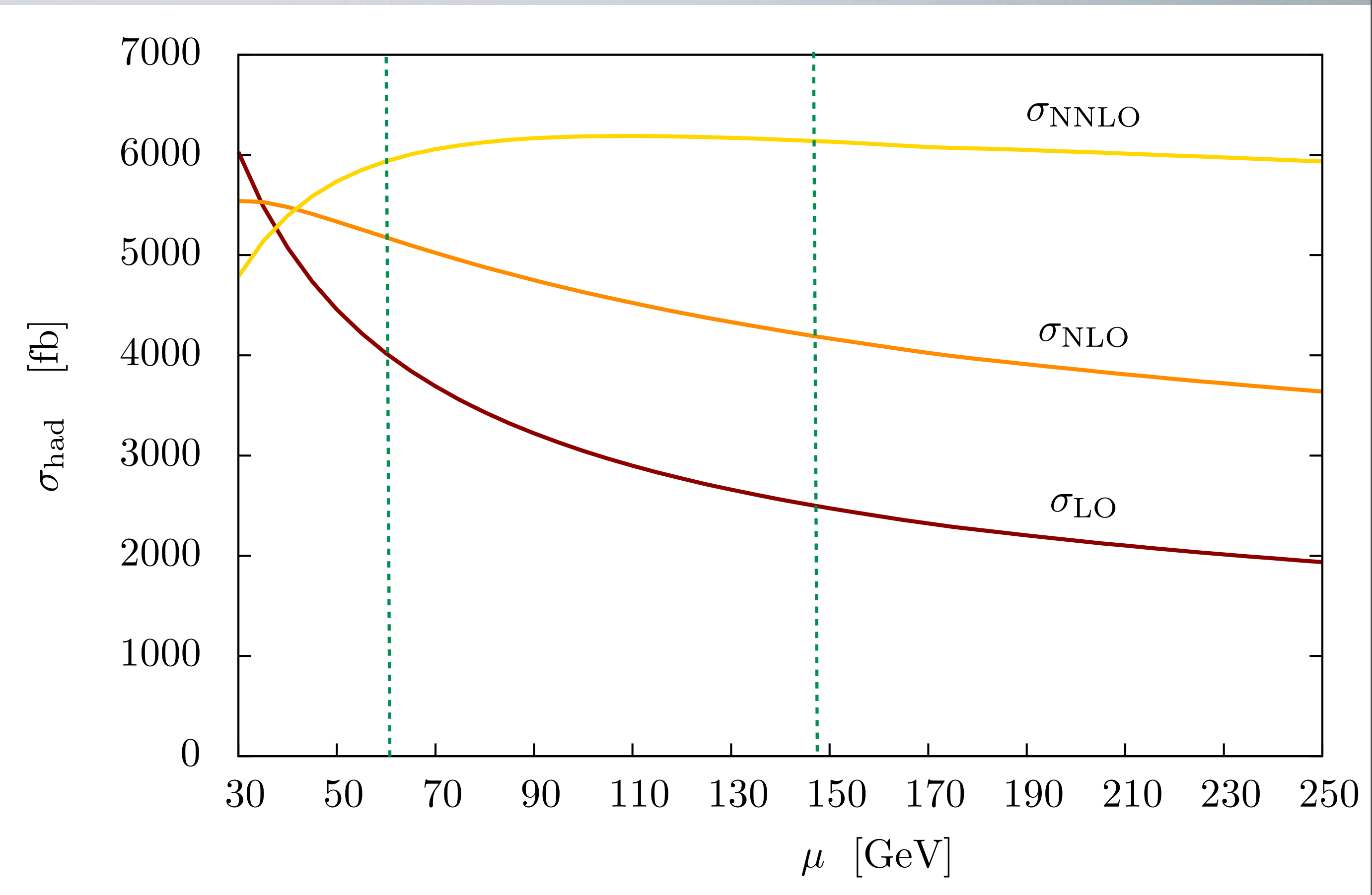
More on scale dependence

dijet production at NNLO (leading colour) :



