Kinematics of Particle Beams

SPECIAL RELATIVITY

(in less than 60 minutes ...)

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(http://cern.ch/Werner.Herr/CAS/CAS2013_Chavannes/lectures/rel.pdf)

Why Special Relativity ?

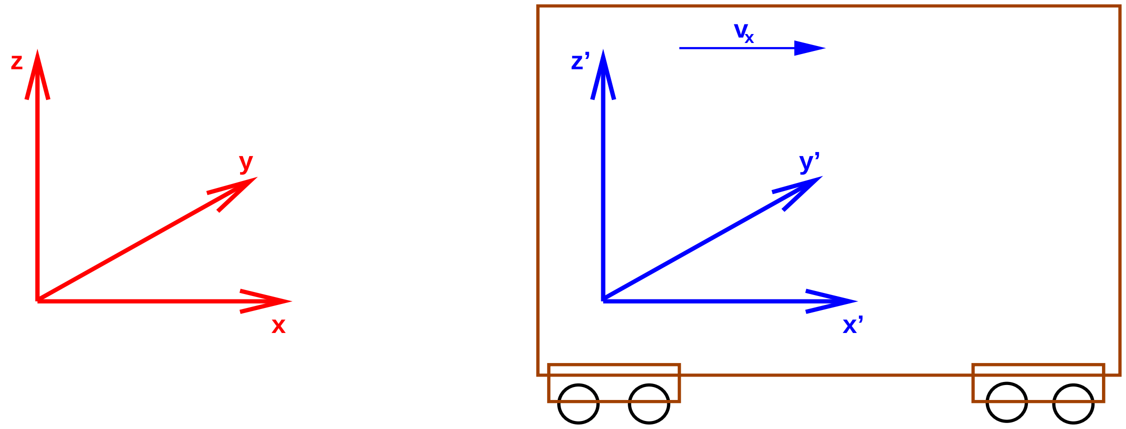
- **Most beams at CERN are relativistic**
- **Strong implications for beam dynamics:**
  - **Transverse dynamics (e.g. momentum compaction, radiation, ...)**
  - **Longitudinal dynamics (e.g. transition, ...)**
  - **Collective effects (e.g. space charge, beam-beam, ...)**
  - **Luminosity in colliders**
  - **Particle lifetime and decay (e.g. μ, π, Z_0, Higgs, ...)**

Small history

- **1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)**
- **1687 (Newton): Principles of Relativity**
- **1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether**
- **1887 (Michelson, Morley): Speed c independent of direction, → ether theory R.I.P.**
- **1904 (Lorentz, Poincaré): Lorentz transformations**
- **1905 (Einstein): Principles of Special Relativity**
- **1907 (Minkowski): Concepts of Spacetime**

Principles of Relativity (Newton)

- **Assume a frame at rest (F) and another frame moving in x-direction (F') with constant velocity $\vec{v} = (v_x, 0, 0)$**



Principles of Relativity (Newton)

- **Assume a frame at rest (F) and another frame moving in x-direction (F') with constant velocity $\vec{v} = (v_x, 0, 0)$**
  - **Classical laws (mechanics) are the same in all frames (Newton, Poincaré)**
  - **No absolute space possible, but absolute time**

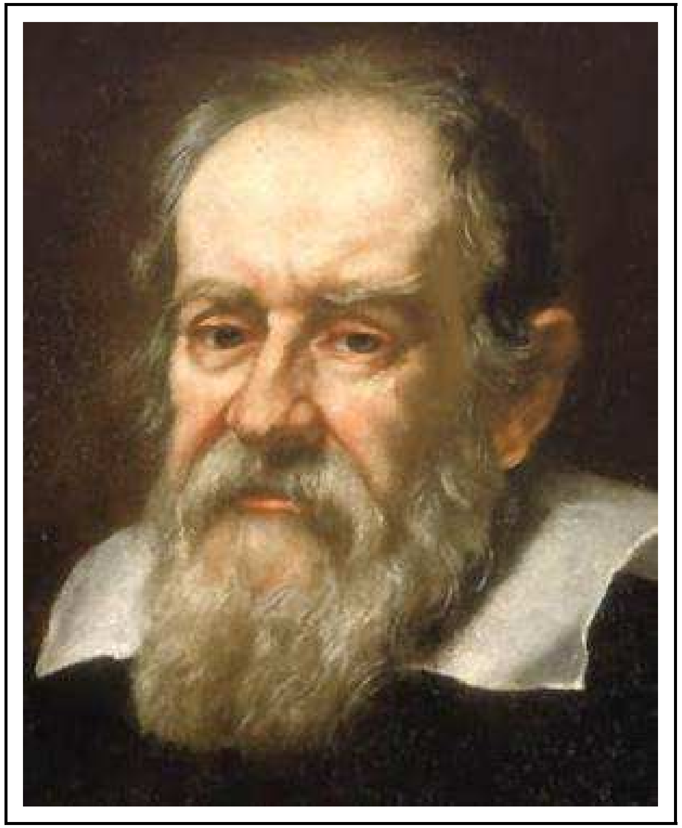
    Time is the same in all frames
  - **Physical laws are invariant ...**

What does it mean ?

- **Invariance of physical laws:**
  - **How is a physical process observed in F described (observed) in the moving frame F' ?**
  - **Need transformation of coordinates (x, y, z) to describe (translate) results of measurements and observations to the moving system (x', y', z').**
  - **For Poincaré's, Newton's principle of relativity need Galilei transformation for**

    $(x, y, z) \rightarrow (x', y', z')$

Galilei transformation



$$
\begin{aligned}
x' &= x - v_x t \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Consequences of Galilei transformation

- **Velocities can be added**

- **From Galilei transformation, take derivative:**

$$x' = x - v_x t$$

$$\dot{x}' = \dot{x} - v_x \quad \rightarrow \quad v' = v - v_x$$

- **A car moving with speed v' in a frame moving with speed v_x we have in rest frame $v = v' + v_x$**

- **But: if $v' = 0.75$c and $v_x = 0.75$c do we get $v =$1.5c ?**

Problems with Galilei transformation

- **Maxwell's equations are wrong when Galilei transformations are applied (because they predict the speed of light, see later)**
  - **First solution: introduction of "ether"**
  - **But: speed of light the same in all frames and all directions (no "ether")**
  - **Need other transformations for Maxwell's equations**
- **Introduced principles of special relativity**

Principles of Special Relativity (Einstein)

- **A frame moving with constant velocity is called an "inertial frame"**
- **All (not only classical) physical laws in related frames have equivalent forms, in particular:**

  speed of light c the same in all frames

- **Cannot distinguish between inertial frames, in particular:**
  - **Cannot determine absolute speed of an inertial frame**
  - **No absolute space, no absolute time**

All you need to know !

Principles of Invariance (Poincaré)

Concept of Invariance:

The laws of Physics are invariant under a transformation between two coordinate frames moving at constant velocities with respect to each other

(The world is not invariant, but the laws of physics are !)

Poincaré + Einstein:

Need Transformations (not Galileaen) which make the physics laws the same everywhere !

Coordinates must be transformed differently

- **Transformation must keep speed of light constant**
- **Time must be changed by transformation as well as space coordinates**
- **Transform** $(x, y, z),\ t \quad \rightarrow \quad (x', y', z'),\ t'$

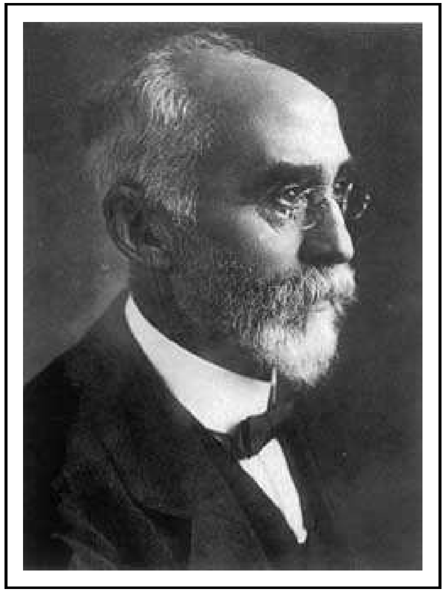
Constant speed of light requires:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \Rightarrow \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

(front of a light wave)

- **Defines the Lorentz transformation (but established by Poincaré !)**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

- **Transformation for constant velocity along x-axis**

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

- β_r **relativistic speed:** β_r **= [0, 1]**
- γ **relativistic factor:** γ **= [1, ∞]**

(unfortunately, you will also see other β and γ ... !)

