Short Introduction to

(Classical) Electromagnetic Theory

(.. and applications to accelerators)

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(http://cern.ch/Werner.Herr/CAS/CAS2013_Chavannes/lectures/em.pdf)

Why electrodynamics ?

- **Accelerator physics relies on electromagnetic concepts:**
  - **Beam dynamics**
  - **Magnets, cavities**
  - **Beam instrumentation**
  - **Powering**
  - **...**

Contents

- **Some mathematics (intuitive, mostly illustrations)**
- **Review of basics and Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**

Small history

- **1785 (Coulomb): Electrostatic field**
- **1820 (Biot-Savart): Field from line current**
- **1826 (Ampere): Field from line current**
- **1831 (Faraday): Law of induction**
- **1835 (Gauss): Flux theorem**
- **1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether**
- **1865 (Maxwell, Lorentz, Heaviside): Lorentz force**
- **1905 (Einstein): Special relativity**

Reading Material

- **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**
- **L. Landau, E. Lifschitz,** *KlassischeFeldtheorie*, **Vol2. (Harri Deutsch, 1997)**
- **W. Greiner,** *Classical Electrodynamics*, **(Springer, February, 22nd, 2009)**
- **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**
- **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

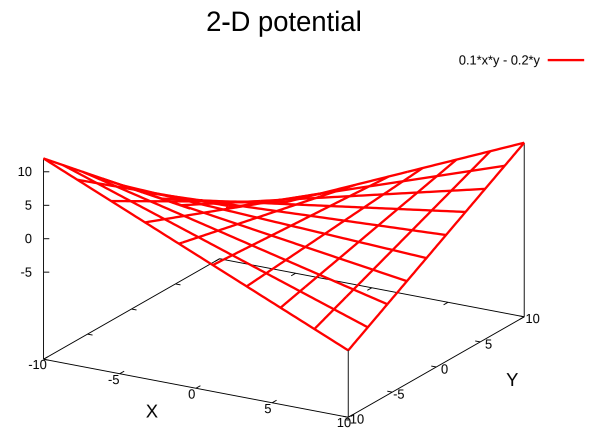
First some mathematics (vectors, potential, calculus ....)

Don't worry ...

- **Not strictly required for understanding**
- **For those interested or a reminder !**
- **I shall cover:**
  - **Potentials and fields**
  - **Calculation on fields (vector calculus)**
  - **Illustrations and examples ...**

(Apologies to mathematicians ...)

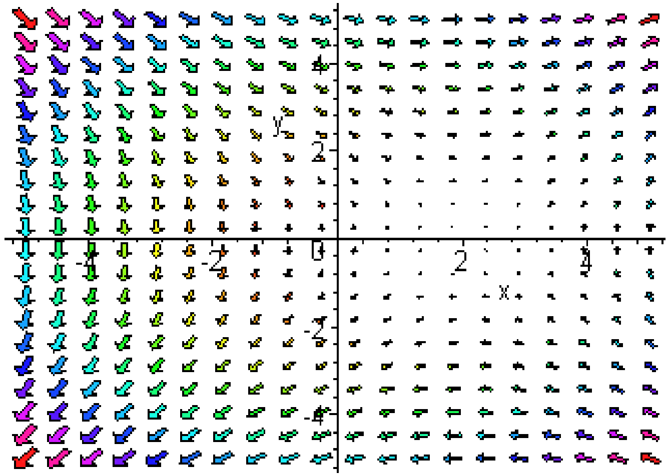
A bit on (scalar) fields (potentials)



- **At each point in space (or plane): a quantity with a value**
- **Described by a scalar $\phi(x, y, z)$ (here in 2-D: $\phi(x, y)$)**
- **Example:** $\phi(x, y) = 0.1x \cdot y - 0.2y$

➡ **We get (for** $x = -4,\ y = 2$**):** $\phi(-4, 2) = -1.2$

A bit on (vector-) fields ...



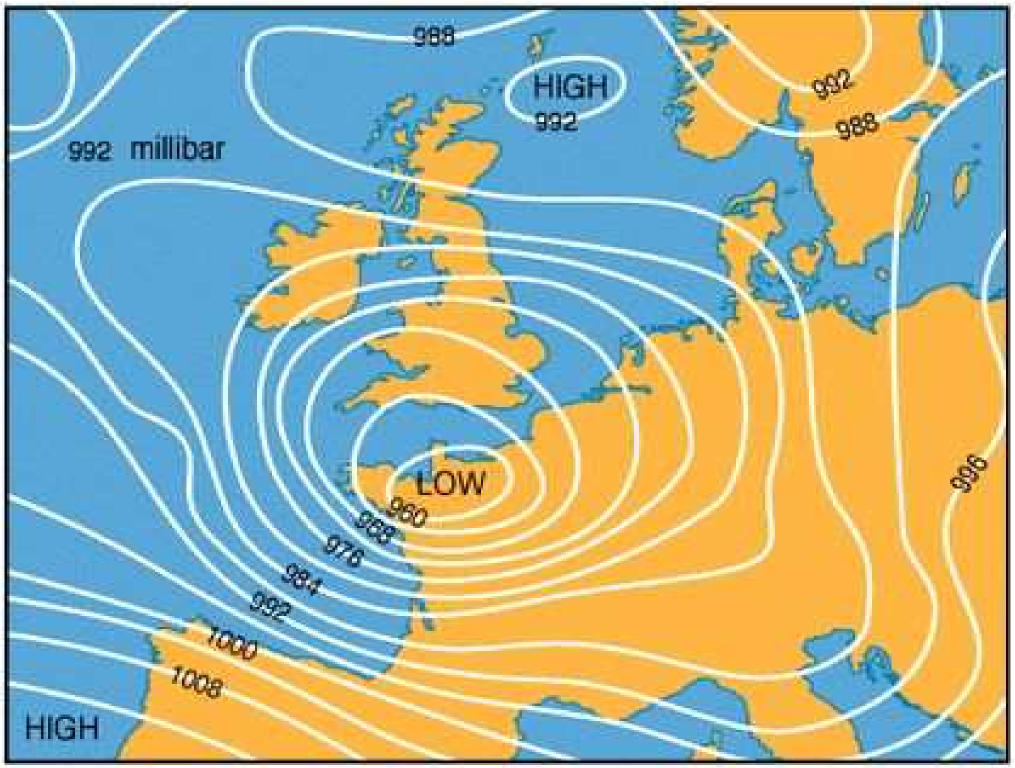
- **At each point in space (or plane): a quantity with a length and direction**
- **Described by a vector** $\vec{F}(x, y, z)$ **(here in 2-D:** $\vec{F}(x, y)$**)**
- **Example:** $\vec{F}(x, y) = (0.1y, 0.1x - 0.2)$

➡ **We get:** $\vec{F}(-4, 2) = (0.2, -0.6)$

Examples:

- **Scalar fields:**
  - **Temperature in a room**
  - **Atmospheric pressure**
  - **Density of molecules in a gas**
  - **Elevation of earth's surface (2D)**
- **Vector fields:**
  - **Speed and direction of wind ..**
  - **Velocity and direction of moving molecules in a gas**
  - **Slope of earth's surface (2D)**

Example: scalar field/potential ...



Example for a scalar field ..

Example: vector field ...

Example for an extreme vector field ..

Vector calculus ...

Scalar fields and vector fields can be related:

To a scalar function $\phi(x, y, z)$ we can apply the gradient which then becomes a vector field $F(x, y, z)$:

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \quad = \quad \vec{F} = (F_1, F_2, F_3)$$

and get a vector. It is a kind of "slope" ! (example: distance between isobars)

Example (2-D):

$$\phi(x, y) = 0.1x \cdot y - 0.2y \quad \rightarrow \quad \nabla\phi = \vec{F}(x, y) = (0.1y, 0.1x - 0.2)$$

Operations on (vector-) fields ...

We can define operations on vectors fields:

Divergence (scalar product of gradient with a vector):

$$\mathsf{div}(\vec{F}) \quad = \quad \nabla \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

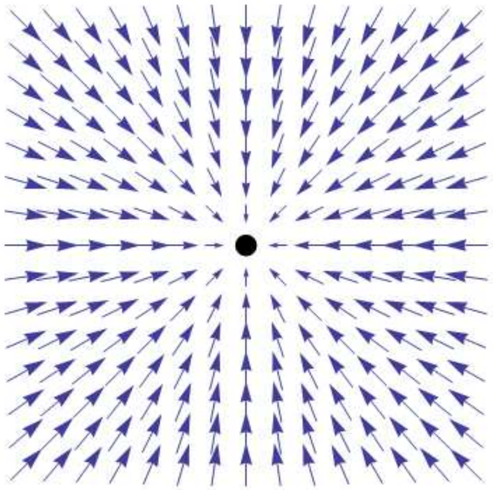
Physical significance: "amount of density", (see later)

Curl (vector product of gradient with a vector):

$$\mathsf{curl}(\vec{F}) \quad = \quad \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

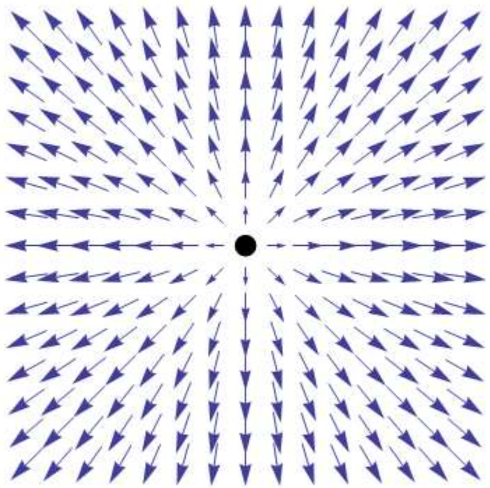
Physical significance: "amount of rotation", (see later)

Divergence of fields ...