Gehrmann-De Ridder, Gehrmann, Glover, Pires 2013

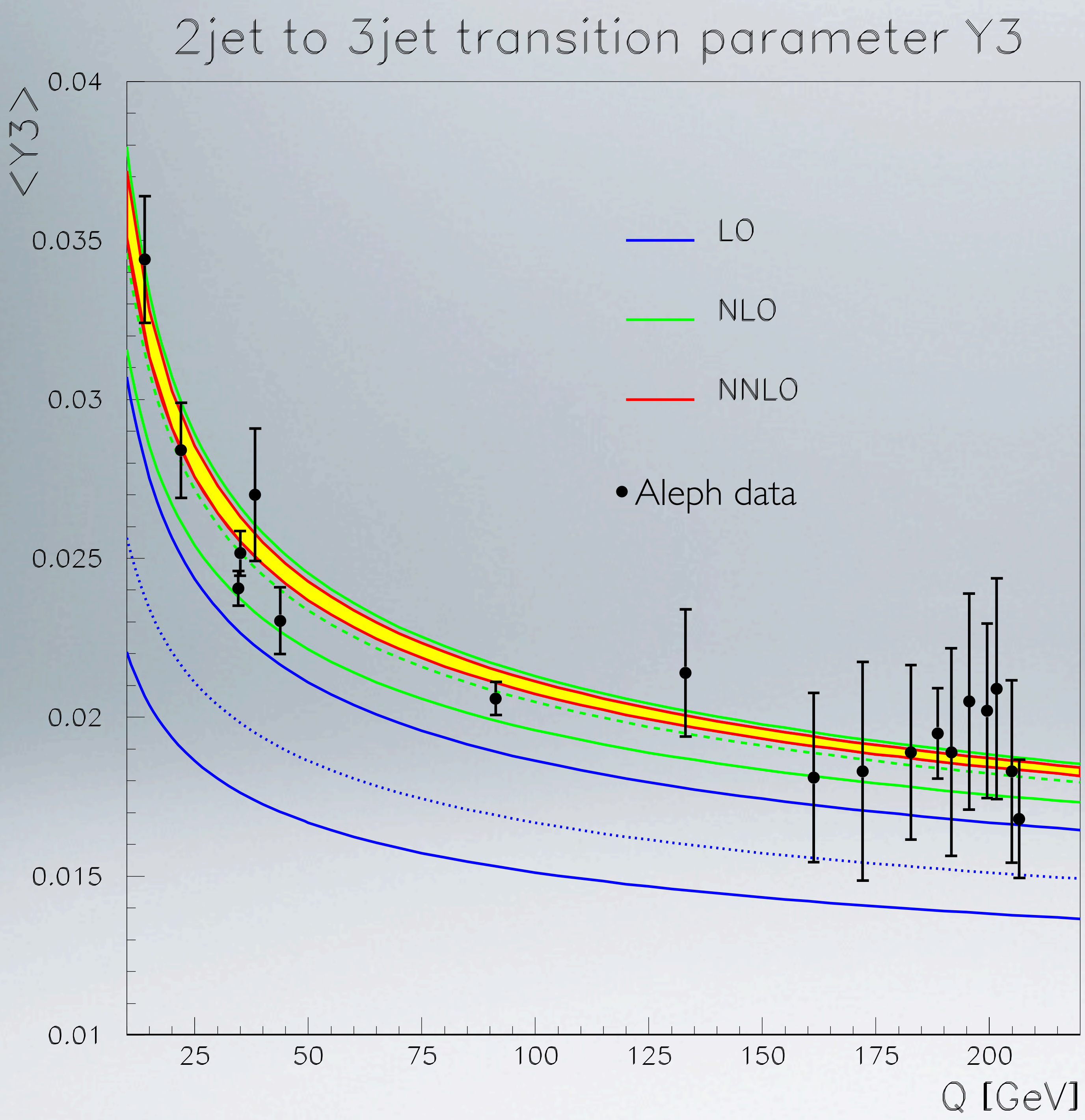
Higgs + jet production (pure gluon only):