Einstein's contributions

$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$$(x, y, z) \quad \rightarrow \quad (x, y, z, ct)$$

- **Time has no absolute meaning**
- **Simultaneity has no absolute meaning**
- **Combine time with the 3 dimensions of space**
- **Energy and mass equivalence**

Consequences of Einstein's interpretation

- **Relativistic phenomena:**
  - **(Non-) Simultaneity of events in independent frames**
  - **Lorentz contraction**
  - **Time dilation**
- **Formalism with four-vectors introduced**
  - **Invariant quantities**
  - **Mass - energy relation**

Simultaneity between moving frames

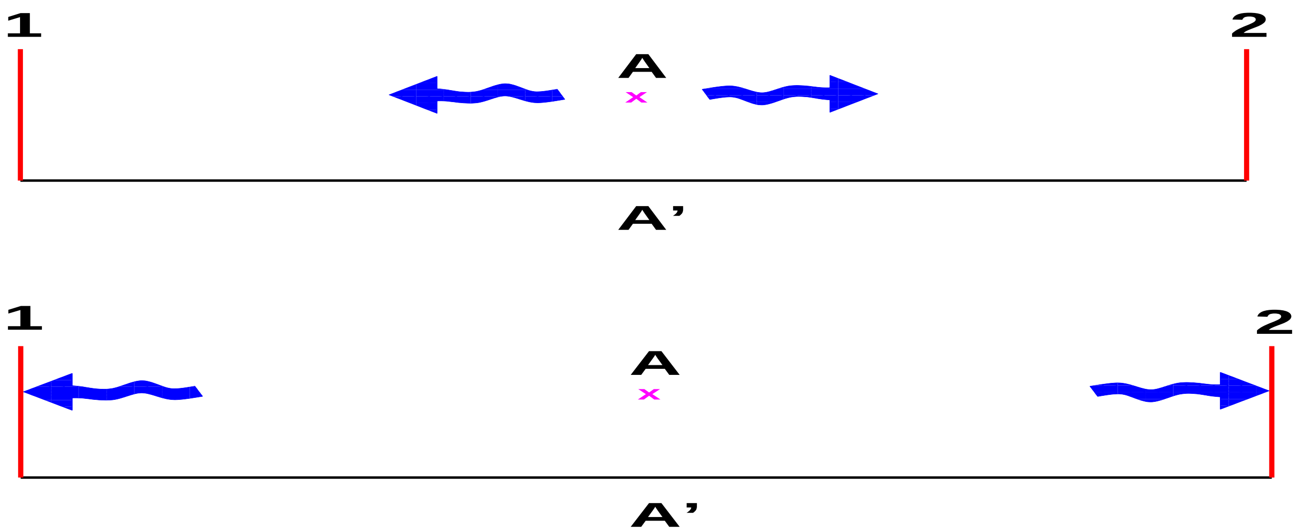
- **Assume two events in frame F at positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$:**

$$t'_1 = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \qquad \text{and} \qquad t'_2 = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

implies that $t'_1 \neq t'_2$ in frame F' !!

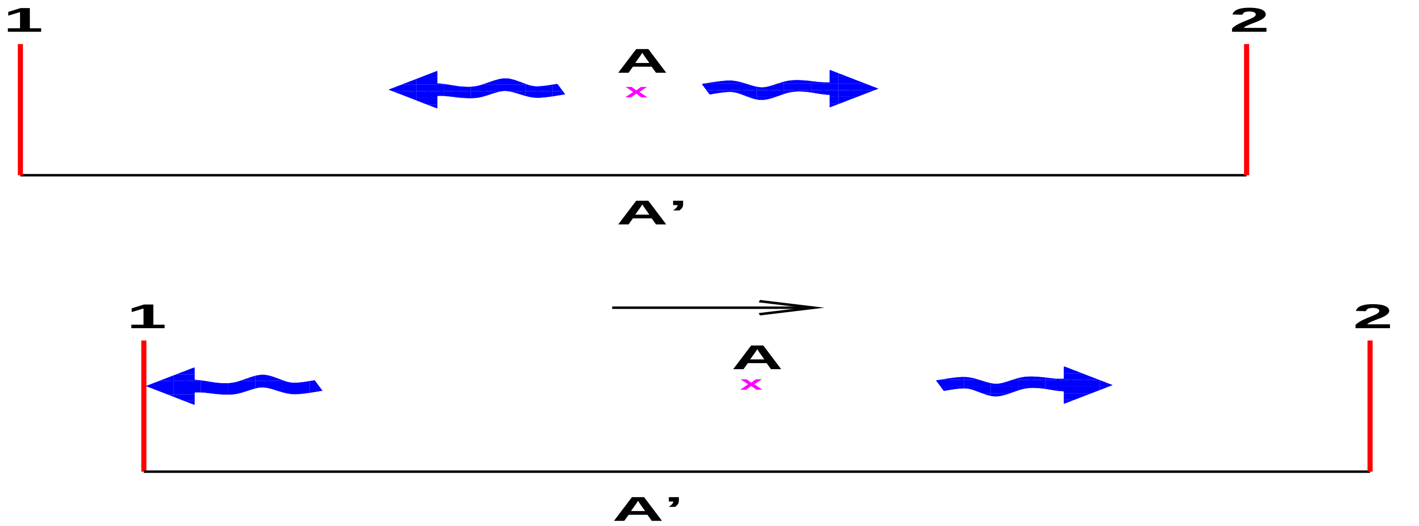
- **Two events simultaneous at positions x_1 and x_2 in F are not simultaneous in F'**

Simultaneity between moving frames



- System with a light source (x) and detectors (1, 2) and one observer (A) in this frame, another (A') outside
- System at rest → observation the same in A and A'
- What if system with A is moving ?

Simultaneity between moving frames



- For A: both flashes arrive simultaneously in 1,2
- For A': flash arrives first in 1, later in 2
- A simultaneous event in F is not simultaneous in F'
- Why do we care ??

Why care about simultaneity ?

- Simultaneity is not frame independent
- This is a key in special relativity
- Most paradoxes are explained by that (although not the twin paradox) !
- Different observers see a different reality

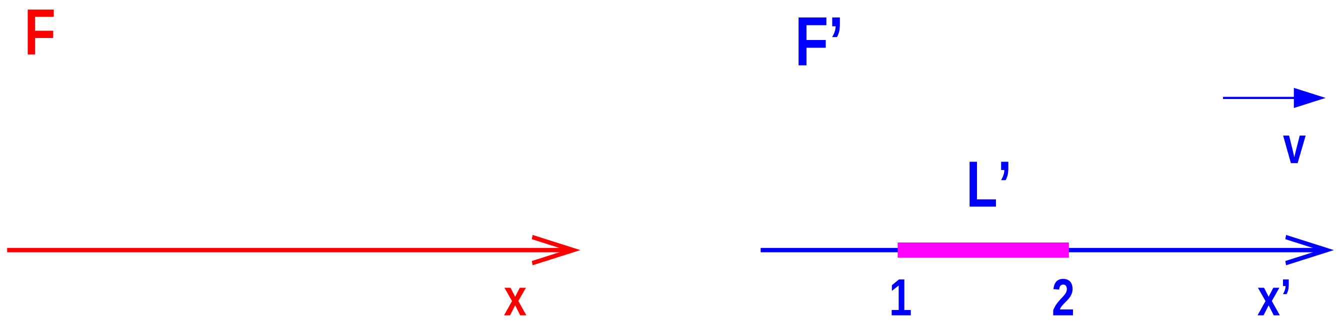
→ relativity is not a spectator sport ...

Why care about simultaneity ?

- **Simultaneity is not frame independent**
- **This is a key in special relativity**
- **Most paradoxes are explained by that (although not the twin paradox) !**

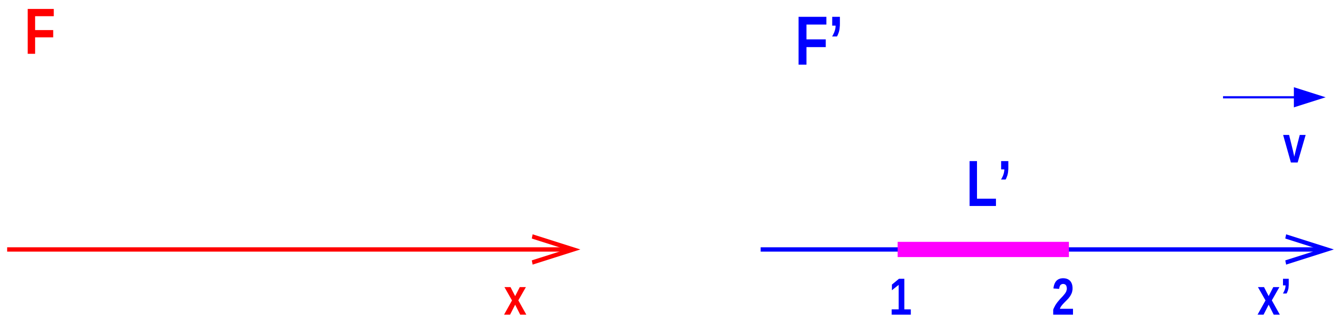
- **More important: sequence of events can change !**
- **For $t_1 < t_2$ we may find (not always !) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)**
- **Requires introduction of "antiparticles" in relativistic quantum mechanics**

Consequences: length measurement



Length of a rod in F' is $L' = x'_2 - x'_1$, measured simultaneously (!) at a fixed time t' in frame F', what is the length L seen in F ??