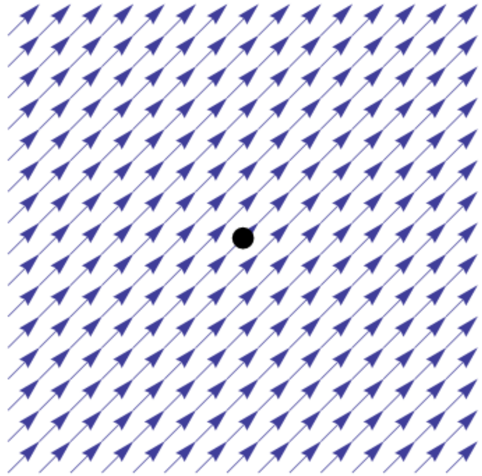
$\nabla \vec{F} < 0$

(sink)



$\nabla \vec{F} > 0$

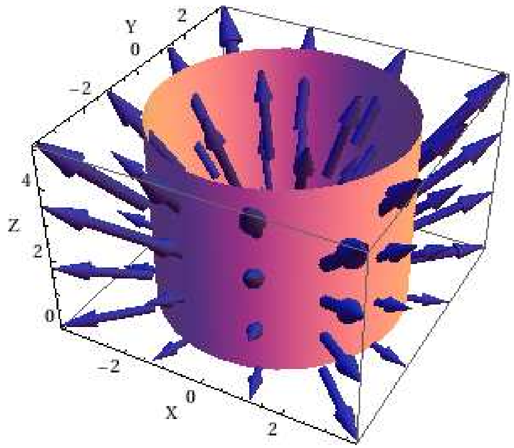
(source)



$\nabla \vec{F} = 0$

(fluid)

Integration of (vector-) fields ...

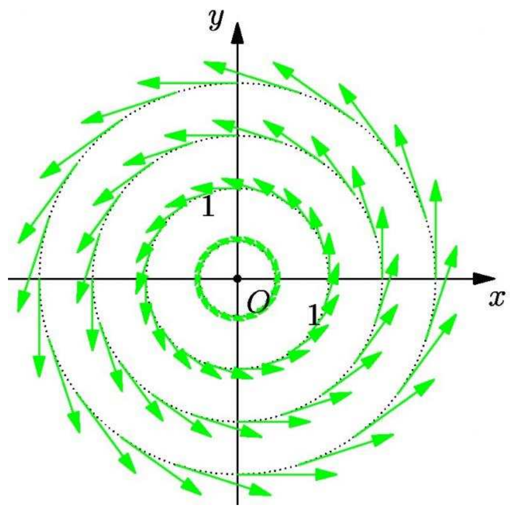


Surface integrals: integrate field vectors passing (perpendicular) through a surface S:

$$\int\int_S \vec{F} \cdot d\vec{S}$$

"count" number of field lines through the surface ...

Curl of fields ...



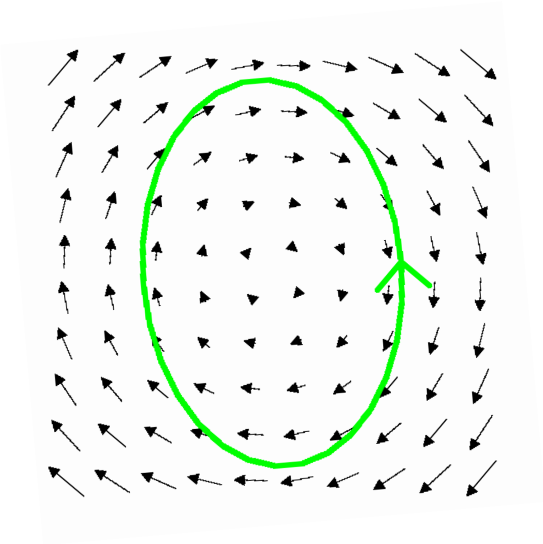
Here we have a field:

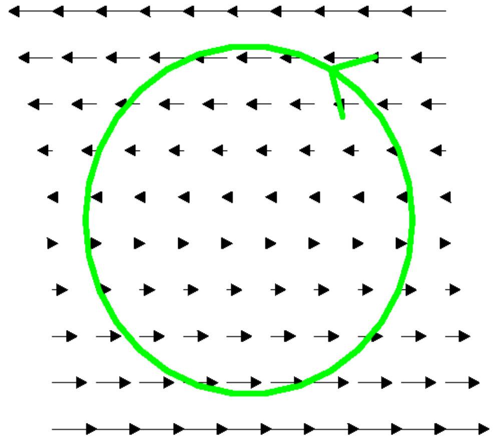
$$\vec{F} = (-y, x, 0)$$

$$\nabla \times \vec{F} \;=\; \mathsf{curl}\vec{F} \;=\; (0, 0, 2)$$

This is a vector in z-direction, perpendicular to plane ...

Integration of (vector-) fields ...





Line integrals: integrate field vectors along a line C:

$$\rightarrow \quad \oint_C \vec{F} \cdot d\vec{r}$$

"sum up" vectors (length) in <u>direction</u> of line C

Integration of (vector-) fields ...

For computations we have important relations:

For any vector $\vec{F}$:

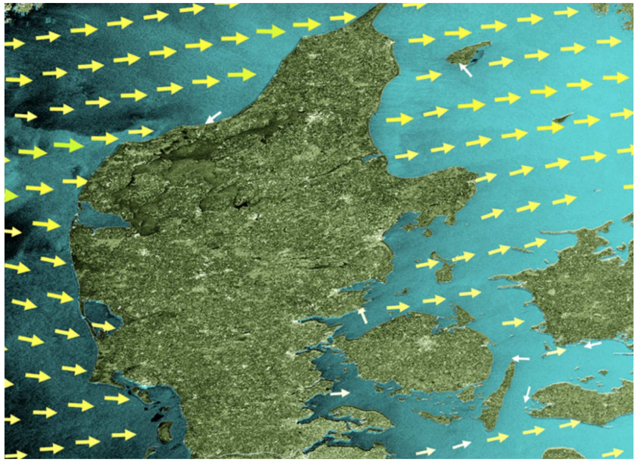
Stokes' Theorem (relates line integral to surface integral):

$$\oint_C \vec{F} \cdot d\vec{r} \; = \; \int\int_S \nabla \times \vec{F} \cdot d\vec{S}$$

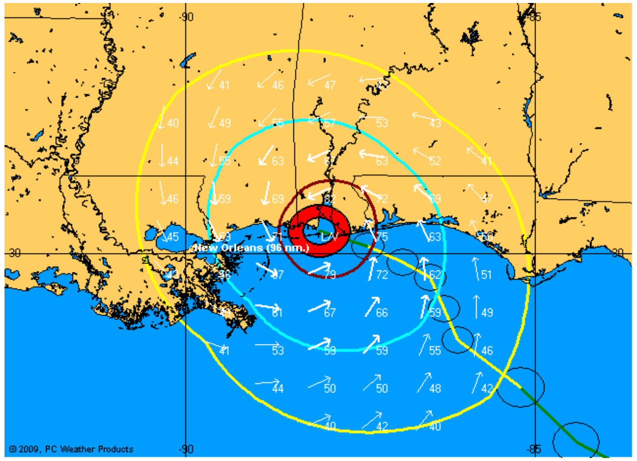
Gauss' Theorem (relates surface integral to volume integral):

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$

Integrating Curl ...



$$\int\int \textbf{curl } \vec{W} = \mathbf{0}$$



$$\int\int \textbf{curl } \vec{W} > \mathbf{0}$$

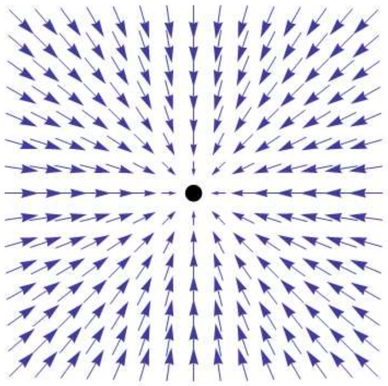
... amount of rotation

What we shall talk about

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

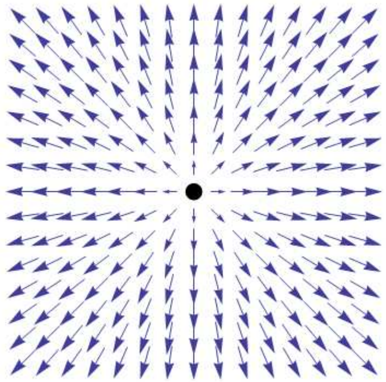
$\vec{E}$	=	**electric field [V/m]**
$\vec{H}$	=	**magnetic field [A/m]**
$\vec{D}$	=	**electric displacement [C/m^2]**
$\vec{B}$	=	**magnetic flux density [T]**
ρ	=	**electric charge density [C/m^3]**
$\vec{j}$	=	**current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**
c	=	**speed of light, 2.99792458** $\cdot 10^{8}$ **[m/s]**

Divergence and charges ..