Boughezal, Caola, Melnikov, Petriello, Schulze 2013

More on scale dependence

e+e- to 3 jets up to NNLO:



bands from scale variations $M_Z/2 \leq \mu \leq 2\,M_Z$

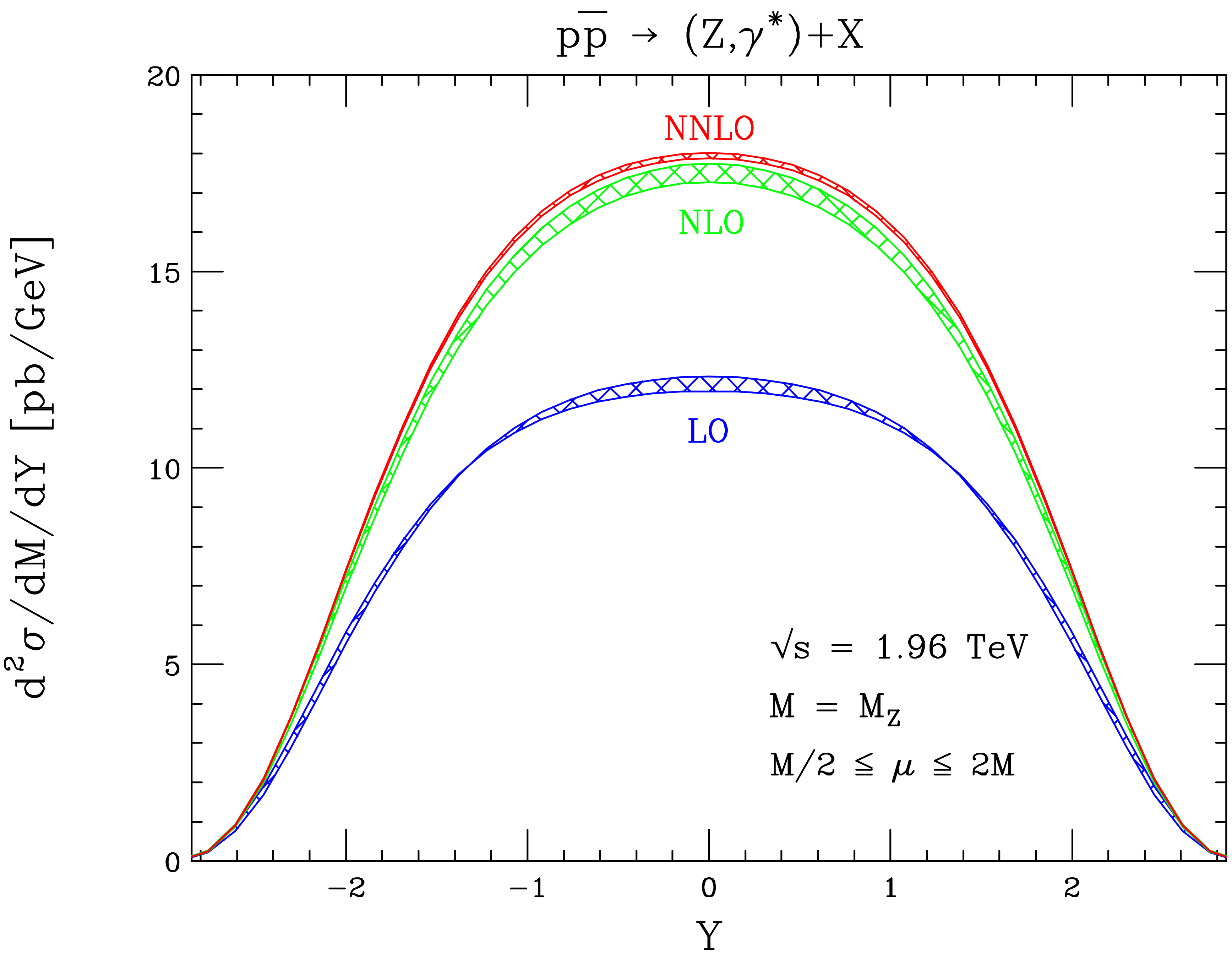
Gehrmann-De Ridder, Gehrmann, Glover, GH, 2008

- reduction of scale uncertainty
- better description of the data
- NNLO, NLO **not** within LO uncertainty band!