Consequences: length measurement



We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F, i.e.: $L = x_2 - x_1$ ➡

$$x_1' = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x_2' = \gamma \cdot (x_2 - vt)$$

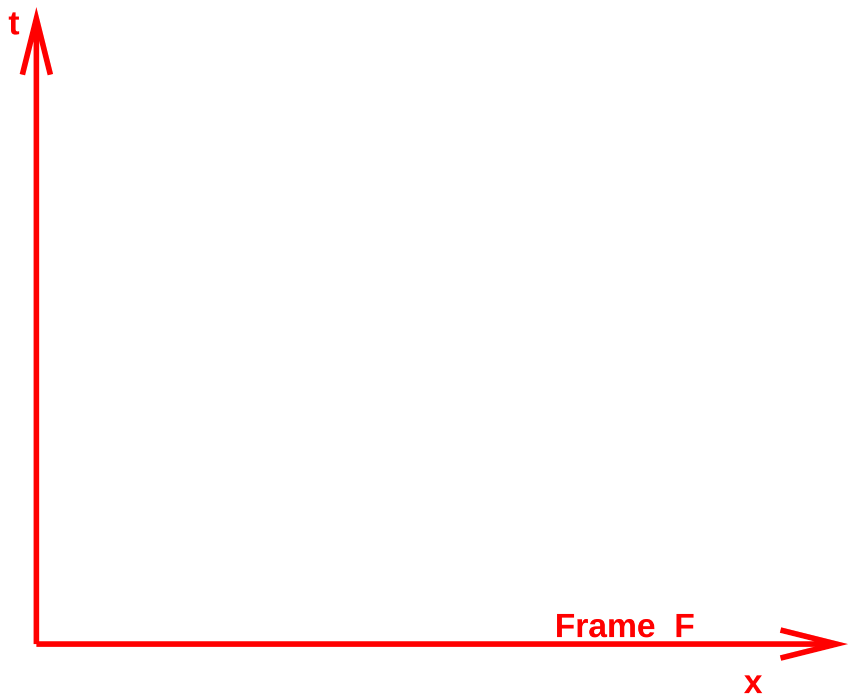
$$L' = x_2' - x_1' = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

➡ $L = L'/\gamma$

Lorentz contraction

- **In moving frame an object has always the same length (it is invariant, our principle !)**
- **From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)**
- **Why do we care ?**
- **Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?**
  - **At 5 GeV ($\gamma \approx 5.3$) $\rightarrow$ L' = 0.53 m**
  - **At 450 GeV ($\gamma \approx 480$) $\rightarrow$ L' = 48.0 m**

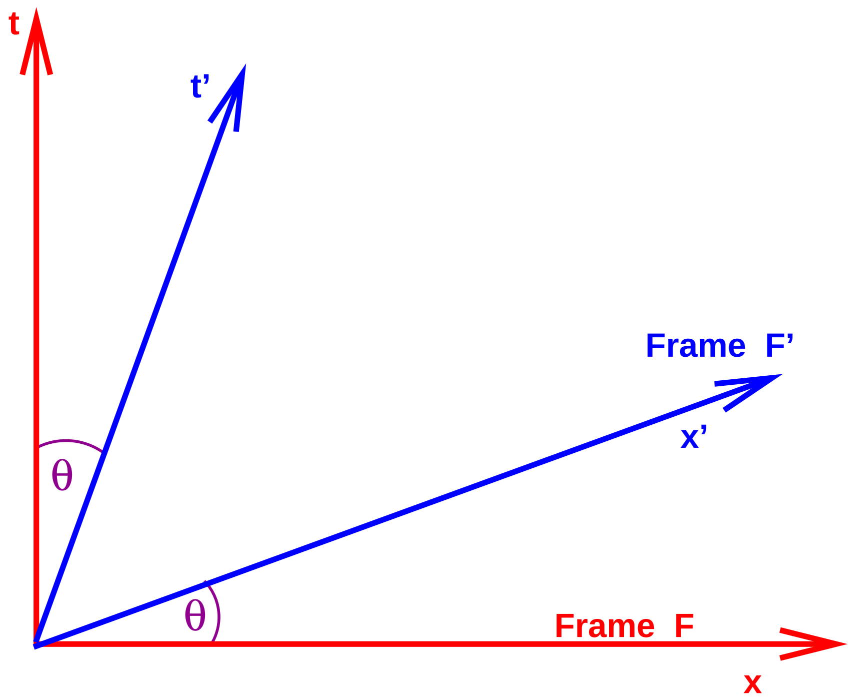
Lorentz transformation - schematic



- Rest frame (x only, difficult to draw many dimensions)

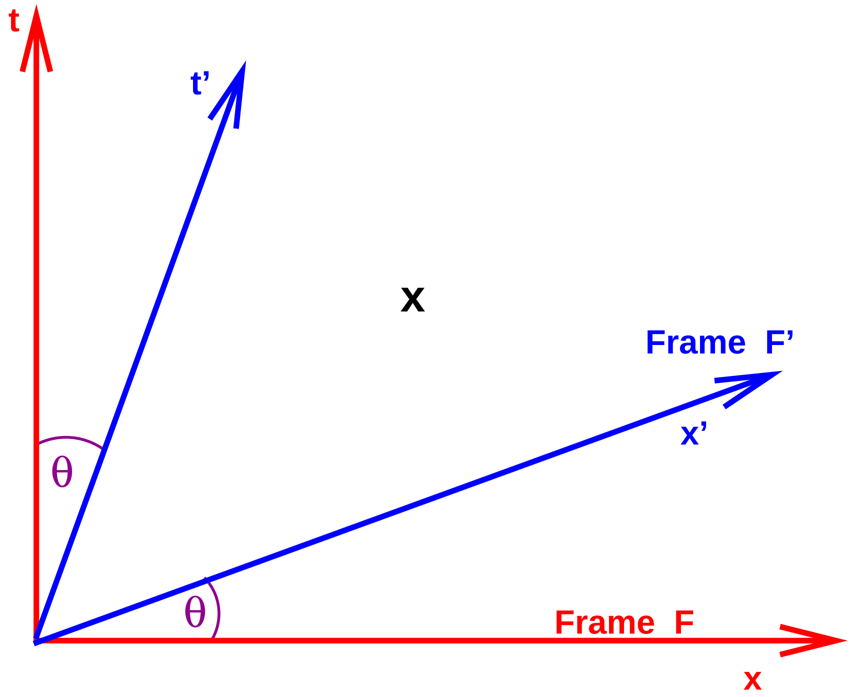
  y and z coordinates are not changed (transformed)

Lorentz transformation - schematic



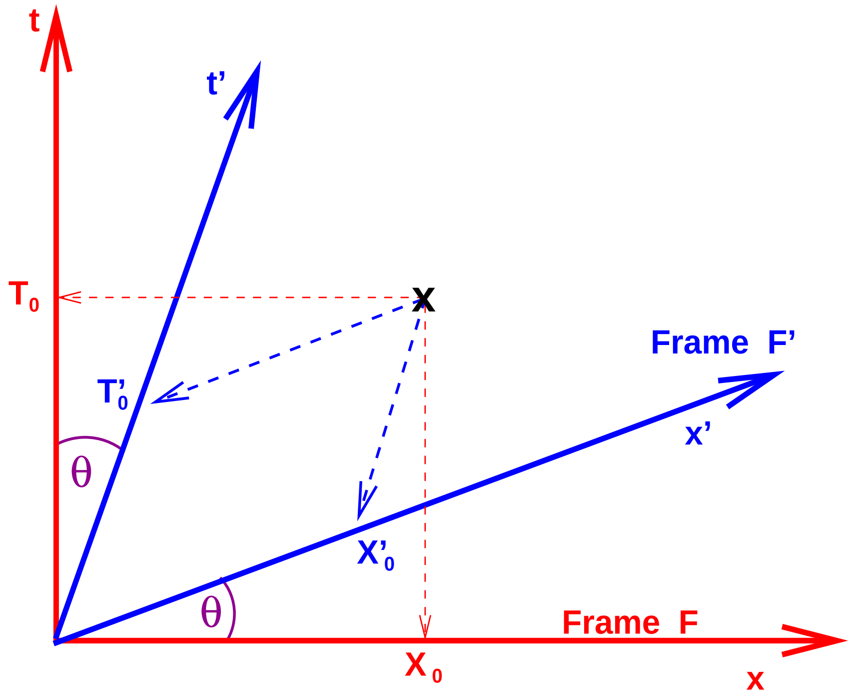
- Rest frame and moving frame
- $\tan(\theta) = \frac{v}{c}$