$\nabla \vec{F} < 0$

(negative charges)



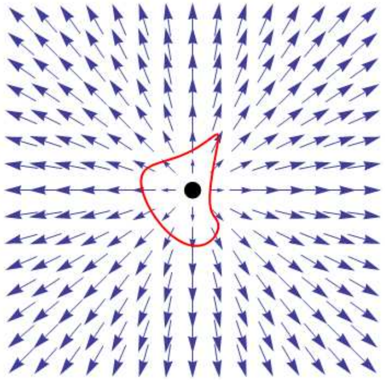
$\nabla \vec{F} > 0$

(positive charges)

- **Large charge** → **large number (or longer) field lines**
- **Small charge** → **small number (or shorter) field lines**

→ **Formal "counting"** $\Longrightarrow$

Divergence and charges ..



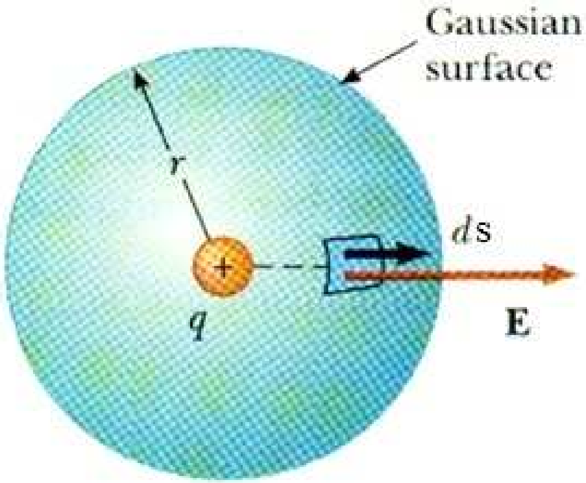
- Put ANY closed surface around charges (sphere, box, ...)
- Add field lines coming out (as positive) and going in (as negative)

→ If positive: total net charge enclosed positive

→ If negative: total net charge enclosed negative

Gauss's Theorem

(Maxwell's first equation ...)



$\frac{1}{\epsilon_0}\int\int_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int\int\int_V \nabla\vec{E}\cdot dV = \frac{Q}{\epsilon_0}$

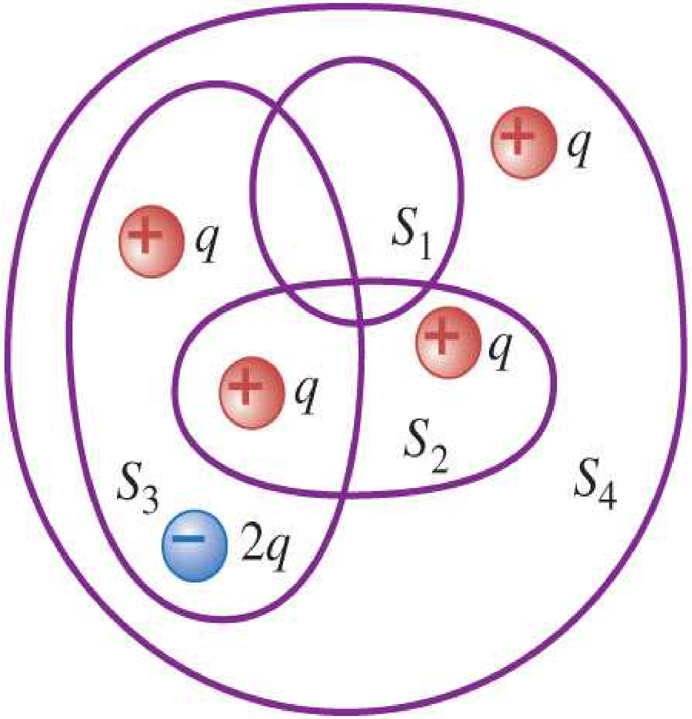
$\nabla\vec{E} = \frac{\rho}{\epsilon_0}$

Flux of electric field $\vec{E}$ through a closed region proportional to net electric charge Q enclosed in the region (Gauss's Theorem). Written with charge density ρ we get Maxwell's first equation:

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

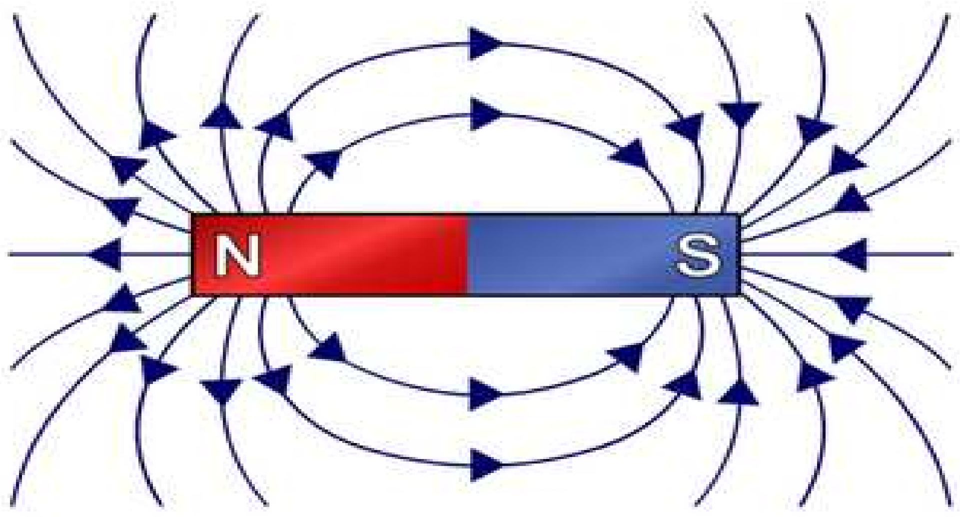
Gauss's Theorem

(Maxwell's first equation ...)



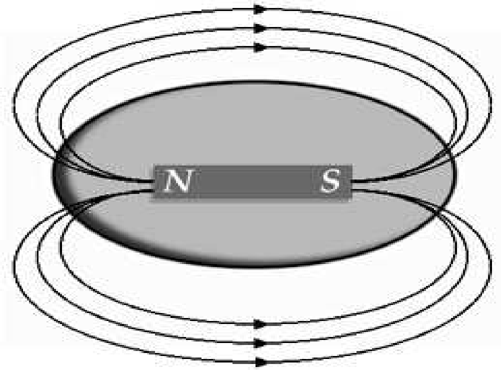
- **Exercise: what are the values of the "integrals" over the surfaces S_1, S_2, S_3, S_4 ? (here in 2D)**

Definitions



- **Magnetic field lines from North to South**
- **Q: which is the direction of the earth magnetic field lines ?**

Maxwell's second equation ...



$$\int\int_S \vec{B} \; d\vec{S} = \int\int\int_V \nabla\vec{B} \; dV = 0$$

$$\nabla\vec{B} = 0$$

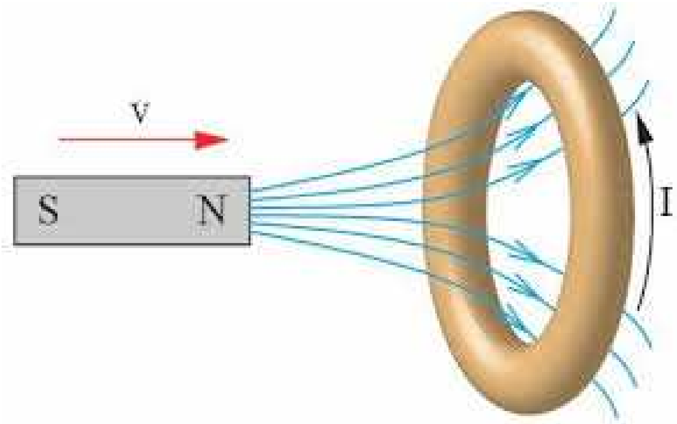
Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

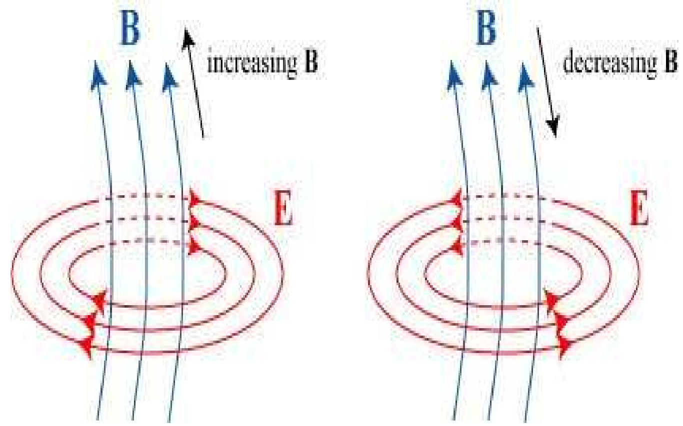
$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

→ **Physical significance: no Magnetic Monopoles**

Maxwell's third equation ...