$\Rightarrow$

- scale variations of LO result do not necessarily give a realistic error estimate
- choice of a convenient central scale is important (and not straightforward)
- NLO does a reasonable job, LO does not

More on scale dependence

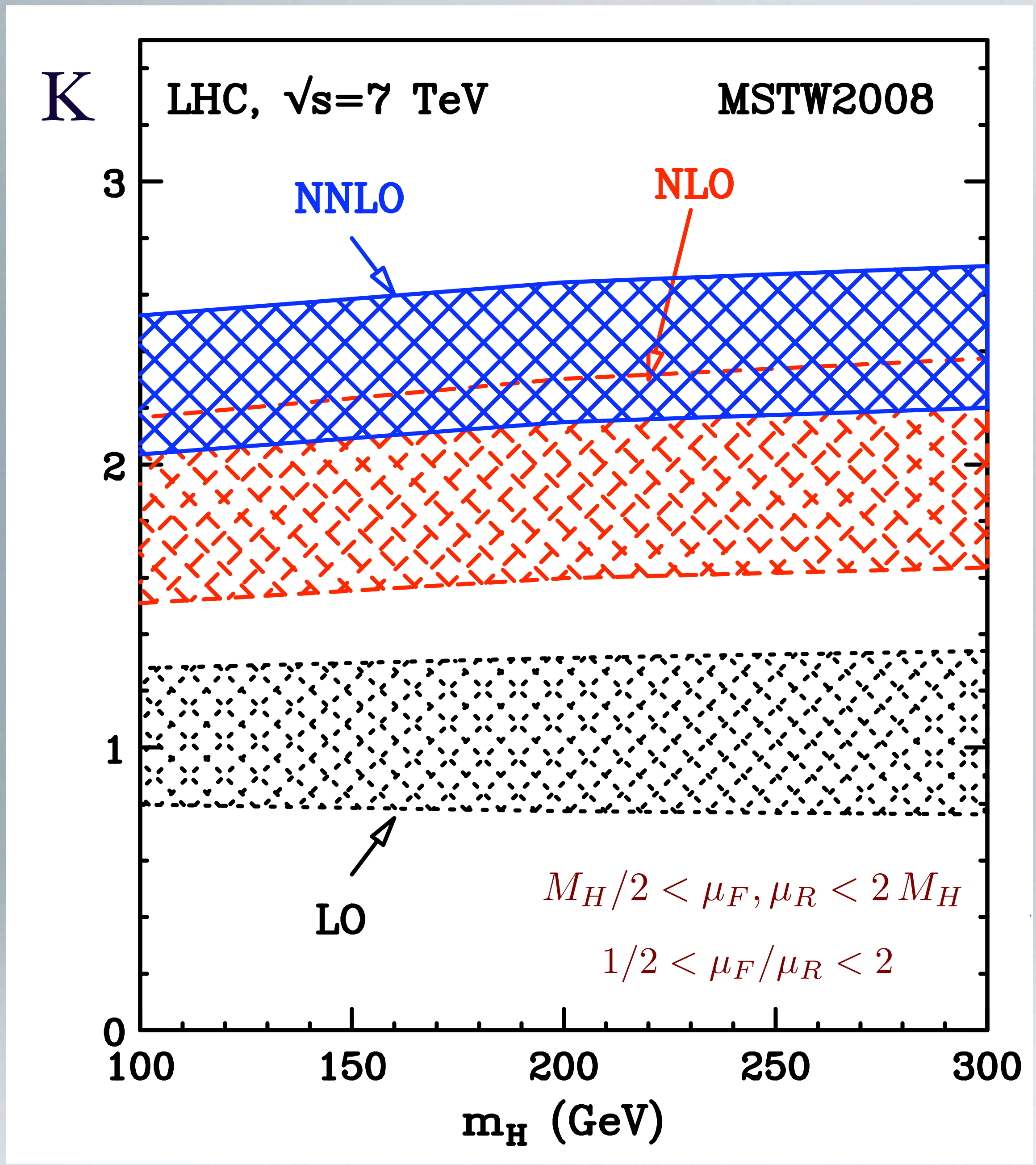


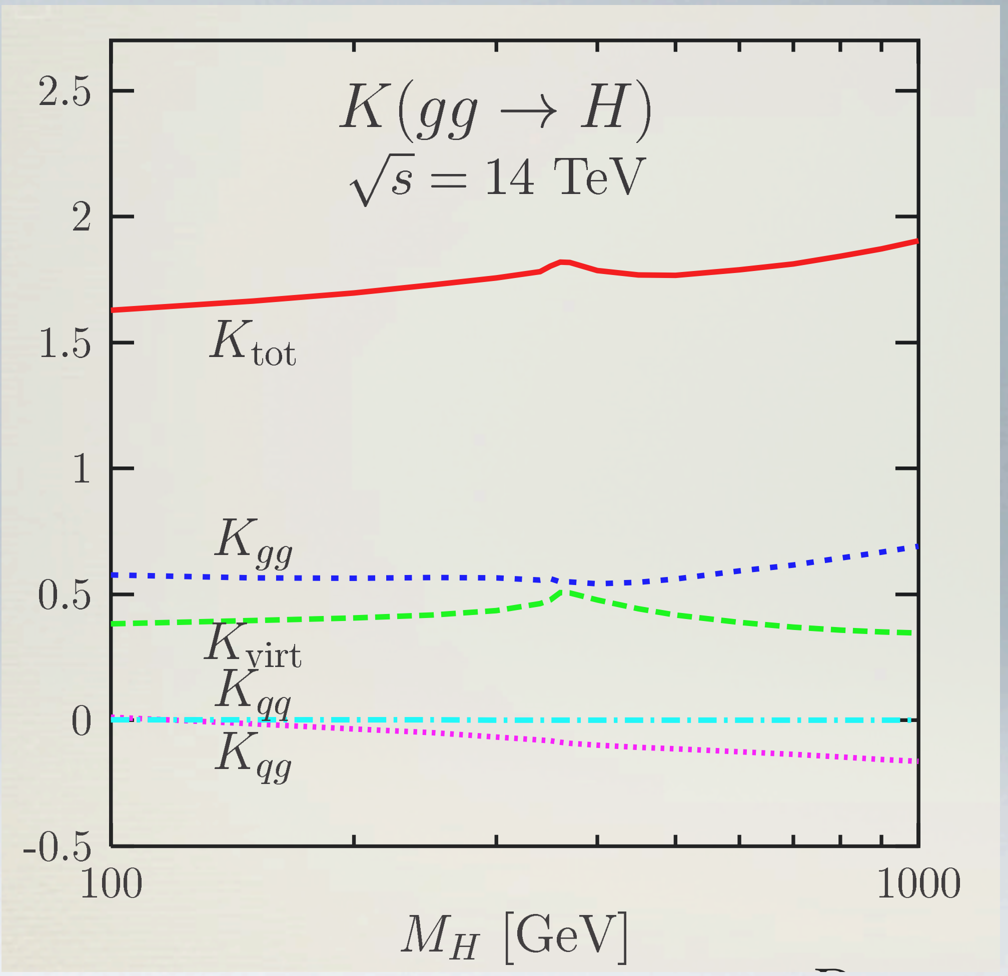
[Anastasiou et al. 04]