Lorentz transformation - schematic



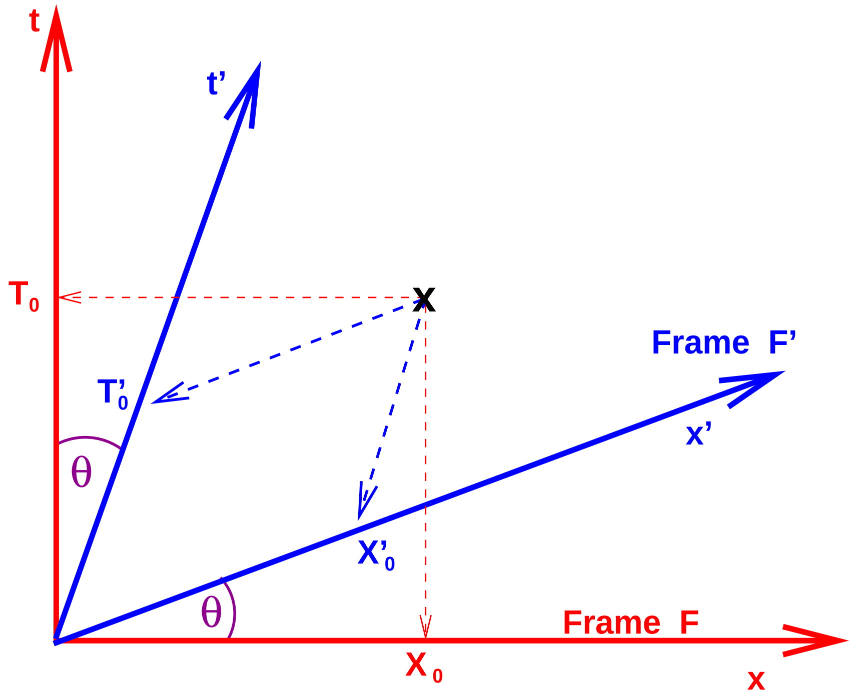
- **An event X**

Lorentz transformation - schematic



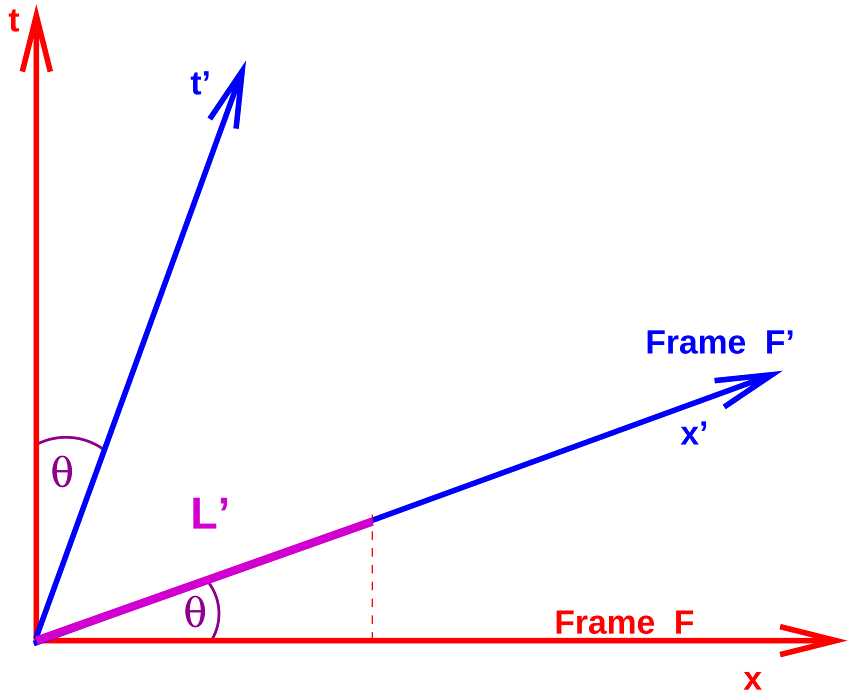
- Event X seen at different time and location in the two frames, projected on axes of F and F'

Lorentz transformation - schematic



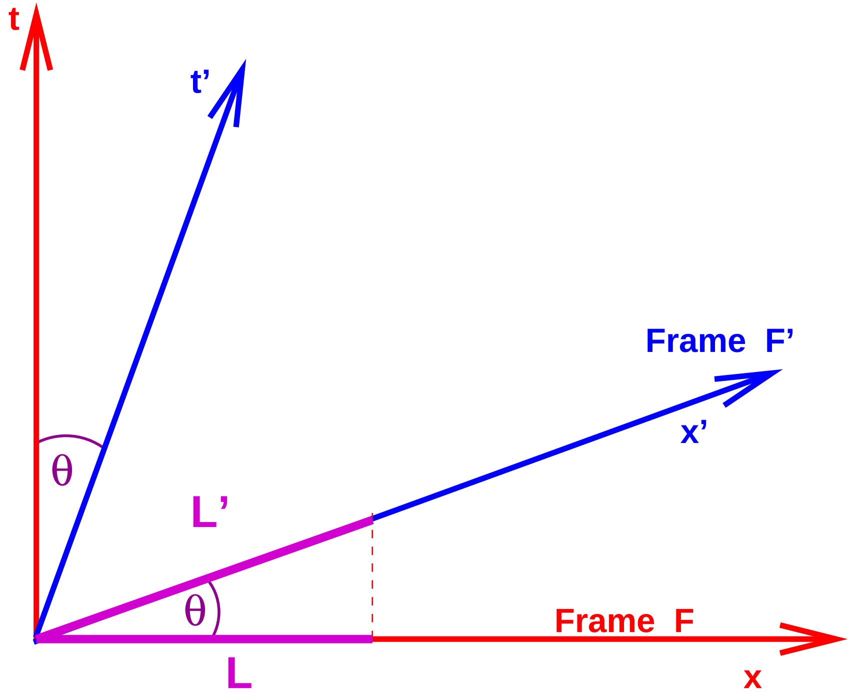
- Q: How would a Galilei-transformation look like ??

Lorentz contraction - schematic



- Length L' as measured in moving frame

Lorentz contraction - schematic



- From moving frame: L appears shorter in rest frame
- Length is maximum in frame (F') where object is at rest

Lorentz contraction

For the coffee break and lunch:

Could you "see" (visually) a Lorentz contraction ??
(if you run fast enough ...)

Time dilation

A clock measures time difference $\Delta t = t_2 - t_1$ **in frame F, measured at fixed position x, what is the time difference** $\Delta t' = t'_2 - t'_1$ **as measured from the moving frame F' ??**

For Lorentz transformation of time in moving frame we have:

$$t'_1 = \gamma(t_1 - \frac{v \cdot x}{c^2}) \quad \text{and} \quad t'_2 = \gamma(t_2 - \frac{v \cdot x}{c^2})$$

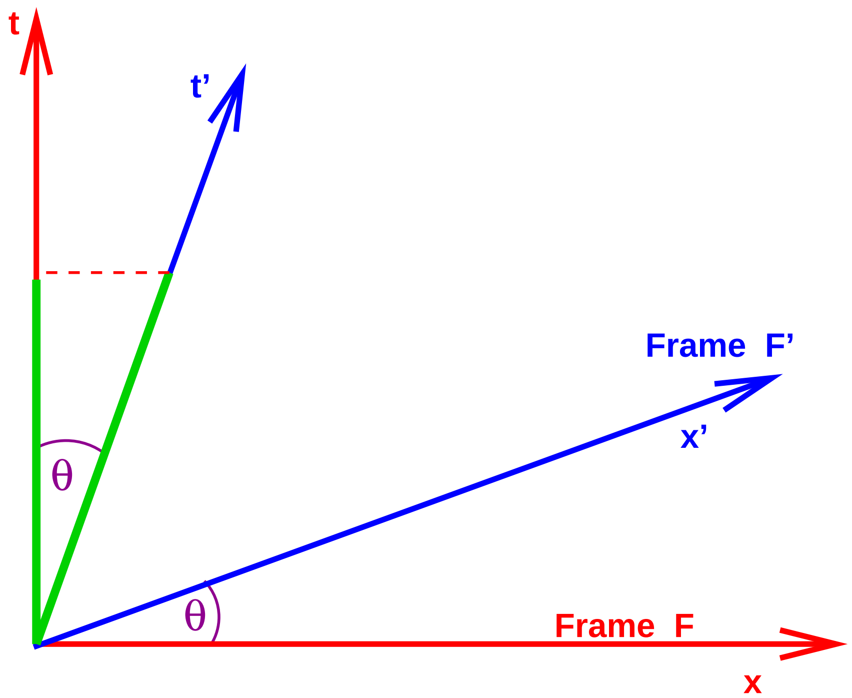
$$\Delta t' = t'_2 - t'_1 = \gamma \cdot (t_2 - t_1) = \gamma \cdot \Delta t$$

$$\rightarrow \Delta t' = \gamma \Delta t$$

Time dilation

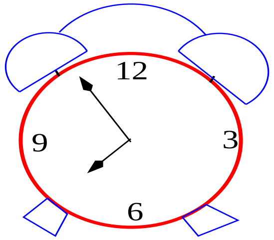
- **In moving frame time appears to run slower**
- **Why do we care ?**
  - μ **have lifetime of 2** μ**s (**$\equiv$ **600 m)**
  - **For** $\gamma \geq$ **150, they survive 100 km to reach earth from upper atmosphere**
  - **They can survive more than 2** μ**s in a** μ**-collider**
  - **Generation of neutrinos from the SPS beams**