Faradays law:

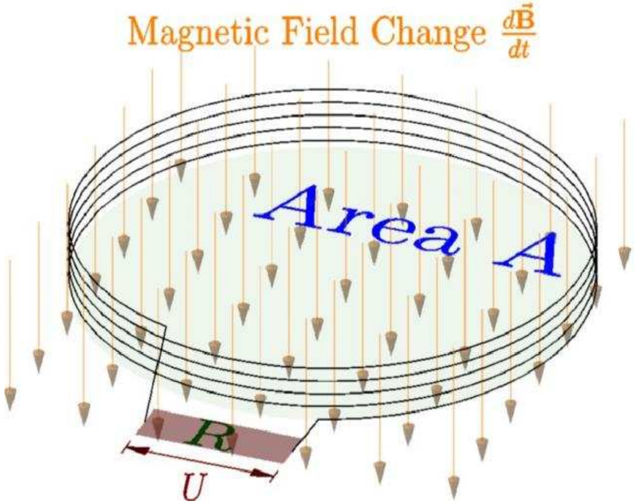




Changing magnetic field introduces electric current in a coil

- **Can move magnet towards/away from coil**
- **Can move coil towards/away from magnet**

Maxwell's third equation ...



$$-\int_S \frac{\partial \vec{B}}{\partial t} d\vec{S} = \int_S \nabla \times \vec{E}\ d\vec{S} = \oint_C \vec{E} \cdot d\vec{r}$$

$$\Phi \ = \ \int\int_S \vec{B} \cdot \vec{S} \quad \text{magnetic flux}$$

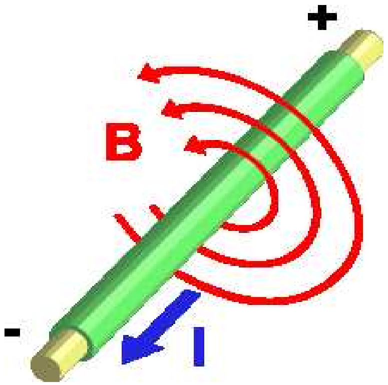
Changing magnetic field through an area induces electric field in coil in the area (Faraday)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = - \mu_0 \frac{\partial \vec{H}}{\partial t}$$

→ **bicycle dynamo, generators, inductors, transformers .....**

Maxwell's fourth equation ...

From Ampere's law, for example current density $\vec{j}$:



<u>Static</u> electric current induces magnetic field

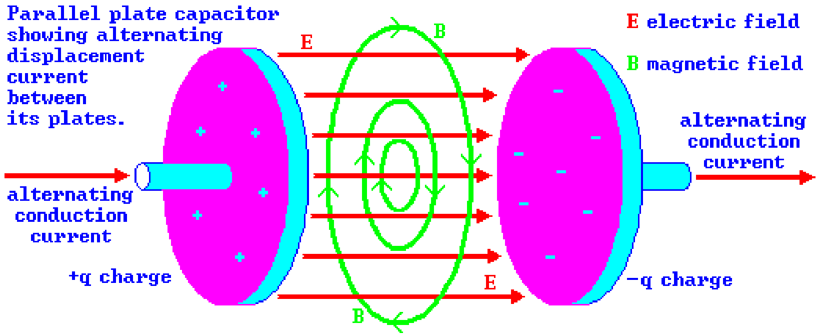
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

or if you prefer:

$$\oint_C \vec{B} \; d\vec{r} = \int \int_S \nabla \times \vec{B} \; d\vec{S} = \mu_0 \int \int_S \vec{j} \; d\vec{S} = \mu_0 \vec{I}$$

Maxwell's fourth equation ...

From displacement current, for example charging capacitor $\vec{j}_d$:



<u>Changing</u> electric field induce magnetic field

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's fourth equation ...

From Ampere's law and displacement current, complete fourth Maxwell equation:

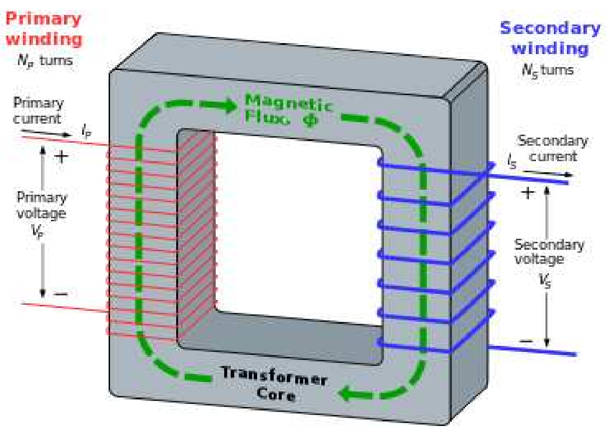
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

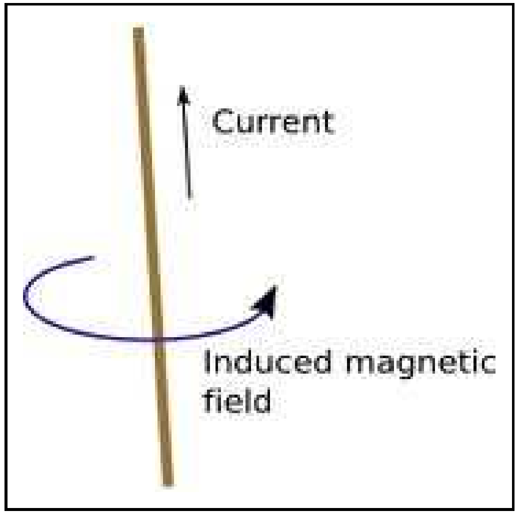
Example: transformer



- **Transforms A.C. electric energy from one circuit into another, using magnetic induction**
  - **Changing primary current induces changing magnetic flux in core**
  - **Changing of flux induce secondary alternating Voltage**
  - **Voltage ratio determined by number of windings**

Maxwell's fourth equation - application

Without changing electric field, i.e. $\nabla \times \vec{B} = \mu_0 \vec{j}$ we get Biot-Savart law. For a straight line current (uniform and constant) we have then (that's why *curl* is interesting):



$$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{r}$$

For magnetic field calculations in electromagnets

Maxwell's Equations in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

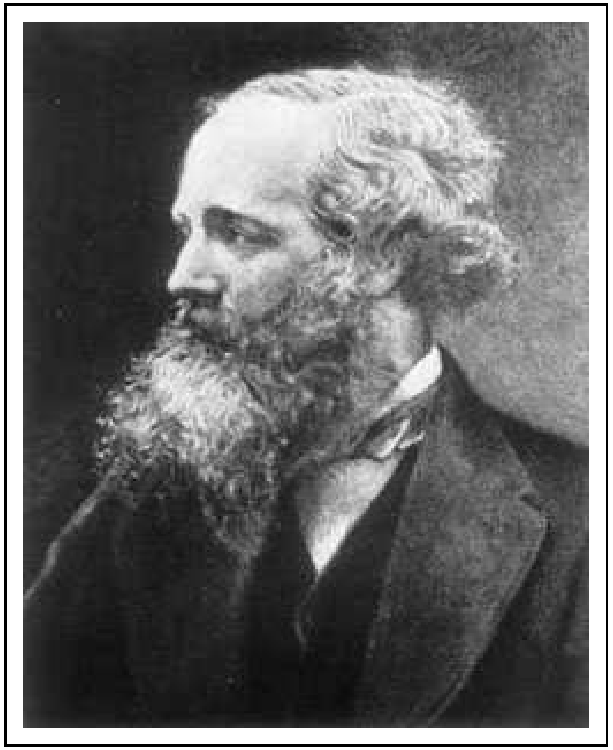
In a material:

$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is <u>relative</u> permittivity** $\approx \quad [1 - 10^5]$

μ_r **is <u>relative</u> permeability** $\approx \quad [0(!) - 10^6]$

Summary: Maxwell's Equations



$$\int_S \vec{D} \cdot d\vec{S} = Q$$

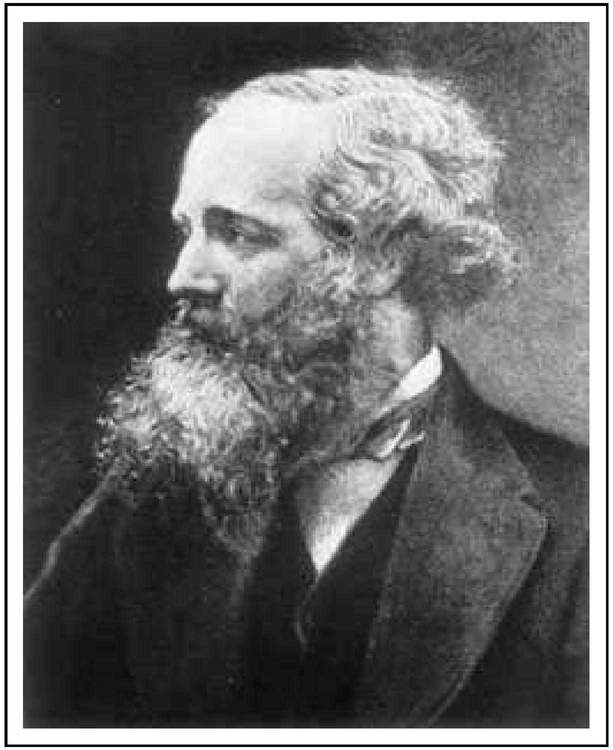
$$\int_S \vec{B} \cdot d\vec{S} = 0$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

$$\oint_C \vec{H} \cdot d\vec{r} = \vec{j} + \frac{d}{dt} \int_S \vec{D} \cdot d\vec{S}$$

Written in Integral form

Summary: Maxwell's Equations



$$\nabla \vec{D} = \rho$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Written in Differential form

Some popular confusion ..