K-factors

$$K = \frac{\sigma^{N(N)LO}}{\sigma^{LO}}$$

example Higgs production:





NNLO: Harlander, Kilgore '02

Anastasiou Melnikov '02

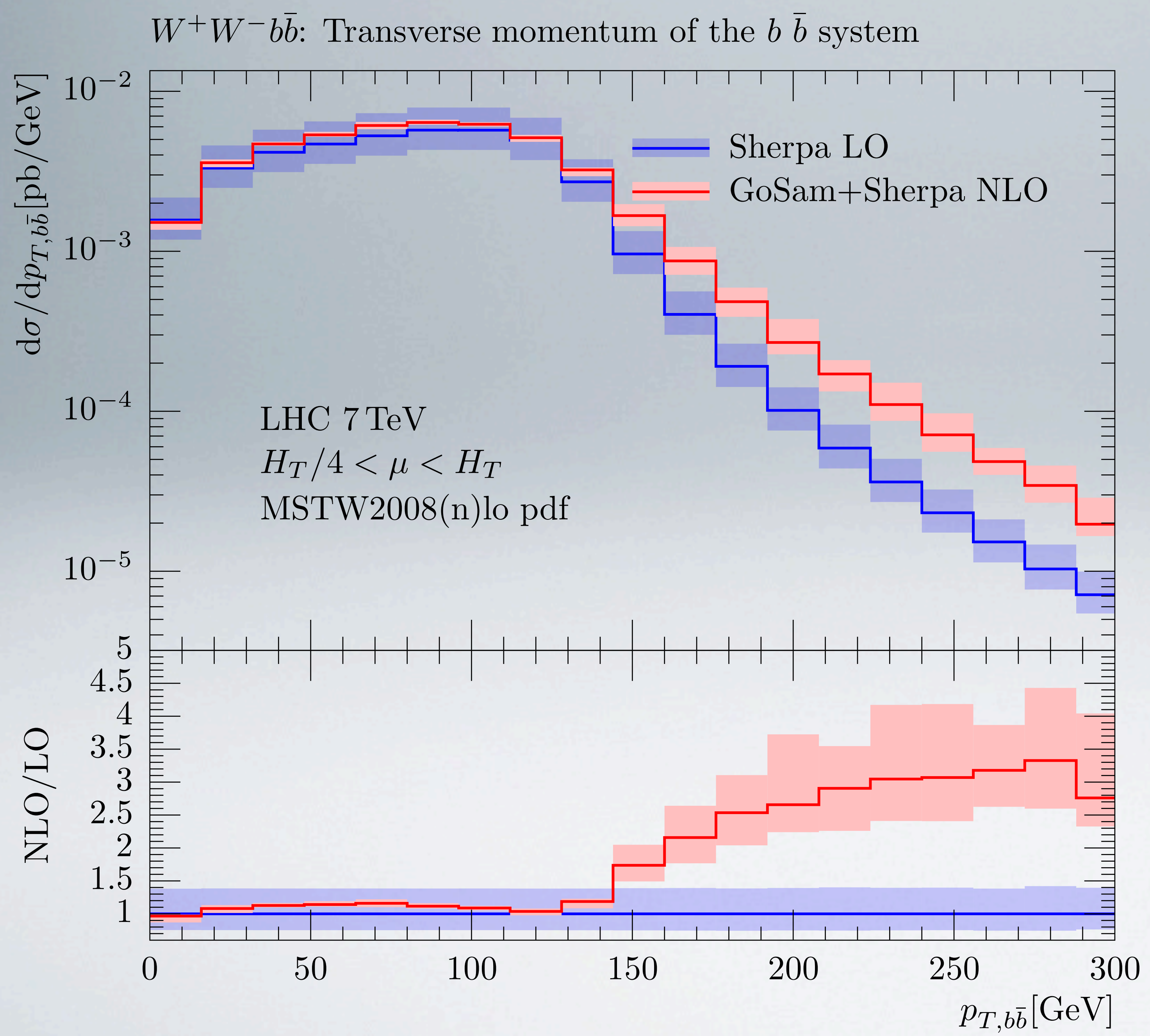
Ravindran, Smith van Neerven, '03

K-factors very large !