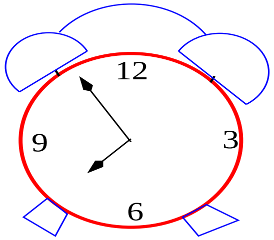
Time dilation - schematic



- From moving frame: time goes slower in rest frame
- Time shortest in frame (F') where object is at rest

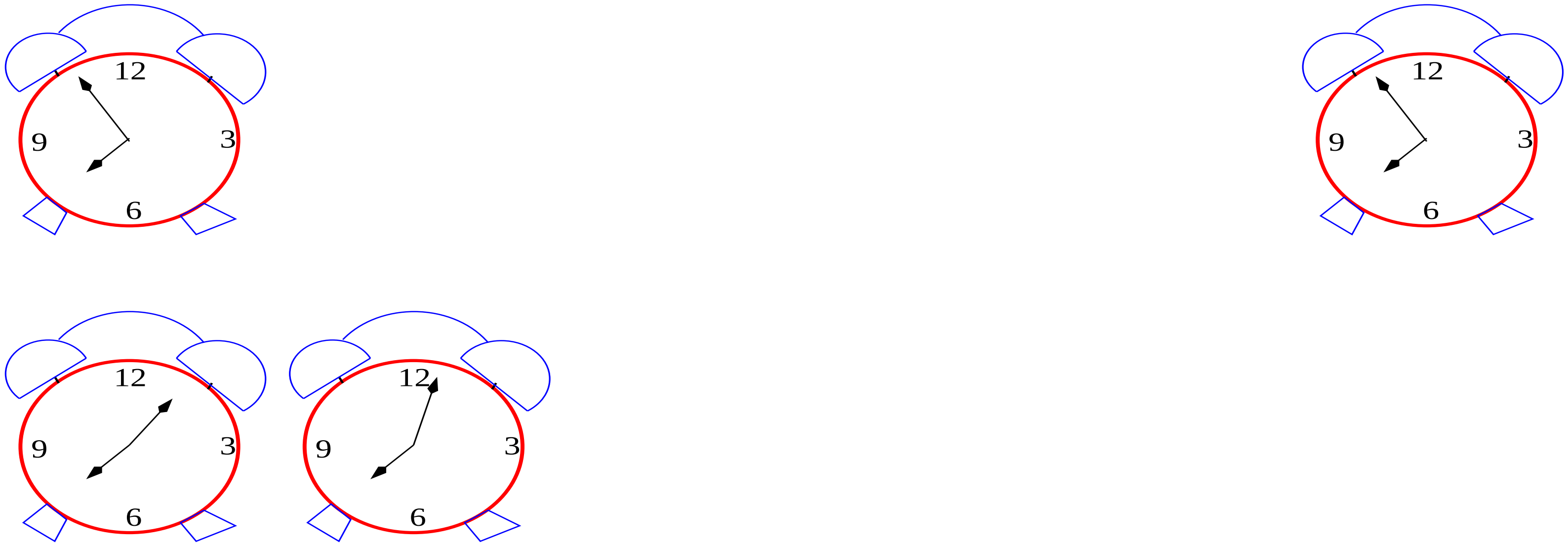
Moving clocks go slower





Moving clocks go slower

Ten minutes later ...



Travel by airplane:

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns !*)

*) **(unfortunately there is a catch 22 ...)**

Addition of velocities

- **Galilei:** $v = v_1 + v_2$
- **With Lorentz transform we have:**

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \qquad \text{or equivalently :} \qquad \beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

for $\beta = 0.5$ **we get:**

0.5c + 0.5c = 0.8c

0.5c + 0.5c + 0.5c = 0.93c

0.5c + 0.5c + 0.5c + 0.5c = 0.976c

0.5c + 0.5c + 0.5c + 0.5c + 0.5c = 0.992c

→ **Nothing can go faster than the speed of light ...**

First summary

- Physics laws the same in different moving frames ...
- Speed of light is maximum possible speed
- Constant speed of light requires Lorentz transformation
- Moving objects appear shorter
- Moving clocks seem to go slower
- No absolute space or time !

→ Now: applications and how to calculate something ...

Introducing four-vectors

Four-vector: $F = (f_1, f_2, f_3, f_4)$

a vector with four components

Example: position four-vector $X = (ct, x, y, z) = (ct, \vec{x})$

This mathematical setting is called Minkowski space and

Lorentz transformation can be written in matrix form:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \frac{-\gamma v}{c} & 0 & 0 \\ \frac{-\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$X' = M_L \circ X$$

Introducing four-vectors

Define a scalar product[*)] **like:** $X \diamond Y$

$$X = (x_0, \vec{x}), \qquad Y = (y_0, \vec{y}) \quad \longrightarrow \quad X \diamond Y = x_0 \cdot y_0 - \vec{x} \cdot \vec{y}$$

For example try $X \diamond X \quad (ct, \vec{x}) \diamond (ct, \vec{x})$**:**

$$X \diamond X = c^2t^2 - x^2 - y^2 - z^2$$

This product is an invariant, i.e.:

$$X \diamond X = c^2t^2 - x^2 - y^2 - z^2 = X' \diamond X' = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Invariant Quantities have the same value in all inertial frames

*) **definition of product not unique ! (I use PDG 2008)**

Why bother about four-vectors ?

- We have seen the importance of invariants:
- Ensure equivalence of physics laws in different frames

→ The solution: write the laws of physics in terms of four vectors

→ Any four-vector (scalar) product $F \diamond F$ has the same value in all coordinate frames moving at constant velocities with respect to each other ...

(remember that phrase ?)

Using four-vectors

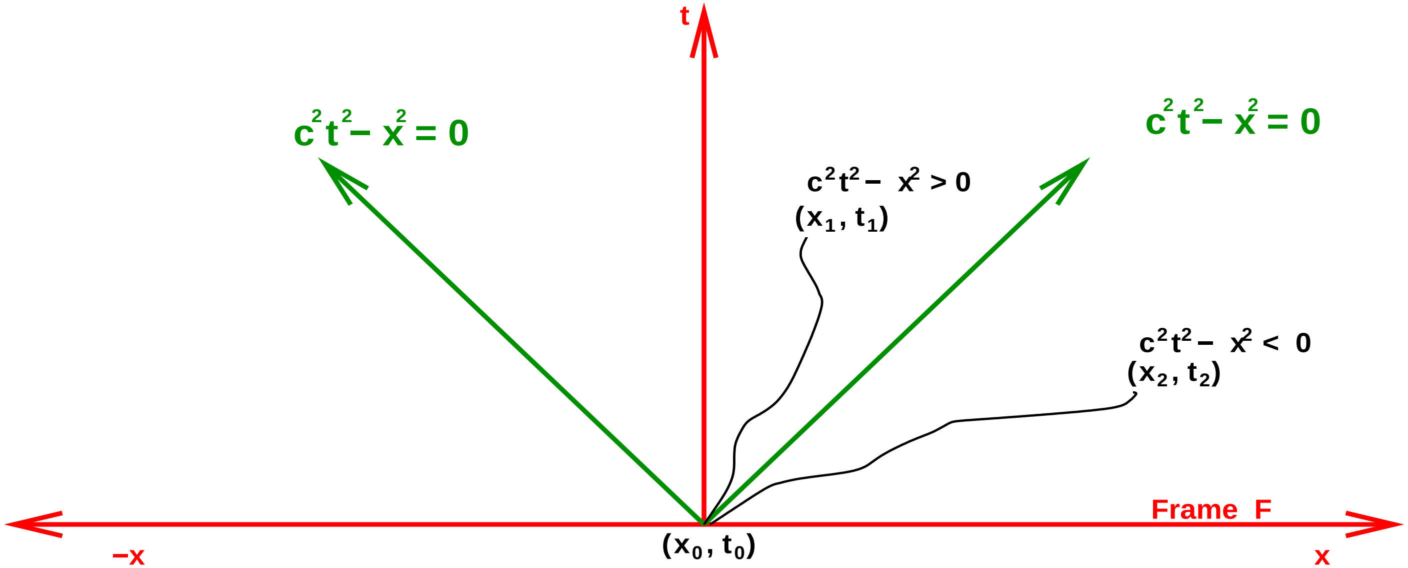
We can describe a distance in the spacetime between two points X_1 and X_2:

$$\Delta X = X_2 - X_1 = (ct_2 - ct_1,\ x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

$$\Delta s^2 \ = \ \Delta X \diamond \Delta X \ = \ c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Δs^2 **can be positive (timelike) or negative (spacelike)**