V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$\begin{aligned}\vec{H} &= \frac{\vec{B}}{\mu_0} - \vec{M} \\ \vec{D} &= \epsilon_0\vec{E} + \vec{P}\end{aligned}$$

$\vec{M}$ **and** $\vec{P}$ **are** *Magnetization* **and** *Polarisation* **in material**

Applications of Maxwell's Equations

- **Lorentz force, motion in EM fields**
  - **- Motion in electric fields**
  - **- Motion in magnetic fields**
- **EM waves (in vacuum and in material)**
- **Boundary conditions**
- **EM waves in cavities and wave guides**

Lorentz force on charged particles

Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ like (Lorentz force):

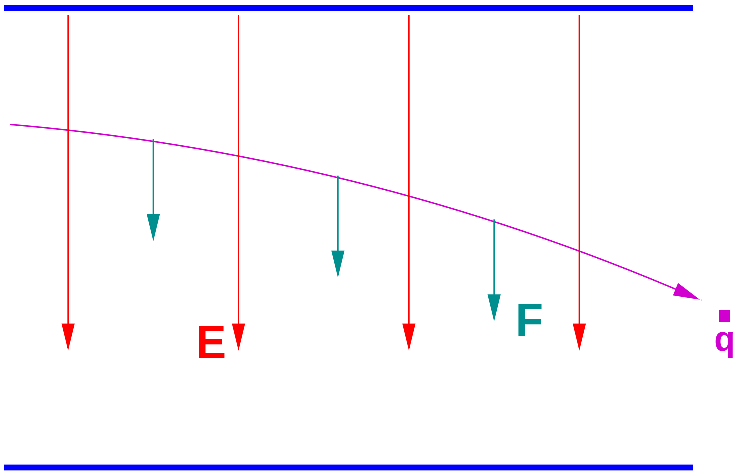
$$\vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

for the equation of motion we get (using Newton's law and relativistic γ);

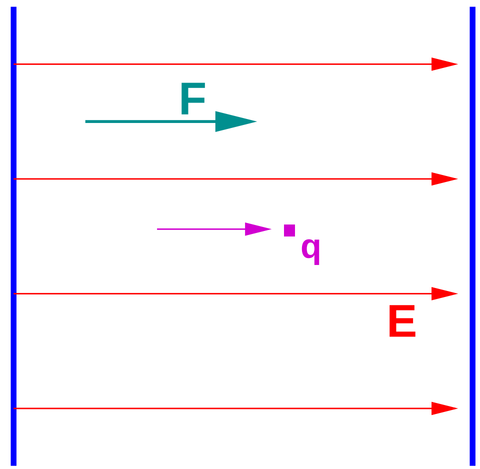
$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

(More complicated for quantum objects, but not relevant here)

Motion in electric fields



$\vec{v} \perp \vec{E}$



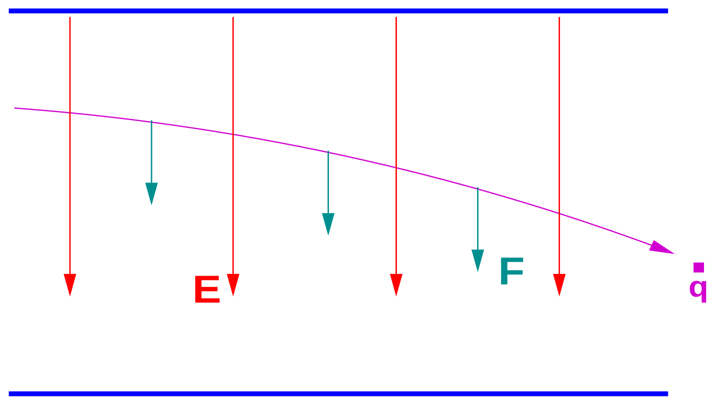
$\vec{v} \parallel \vec{E}$

Assume no magnetic field:

$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in electric fields



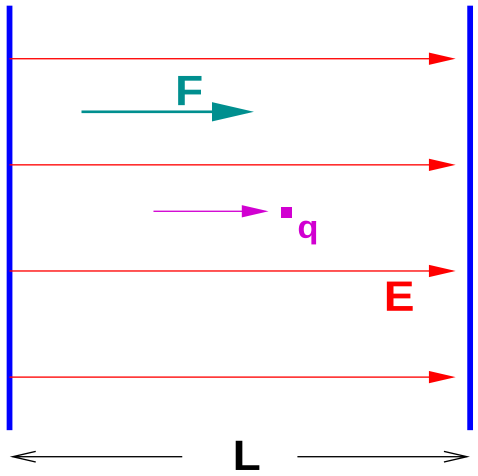
$$\frac{d}{dt}(m_0\gamma\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

The solution is:

$$\vec{v} \quad = \quad \frac{q \cdot \vec{E}}{\gamma \cdot m_0} \cdot \ t \qquad \Rightarrow \qquad \vec{x} \quad = \quad \frac{q \cdot \vec{E}}{\gamma \cdot m_0} \cdot \ t^2 \qquad (\textbf{parabola})$$

Constant E-field deflects beams: TV, electrostatic separators (SPS,LEP)

Motion in electric fields



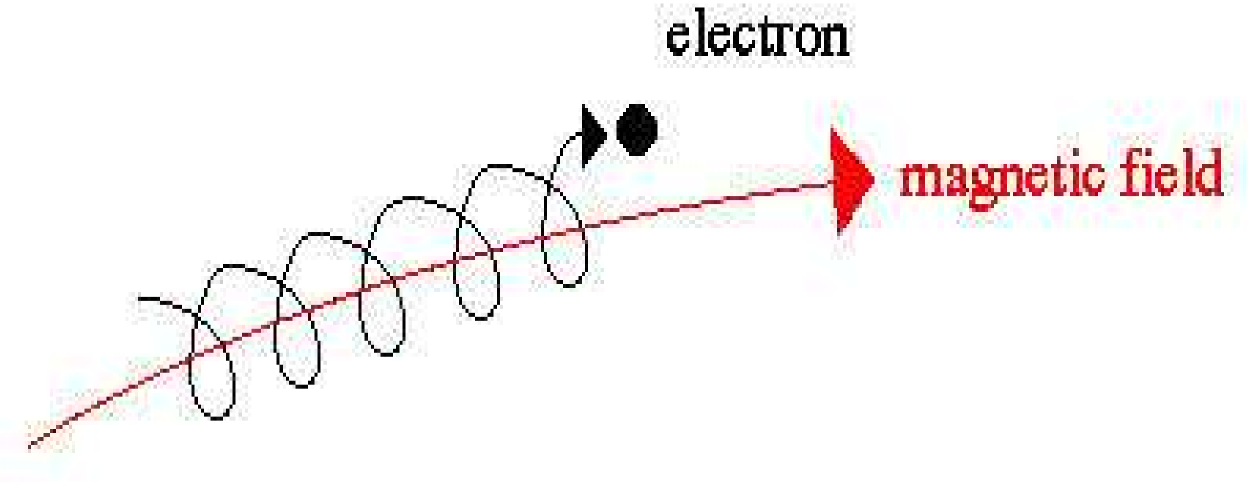
$$\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{f} = q\cdot\vec{E}$$

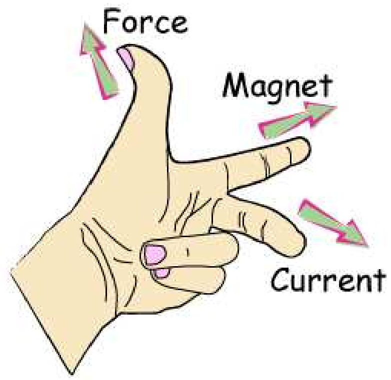
For constant field $\vec{E} = (E,0,0)$ in x-direction the energy gain is:

$$m_0c^2(\gamma - 1) = qEL$$

Constant E-field gives uniform acceleration over length L

Motion in magnetic fields





Assume first no electric field:

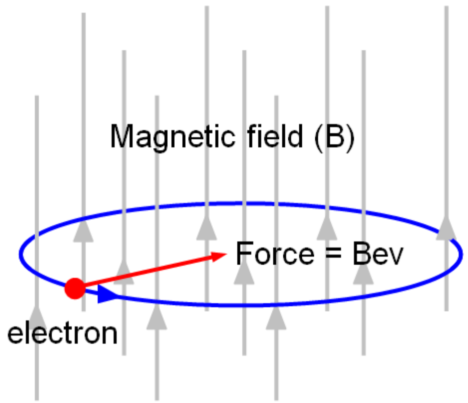
$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$$

Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

No forces on particles at rest !

Particles will spiral around the magnetic field lines ...

Motion in magnetic fields



We get a circular motion with radius ρ:

$$\rho = \frac{m_0 \gamma v_\perp}{q \cdot B}$$

defines the Magnetic Rigidity: $B \cdot \rho = \frac{m_0 \gamma v}{q} = \frac{p}{q}$

Magnetic fields deflect particles, but no acceleration (synchrotron, ..)