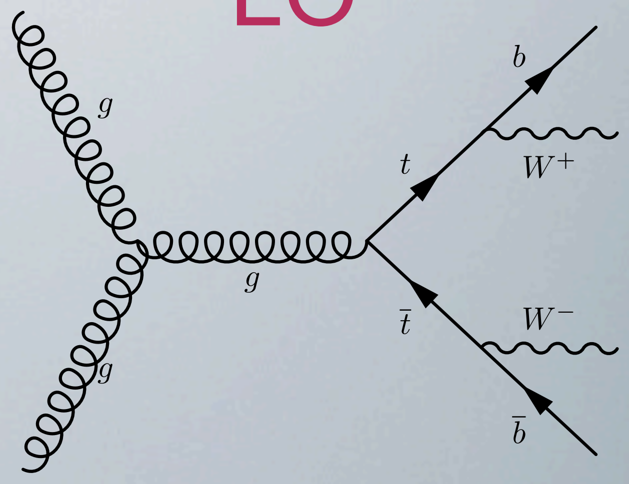
K-factors

Note: K-factors for distributions are in general not constant !

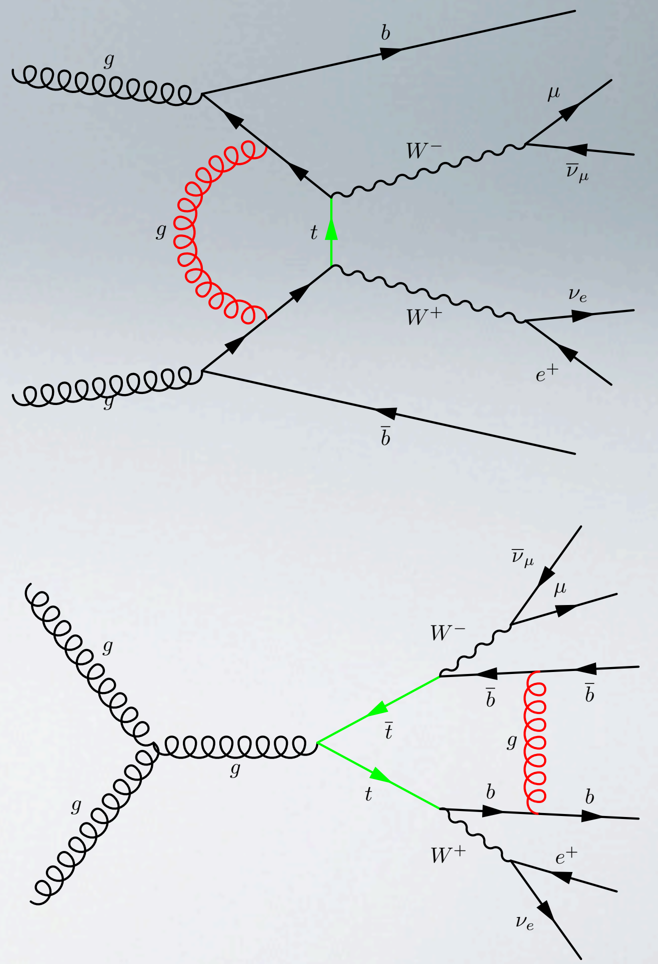
example : $pp \to W^+W^-b\bar{b}$



LO



NLO



GH, J.Schlenk, J.Winter

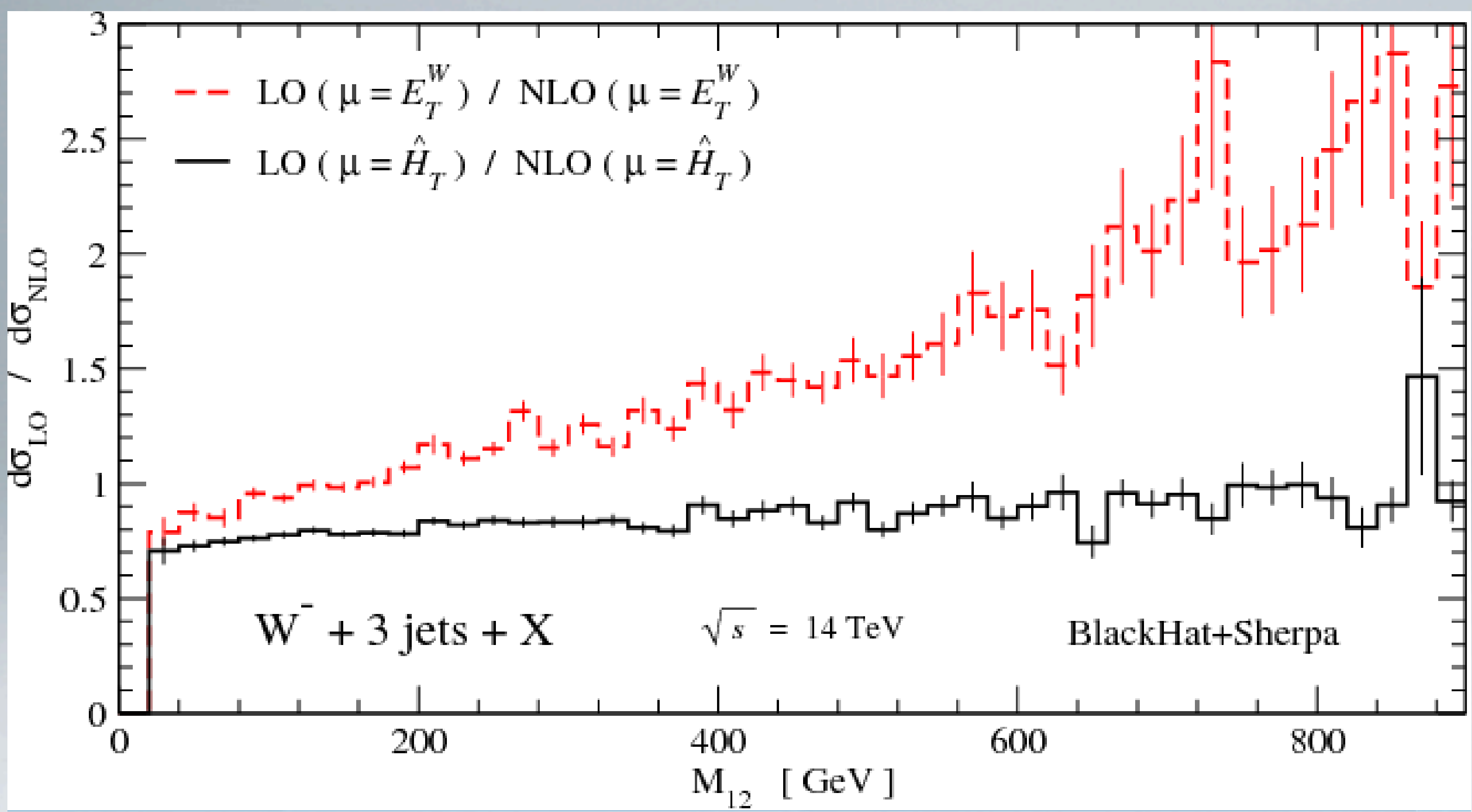
What is a convenient scale choice ?

example from W+3 jets:

possible scale choices:

$$E_T^W = \sqrt{M_W^2 + p_T^2(W)}$$

$$H_T = \sum_{\text{jets}} E_T^{\text{jet}} + E_T^{\text{lepton}} + E_T^{\text{miss}}$$



$$M_{12} = (p_{\text{jet1}} + p_{\text{jet2}})^2$$

C.Berger et al (Blackhat) '09

H_T much better reflects the scale of the hard interaction