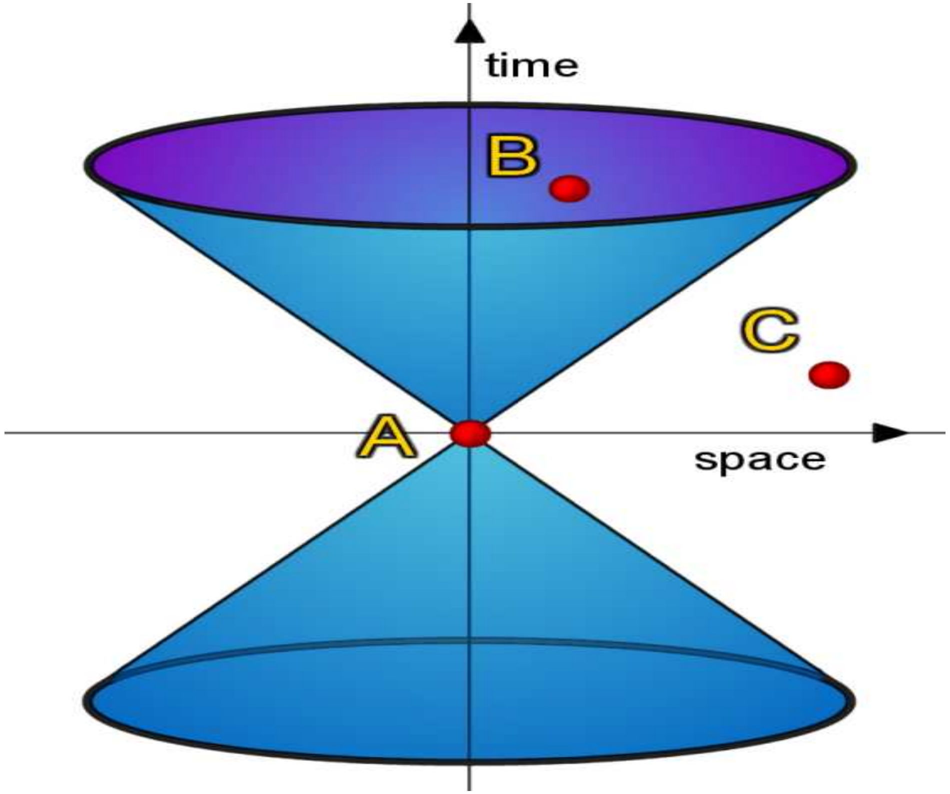
Moving in Minkowski space



- **Light travels with** $c^2 \cdot t^2 - x^2 \; = \; 0$
- **Particle travels with** $c^2 \cdot t^2 - x^2 \; > \; 0$ **(allowed)**
- **Particle travels with** $c^2 \cdot t^2 - x^2 \; < \; 0$ **(not allowed)**

→ **Allowed region defines** light cone

Light cone ...



- **Distances: timelike (AB), spacelike (AC)**

Using four-vectors

Special case (time interval $\vec{x_2} = \vec{x_1} + \vec{v}\Delta t$):

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \;=\; c^2\Delta t^2(1 - \frac{v^2}{c^2}) \;=\; c^2(\frac{\Delta t}{\gamma})^2 \;=\; c^2\Delta\tau^2$$

- $\Delta\tau$ **is the time interval measured in the moving frame**
- τ **is a fundamental time: proper time** τ

The meaning of ”proper time”

$\Delta\tau$ is the time interval measured <u>inside</u> the moving frame

Back to μ-decay

- **μ lifetime is $\approx$ 2 μs**
- **μ decay in $\approx$ 2 μs in their frame, i.e. using the ”proper time”**
- **μ decay in $\approx$ $\gamma\cdot$ 2 μs in the laboratory frame, i.e. earth**
- **μ appear to live longer than 2 μs in the laboratory frame, i.e. earth**

The meaning of ”proper time”

- **How to make neutrinos ?? Let pions decay:** $\pi \rightarrow \mu + \nu_\mu$
  - π**-mesons have lifetime of** $2.6 \cdot 10^{-8}$ **s (i.e. 7.8 m)**
  - **For 40 GeV** π**-mesons:** $\gamma = 288$
  - **In laboratory frame: decay length is 2.25 km (required length of decay tunnel)**
- **VERY intuitive (quote A. Einstein, modified):**
  - **It's 7:00 a.m. in bed, you close your eyes for** $\Delta\tau = 10$ **minutes, it's 9:00 a.m.**
  - **It's 7:00 a.m. in a meeting, you close your eyes for** $\Delta\tau = 10$ **minutes, it's 7:01 a.m.**

More four-vectors

Position four-vector X:

$$X = (ct, x, y, z) = (ct, \vec{x})$$

Velocity four-vector V:

$$V = \frac{dX}{d\tau} = \gamma \frac{dX}{dt} = \gamma \dot{X} = \gamma(\frac{d(ct)}{dt}, \dot{x}, \dot{y}, \dot{z}) = \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{v})$$

Please note that:

$$V \diamond V = \gamma^2(c^2 - \vec{v}^2) = c^2!!$$

- **c is an invariant (of course), has the same value in all inertial frames**

More four-vectors

Momentum four-vector P:

$$P = m_0 V = m_0 \gamma(c, \vec{v}) = (\mathbf{m}c, \vec{p})$$

using:

m_0 **(mass of a particle)**

$\mathbf{m} \equiv m_0 \cdot \gamma$ **(relativistic mass)**

$\vec{p} = \mathbf{m} \cdot \vec{v} = m_0 \gamma \vec{v}$ **(relativistic 3-momentum)**

We can get another invariant: $P \diamond P = m_0^2 (V \diamond V) \;=\; m_0^2 c^2$

Invariant of the four-momentum vector is the mass m_0

→ **The rest mass is the same in all frames (thanks a lot ..)**

Still more four-vectors

Force four-vector F:

$$F = \frac{dP}{d\tau} = \gamma\frac{dP}{dt} = \gamma\frac{d}{dt}(mc, \vec{p}) = \gamma(c\frac{dm}{dt}, \frac{d\vec{p}}{dt}) = \gamma(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt})$$

and we had already more four-vectors:

$$\begin{aligned} \text{Coordinates :} \quad & X = (ct, x, y, z) = (ct, \vec{x}) \\ \text{Velocities :} \quad & V = \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{v}) \\ \text{Momenta :} \quad & P = m_0 V = m_0\gamma(c, \vec{v}) = (\mathbf{m}c, \vec{p}) \end{aligned}$$

$X \diamond X$, $V \diamond V$, $P \diamond P$, $P \diamond X$, $V \diamond F$,... **are ALL invariants**

Dynamics with four-vectors

We compute: $V \diamond F \ = \ 0$

All right, 0 **is the same in all frames, sounds useless, but:**

$$V \diamond F \ = \ 0 \quad \longrightarrow \quad \frac{d}{dt}(mc^2) - \vec{f}\vec{v} = 0$$

Now $\vec{f}\vec{v}$ **is rate of change of kinetic energy** dT/dt

after integration:

$$T = \int \frac{dT}{dt} dt = \int \vec{f}\vec{v} dt = \int \frac{d(mc^2)}{dt} dt = mc^2 + const.$$

$$T = mc^2 + const. = mc^2 - m_0 c^2$$

Relativistic energy

Interpretation:

$$E = mc^2 = T + m_0c^2$$

- **Total energy** E **is** $E = mc^2$
- **Sum of kinetic energy plus rest energy**
- **Energy of particle at rest is** $E_0 = m_0c^2$

$$E = m \cdot c^2 = \gamma m_0 \cdot c^2$$

using the definition of relativistic mass again: $m = \gamma m_0$

Still more four-vectors

Equivalent four-momentum vector (using E instead of m):

$$P = (mc, \vec{p}) \rightarrow (E/c, \vec{p})$$

then:

$$P \diamond P = m_0^2 c^2 = \frac{E^2}{c^2} - \vec{p}^2$$

follows:

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

another familiar expression ...