Motion in magnetic fields

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[ev]}{c[m/s]}$$

Example LHC:

B = 8.33 T, p = 7000 GeV/c → ρ **= 2804 m**

Use of static fields (some examples, incomplete)

- **Magnetic fields**
  - **Bending magnets**
  - **Focusing magnets (quadrupoles)**
  - **Correction magnets (sextupoles, octupoles, orbit correctors, ..)**
- **Electric fields**
  - **Electrostatic separators (beam separation in particle-antiparticle colliders)**
  - **Very low energy machines**

Electromagnetic waves in vacuum

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times (\nabla \times \vec{E}) &= -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) \\
-(\nabla^2 \vec{E}) &= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \\
-(\nabla^2 \vec{E}) &= -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}$$

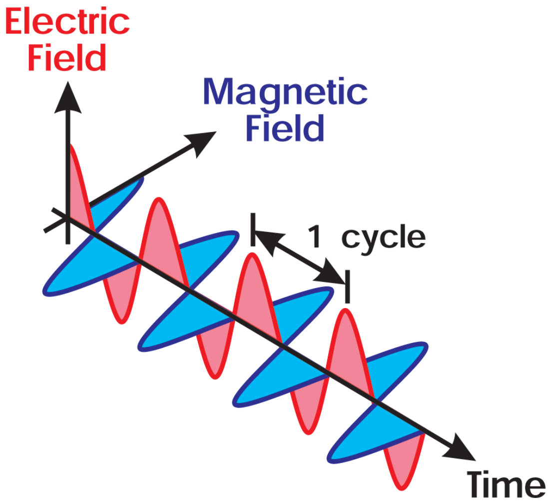
$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu \cdot \epsilon \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similar expression for the magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \mu \cdot \epsilon \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

Equation for a plane wave with velocity in vacuum: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

Electromagnetic waves



$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

$$\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

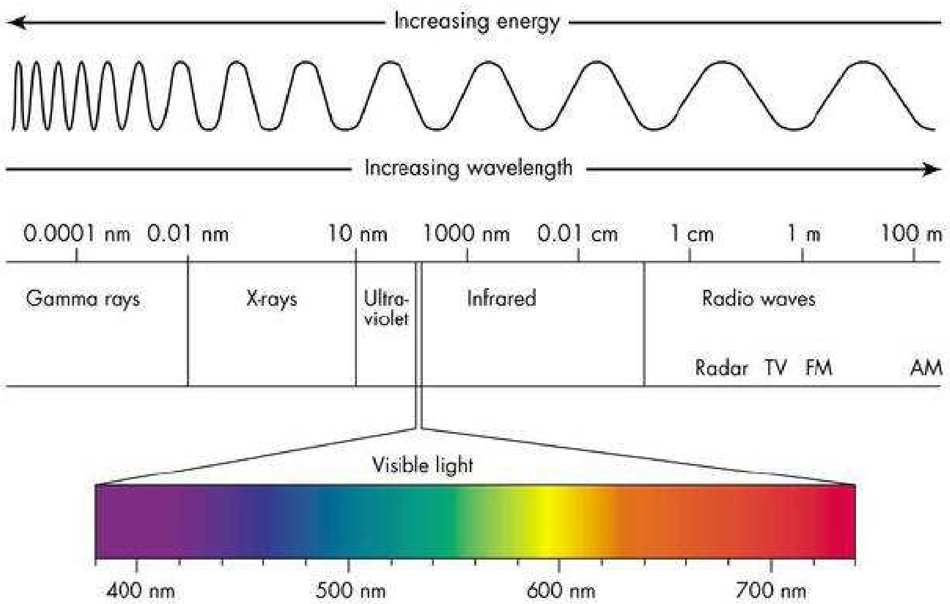
$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad (\textbf{propagation vector})$$

$$\lambda = (\textbf{wave length, 1 cycle})$$

$$\omega = (\textbf{frequency} \cdot 2\pi)$$

Magnetic and electric fields are transverse to direction of propagation: $\vec{E} \perp \vec{B} \perp \vec{k}$

Spectrum of Electromagnetic waves



Example: yellow light ➡ $\approx 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

gamma rays ➡ $\leq 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

LEP (SR) ➡ $\leq 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

Boundary conditions for fields

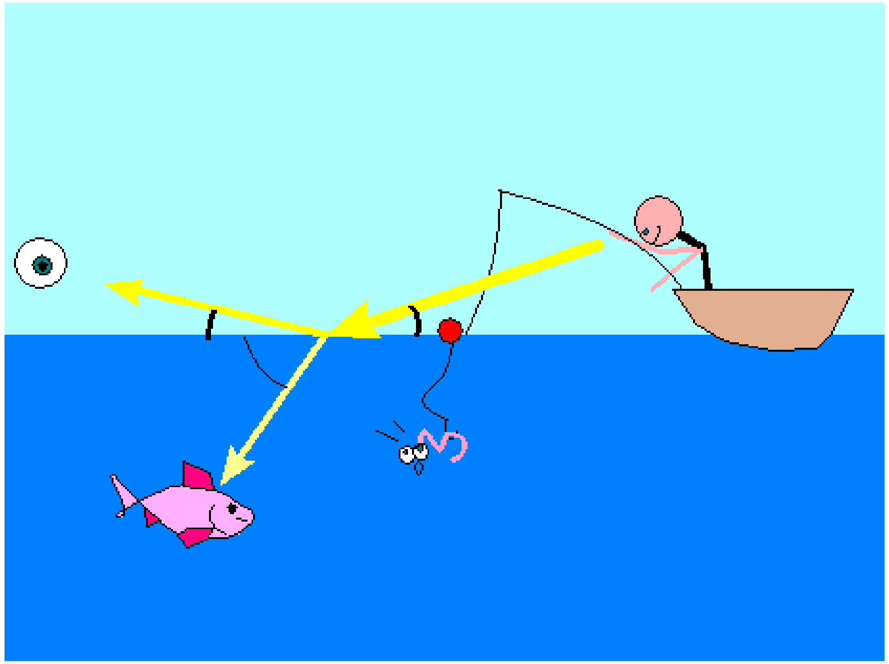
Need to look at the behaviour of electromagnetic fields at boundaries between different materials (air-glass, air-water, vacuum-metal, ...).

Important for highly conductive materials, e.g.:

- **RF systems**
- **Wave guides**
- **Impedance calculations**

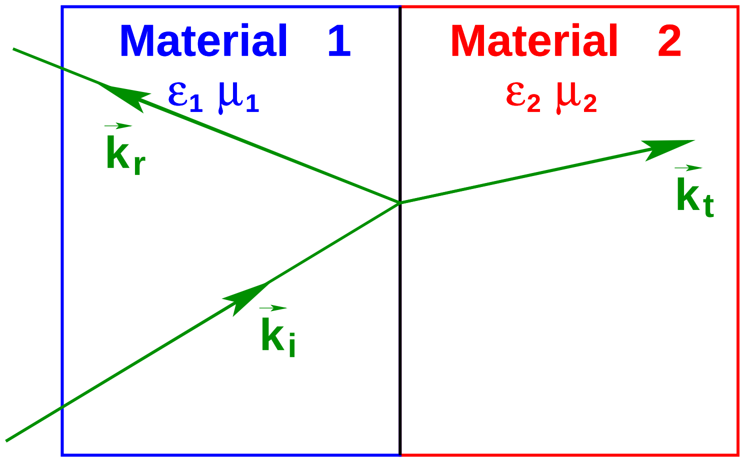
Can be derived from Maxwell's equations, here only the results !

Application and Observation



- **Some of the light is reflected**
- **Some of the light is transmitted and refracted**

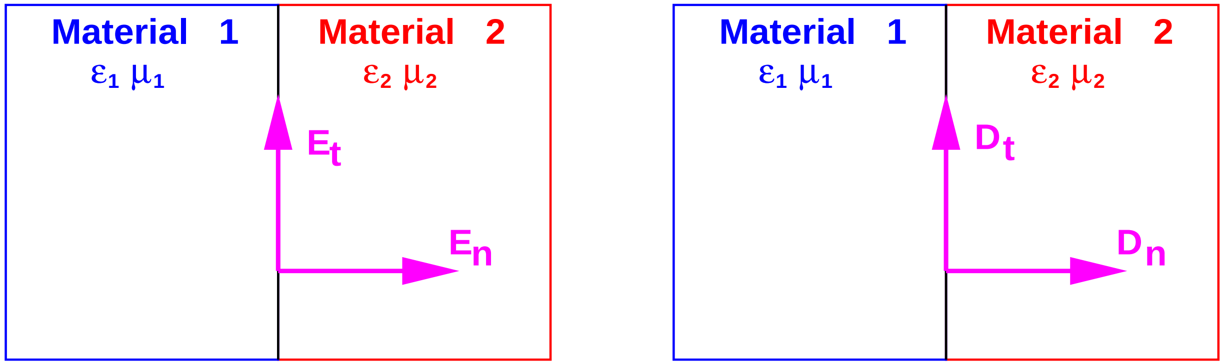
Boundary conditions for fields



What happens when an incident wave ($\vec{K}_i$) encounters a boundary between two different media ?

- **Part of the wave will be reflected ($\vec{K}_r$), part is transmitted ($\vec{K}_t$)**
- **What happens to the electric and magnetic fields ?**

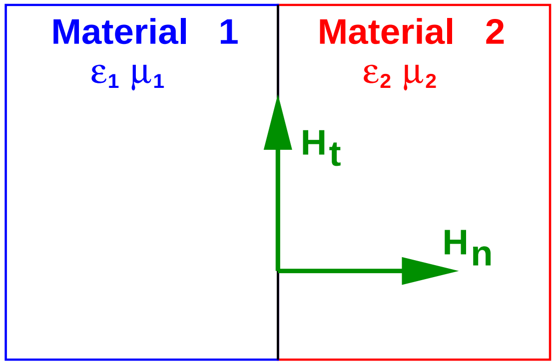
Boundary conditions for fields

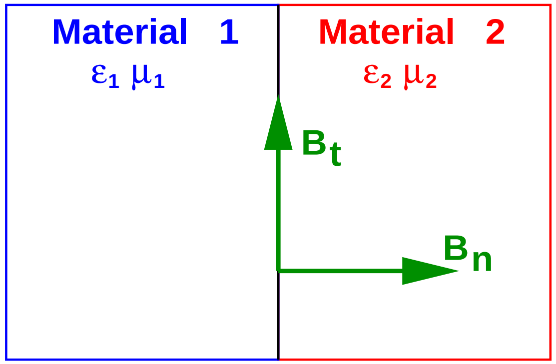


Assuming <u>no</u> surface charges:

- **<u>tangential $\vec{E}$-field</u> constant across boundary** $(E_{1t} = E_{2t})$
- **<u>normal $\vec{D}$-field</u> constant across boundary** $(D_{1n} = D_{2n})$

Boundary conditions for fields





Assuming no surface currents:

- **tangential $\vec{H}$-field constant across boundary** $(H_{1t} = H_{2t})$
- **normal $\vec{B}$-field constant across boundary** $(B_{1n} = B_{2n})$

Extreme case: ideal conductor

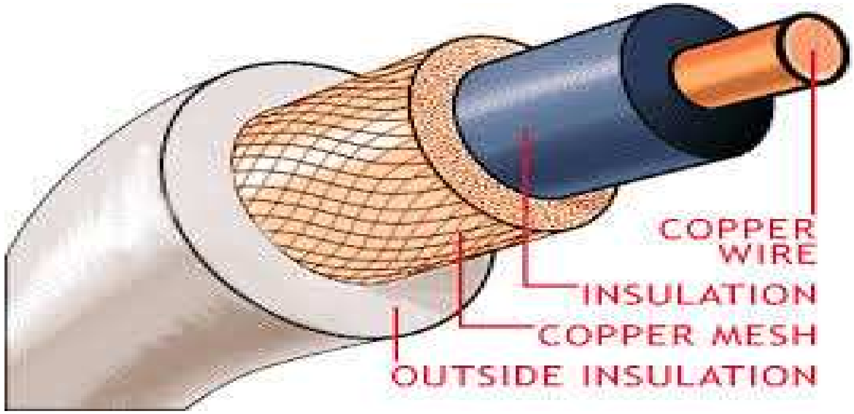
For an ideal conductor (i.e. no resistance) the tangential electric field must vanish, otherwise a surface current becomes infinite. Similar conditions for magnetic fields. We must have:

$$\vec{E}_t = 0, \qquad \vec{B}_n = 0$$

This implies:

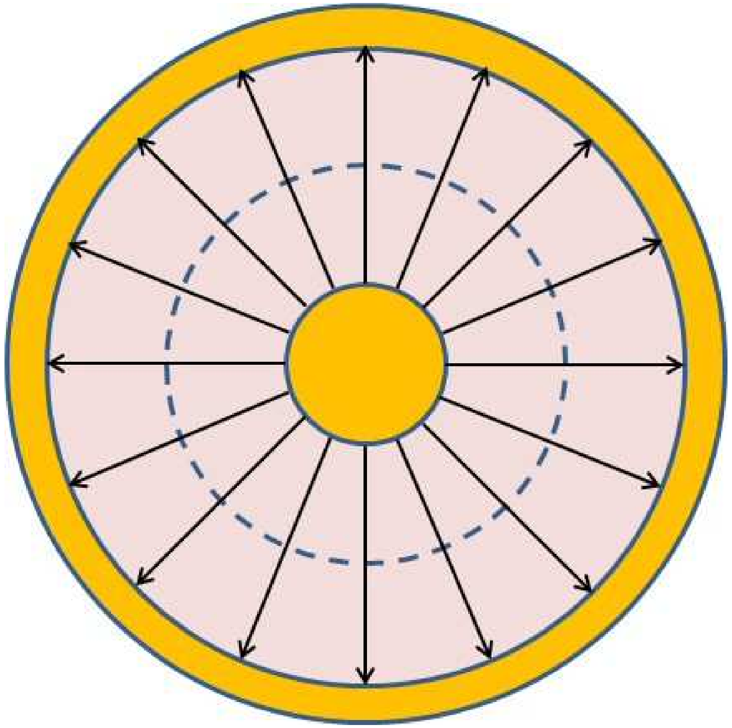
- **All energy of an electromagnetic wave is reflected from the surface.**
- **Fields at any point in the conductor are zero.**
- **Constraints on possible mode patterns in waveguides and RF cavities**

Examples: coaxial cables



- **GHz range, have a cutoff frequency**

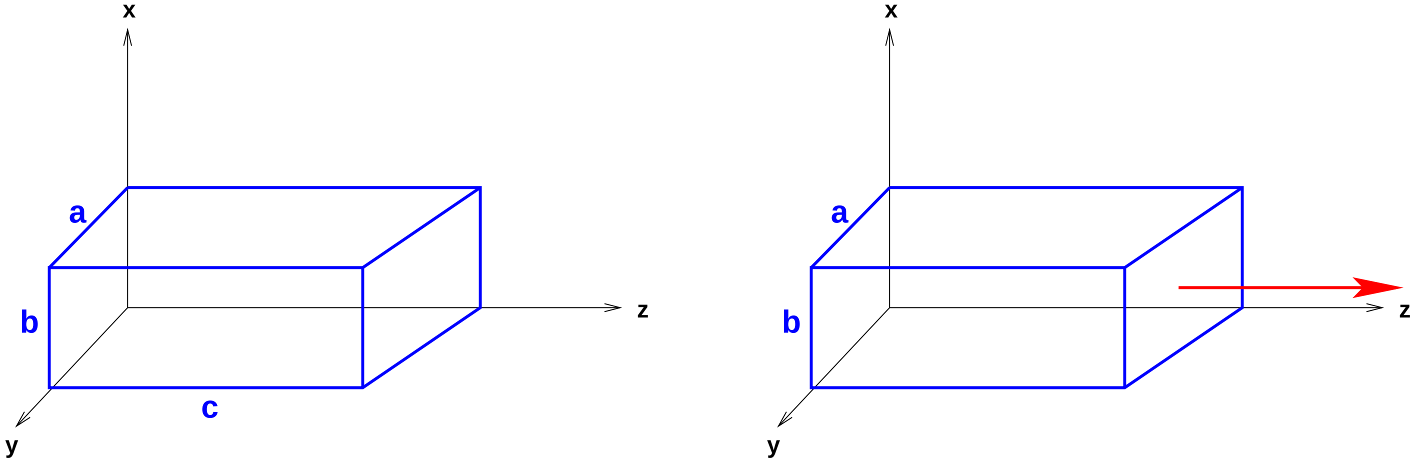
Examples: coaxial cables



- **Mostly TEM modes: electric and magnetic field transverse to direction**

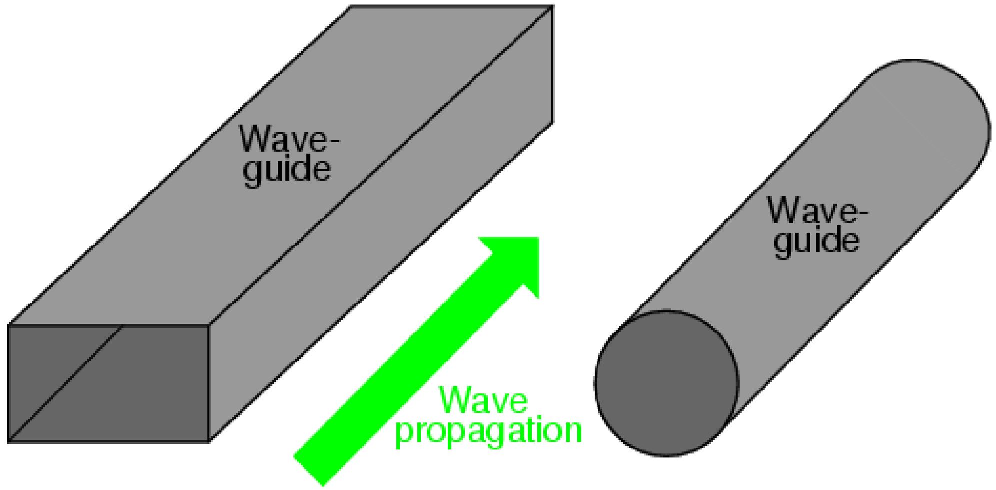
Examples: cavities and wave guides

Rectangular cavity and wave guide (schematic) with dimensions $a \times b \times c$ and $a \times b$:



- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

Examples: wave guides



Consequences for RF cavities

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.
Boundary conditions cannot be fulfilled by wave in free space.
Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

This requires the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and with all boundary conditions:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The numbers m_x, m_y, m_z are called mode numbers, important for shape of cavity !