Relativistic energy

These units are not very convenient:

$m_p = 1.672 \;\cdot\; 10^{-27}$ **Kg**

$\rightarrow\; m_p c^2 = 1.505 \;\cdot\; 10^{-10}$ **J**

$\rightarrow\; m_p c^2 = 938$ **MeV** $\;\rightarrow\; m_p = 938$ $\mathbf{MeV/c^2}$

$\rightarrow\; m_p c^2 \cdot \gamma(7\ TeV) = 1.123 \;\cdot\; 10^{-6}$ **J**

$\rightarrow\; m_p c^2 \cdot \gamma(7\ TeV) \cdot 1.15 \cdot 10^{11} \cdot 2808 = 360 \;\cdot\; 10^{6}$ **J**

Practical units

In particle physics: omit c and dump it into the units:

$$[\mathbf{E}] = \mathbf{eV} \quad [\mathbf{p}] = \mathbf{eV/c} \quad [\mathbf{m}] = \mathbf{eV/c}^2$$

Four-vectors get an easier form:

$$P = (m, \vec{p}) = (E, \vec{p})$$

and from $P \diamond P \ = \ E^2 - p^2 \ = \ m_0^2$ **follows directly:**

$$E^2 = \vec{p}^2 + m_0^2 \quad (= m^2 = \gamma^2 m_0^2)$$

Relativistic energy

Note:

$$E = mc^2 = \gamma \cdot m_0 c^2 \quad \rightarrow \quad E = \gamma m_0$$

$$p = m_0 \gamma v = \gamma m_0 \cdot \beta c \quad \rightarrow \quad p = \gamma m_0 \cdot \beta$$

$$T = m_0(\gamma - 1) \cdot c^2 \quad \rightarrow \quad T = \gamma m_0 - m_0$$

Interpretation of relativistic energy

- **For any object, $m \cdot c^2$ is the total energy**
  - **Object can be composite, like proton ..**
  - **m is the mass (energy) of the object "in motion"**
  - **m_0 is the mass (energy) of the object "at rest"**
- **For discussion: what is the mass of a photon ?**

Relativistic mass

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

assume a 75 kg heavy man:

- **Rocket at 100 km/s, γ = 1.00000001, m = 75.000001 kg**
- **PS at 26 GeV, γ = 27.7, m = 2.08 tons**
- **LHC at 7 TeV, γ = 7642, m = 573.15 tons**
- **LEP at 100 GeV, γ = 196000, m = 14700 tons**

Relativistic mass

- **Why do we care ?**

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- **Particles cannot go faster than c !**
- **What happens when we accelerate ?**

Relativistic mass

When we accelerate:

- **For v $\ll$ c:**
  - **E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **E, m, p increase, but v does not !**
  - **Remember that for later**

Relativistic energy

Since we remember that:

$$T = m_0(\gamma - 1)c^2$$

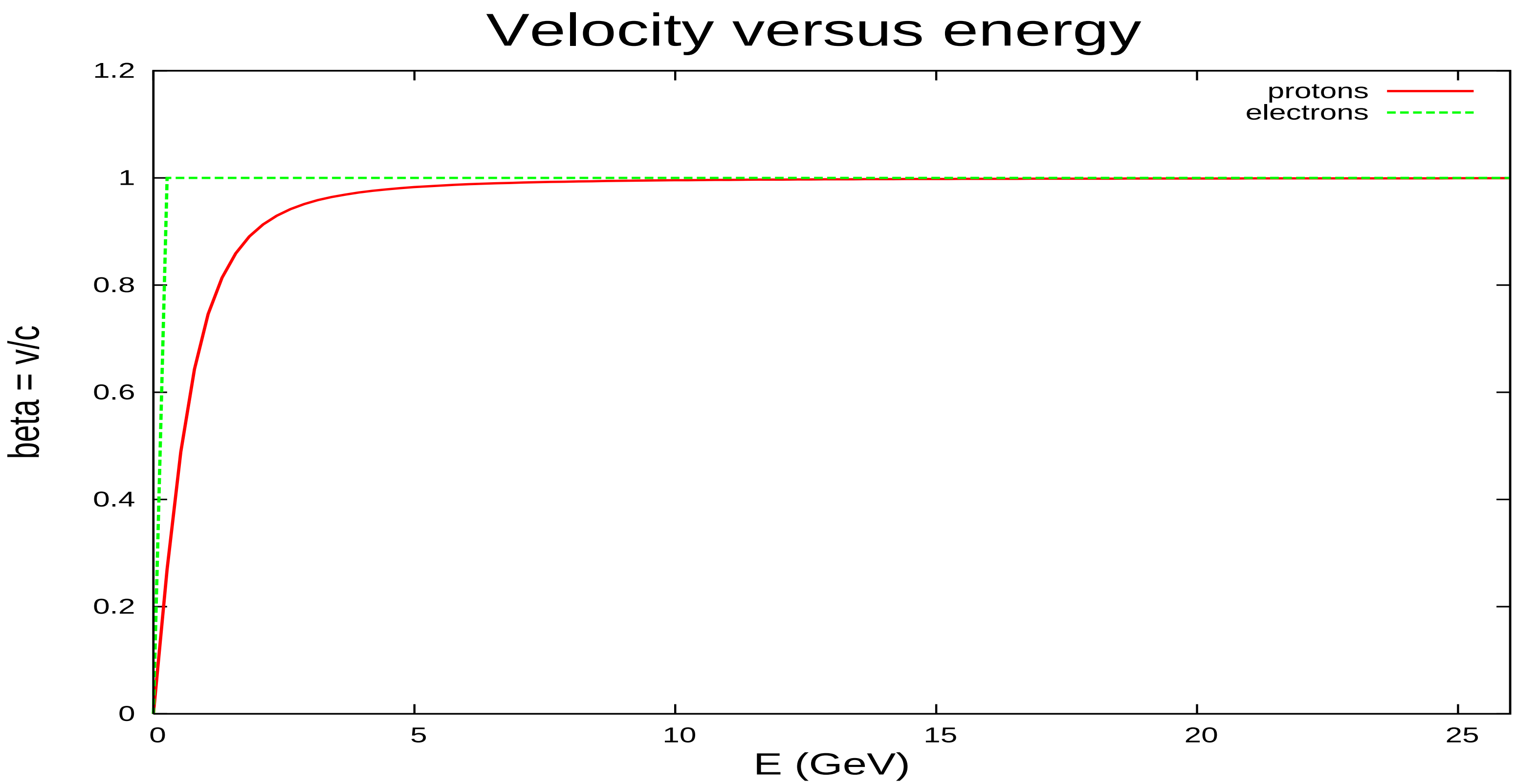
therefore:

$$\gamma = 1 + \frac{T}{m_0 c^2}$$

we get for the speed v**, i.e.** β**:**

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Velocity versus energy (protons)



Why do we care ??

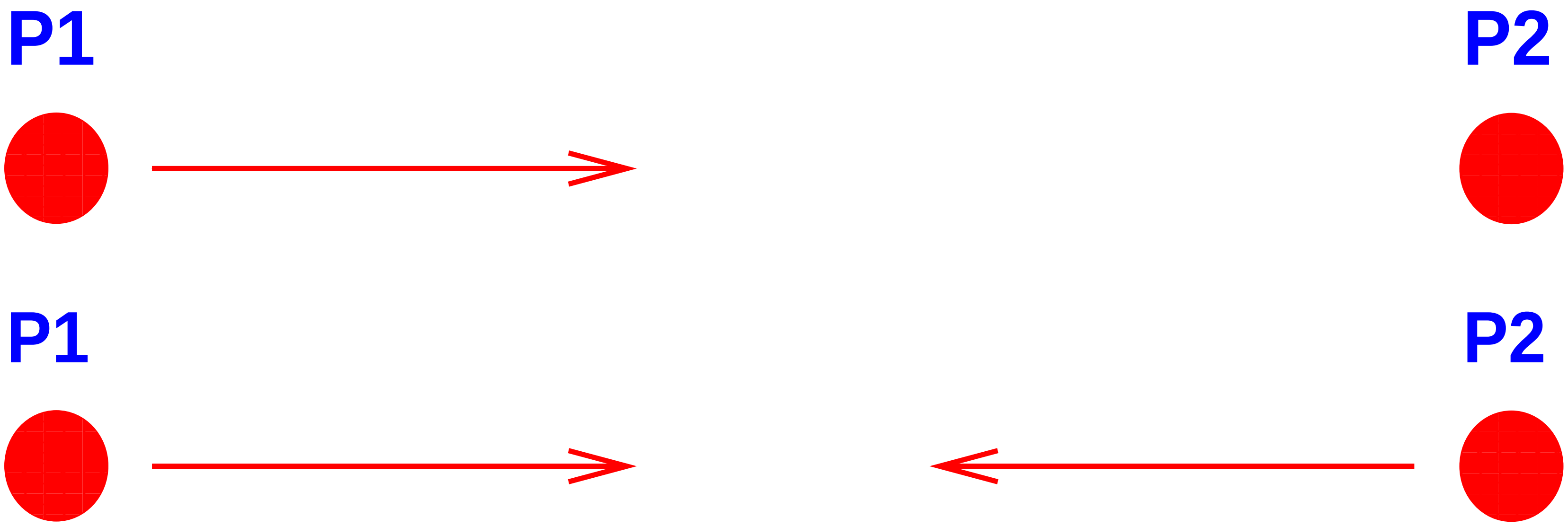
E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μs
7000	299792.455	7462.7	0.99999999	88.92446 μs