Consequences for wave guides

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

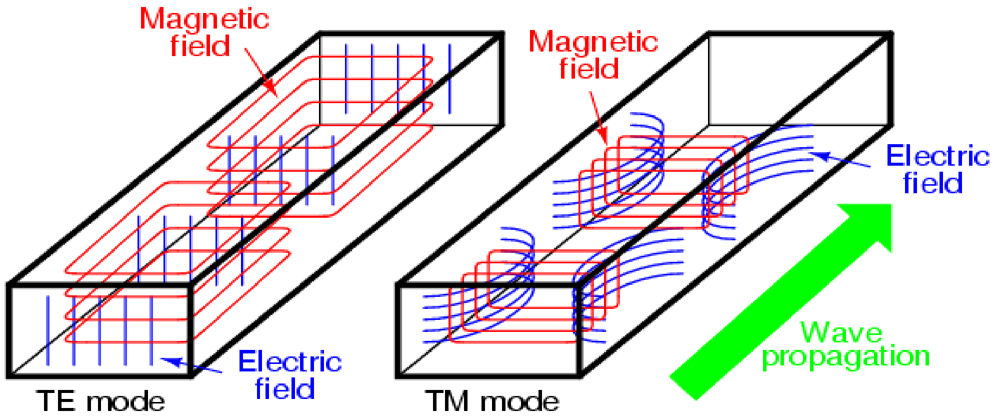
$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

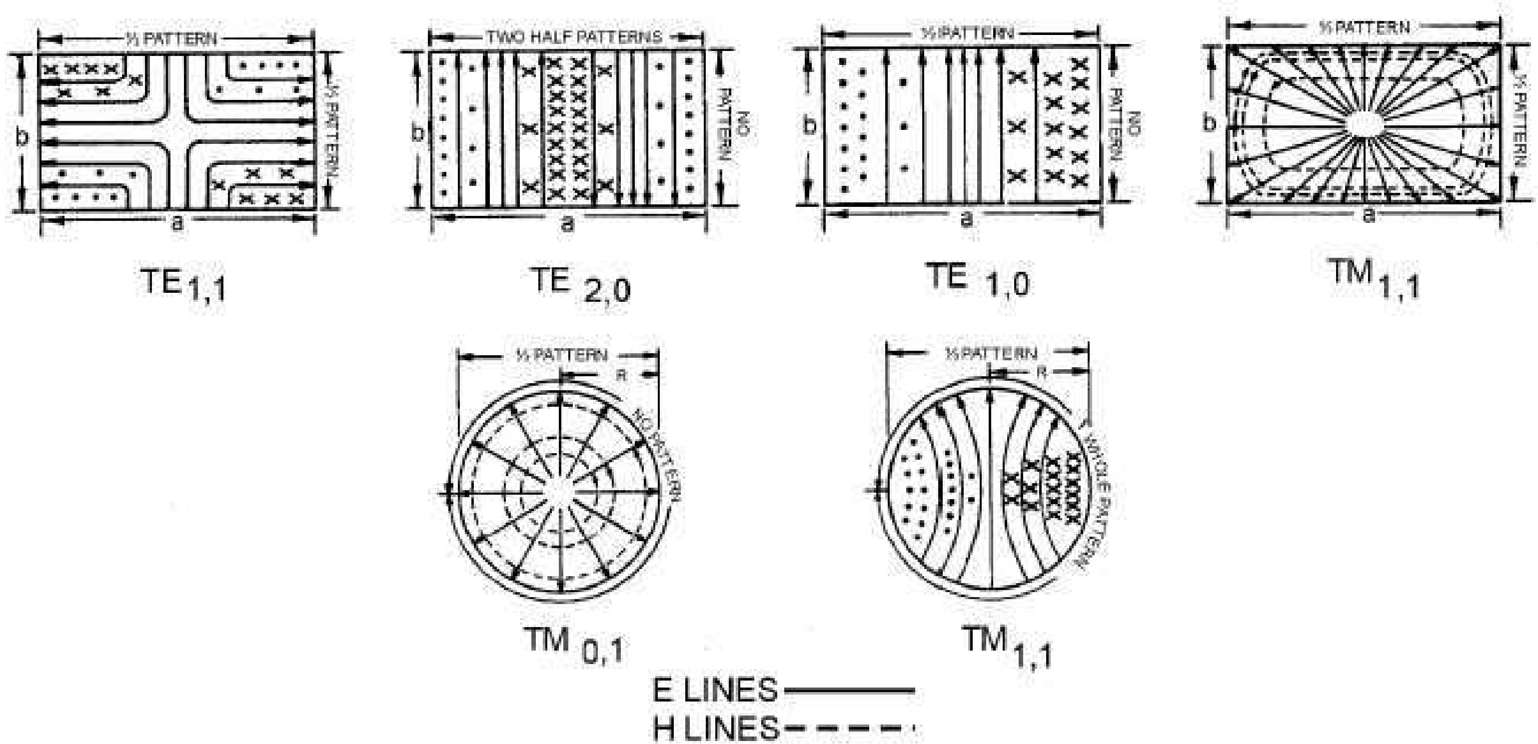
The fields in wave guides



Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

- **Electric and magnetic fields through a wave guide**
- **Shapes are consequences of boundary conditions !**
- **Can be Transverse Electric (TE, no E-field in z-direction) or Transverse Magnetic (TM, no B-field in z-direction)**

Modes in wave guides



- **Modes in wave guides**
- **Field lines, high where density of lines is high**

Consequences for wave guides

We must satisfy again the the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Consequences for wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2$$

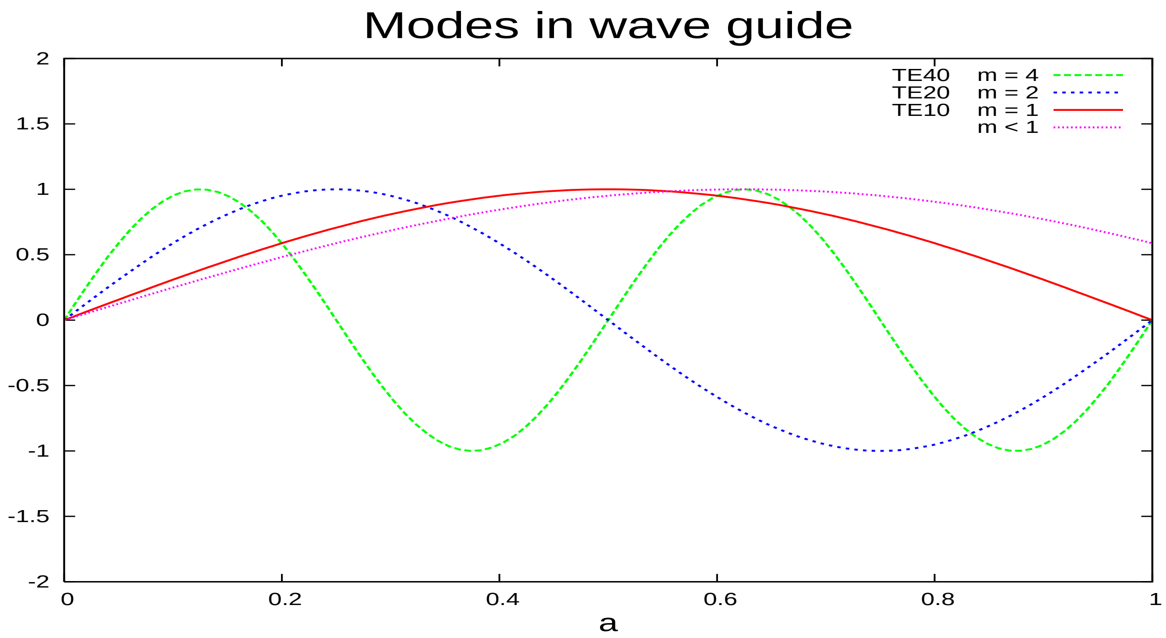
Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c.

- **Above cut-off frequency: propagation without loss**
- **Below cut-off frequency: attenuated wave**

Cut off frequency (1D)



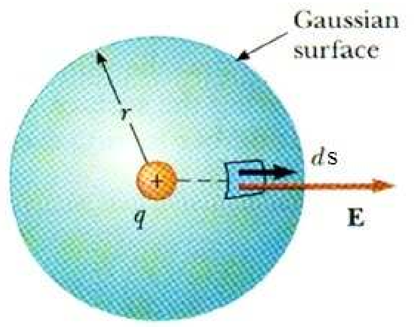
- **Boundary condition** ➡ **E = 0 at: x = 0 and x = a**
- **Requirement for wavelength** $\lambda_x = \frac{2a}{m_x}$, m_x **integer**
- $m_x = 1$ **defines cut off wavelength/frequency**

Done ...

- **Review of basics and Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**

- BACKUP SLIDES -

Maxwell's first equation - example



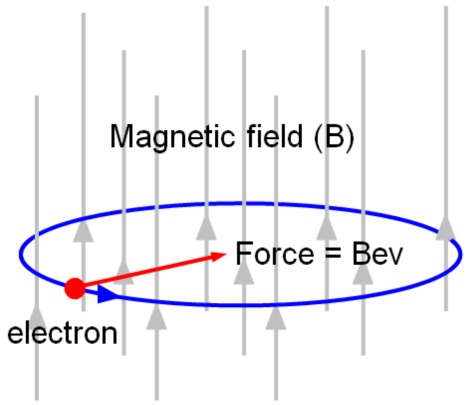
A charge q generates a field $\vec{E}$ according to:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

Surface integral through sphere S is just the charge inside the sphere:

$$\int\int_{sphere} \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\int\int_{sphere}\frac{dS}{r^2} = \frac{q}{\epsilon_0}$$

Is that the full truth ?



If we have a circulating E-field along the circle of radius R ?

➡ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} \quad = \quad -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

$$\rightarrow 2\pi R E_\theta \quad = \quad -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- **This is the principle of a Betatron**
  - **Time varying magnetic field creates circular electric field !**
  - **Time varying magnetic field deflects the charge !**

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \Rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R} \frac{dp}{dt}$$

$$\Rightarrow \quad B(r,t) = \frac{1}{2} \frac{1}{\pi R^2} \int \int B dS$$

B-field on orbit must be half the average over the circle

$\Rightarrow$ **Betatron condition**

Other case: finite conductivity

Assume conductor with finite conductivity ($\sigma_c = \rho_c^{-1}$) , waves will penetrate into surface. Order of the skin depth is:

$$\delta_s = \sqrt{\frac{2\rho_c}{\mu\omega}}$$

i.e. depend on resistivity, permeability and frequency of the waves (ω).
We can get the surface impedance as:

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu\omega}{k}$$

the latter follows from our definition of k and speed of light. Since the wave vector k is complex, the impedance is also complex. We get a phase shift between electric and magnetic field.