- **For identical circumference very small change in revolution time**
- **If path for faster particle slightly longer, the faster particle arrives later !**

Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic relationships, e.g.
  - Particle decay (find mass of parent particle)
  - Particle collisions →

Particle collisions



➤ **What is the available collision energy ?**

Particle collisions - collider

Assume identical particles and beam energies, colliding head-on

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

Particle collisions - collider

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The square of the total available energy s **in the centre of mass system is the momentum invariant:**

$$s = P^* \diamond P^* = 4E^2$$

$$E_{cm} = \sqrt{P^* \diamond P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

Particle collisions - fixed target

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

Particle collisions - fixed target

With the above it follows:

$$P^* \diamond P^* = E^2 + 2m_0 E + m_0^2 - \vec{p}^2$$

since $E^2 - \vec{p}^2 = m_0^2$ **we get:**

$$s = 2m_0 E + m_0^2 + m_0^2$$

if E **much larger than** m_0 **we find:**

$$E_{cm} = \sqrt{s} = \sqrt{2m_0 E}$$

Particle collisions - fixed target

Homework: try for $E1 \neq E2$ **and** $m1 \neq m2$

Examples:

collision	beam energy	$\sqrt{s}$ (collider)	$\sqrt{s}$ (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	7000 (GeV)	14000 (GeV)	114.6 (GeV)
e+e−	100 (GeV)	200 (GeV)	0.320 (GeV)
TLEP	175 (GeV)	350 (GeV)	0.423 (GeV) !

Kinematic invariant

We need to make cross sections (and therefore luminosity) invariant !
This is done by a calibration factor which is (without derivation):

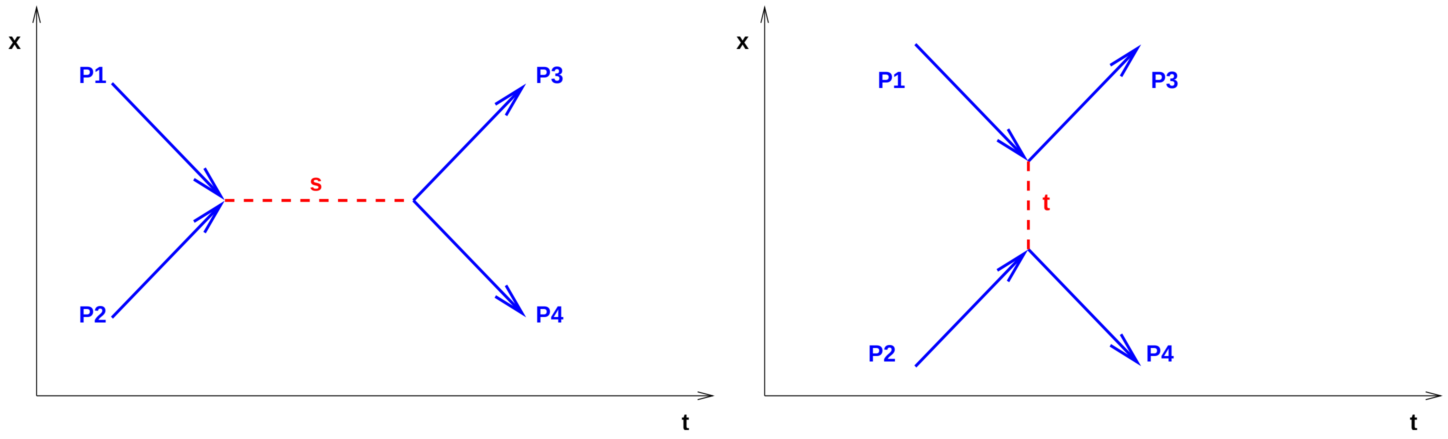
$$K = \sqrt{(\vec{v_1} - \vec{v_2})^2 - (\vec{v_1} \times \vec{v_2})^2/c^2}$$

Here $\vec{v_1}$ and $\vec{v_2}$ are the velocities of the two (relativistic) beams.
For a (symmetric) collider, e.g. LHC, we have:

$$\vec{v_1} = -\vec{v_2}, \qquad \vec{v_1} \times \vec{v_2} \; = \; 0 \quad \mathrm{head-on!}$$

→ $K \; = \; 2 \; \cdot \; c$ **!**

For completeness ...



Squared centre of mass energy:

$$s = (P1 + P2)^2 = (P3 + P4)^2$$

Squared momentum transfer in particle scattering (small t - small angle, see again lecture on Luminosity):

$$t = (P1 - P3)^2 = (P2 - P4)^2$$

Kinematic relations

We have already seen a few, e.g.:

- $T = E - E_0 = (\gamma - 1)E_0$
- $E = \gamma \cdot E_0$
- $E_0 = \sqrt{E^2 - c^2 p^2}$
- **etc. ...**

Very useful for everyday calculations →

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp $=$	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T $=$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

- **Example: CERN Booster**

At injection: T = 50 MeV

- **E = 0.988 GeV, p = 0.311 GeV/c**
- γ **= 1.0533,** β **= 0.314**

At extraction: T = 1.4 GeV

- **E = 2.338 GeV, p = 2.141 GeV/c**
- γ **= 2.4925,** β **= 0.916**

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

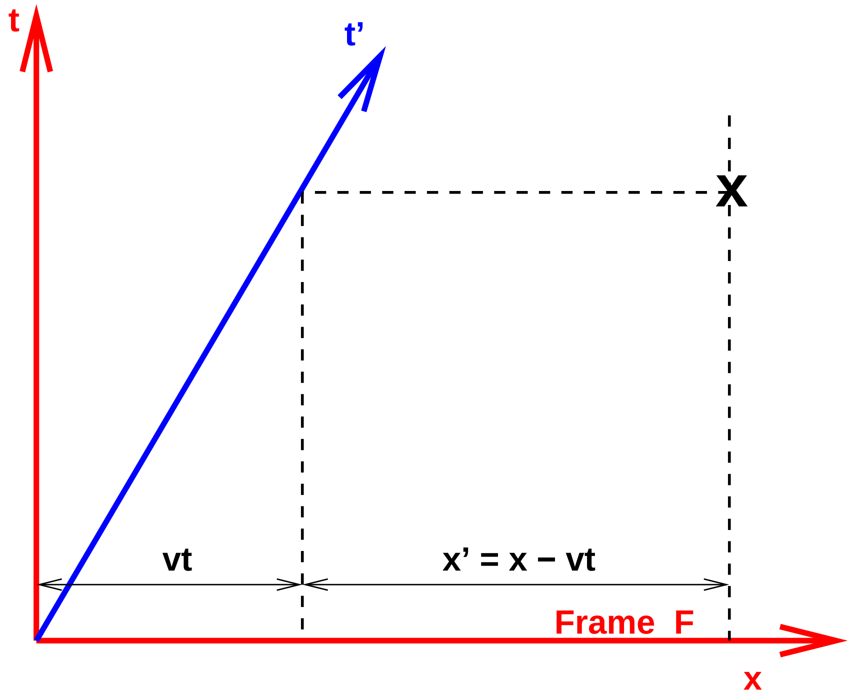
Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4}$ → $\frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

Summary

- Special Relativity is very simple, derived from basic principles
- Relativistic effects vital in accelerators:
  - Lorentz contraction and Time dilation
  - Invariants !
  - Relativistic mass effects
  - Modification of electromagnetic field
- Find back in later lectures ...

- BACKUP SLIDES -

Galilei transformation - schematic



- **Rest frame and Galilei transformation ...**

Forces and fields

Motion of charged particles in electromagnetic fields $\vec{E}, \vec{B}$ determined by Lorentz force

$$\vec{f} = \frac{d}{dt}(m_0 \gamma \vec{v}) = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

or as four-vector:

$$F = \frac{dP}{d\tau} = \gamma \left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right)$$

Field tensor

Electromagnetic field described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

derived from four-vector $A_\mu = (\Phi, \vec{A})$ like:

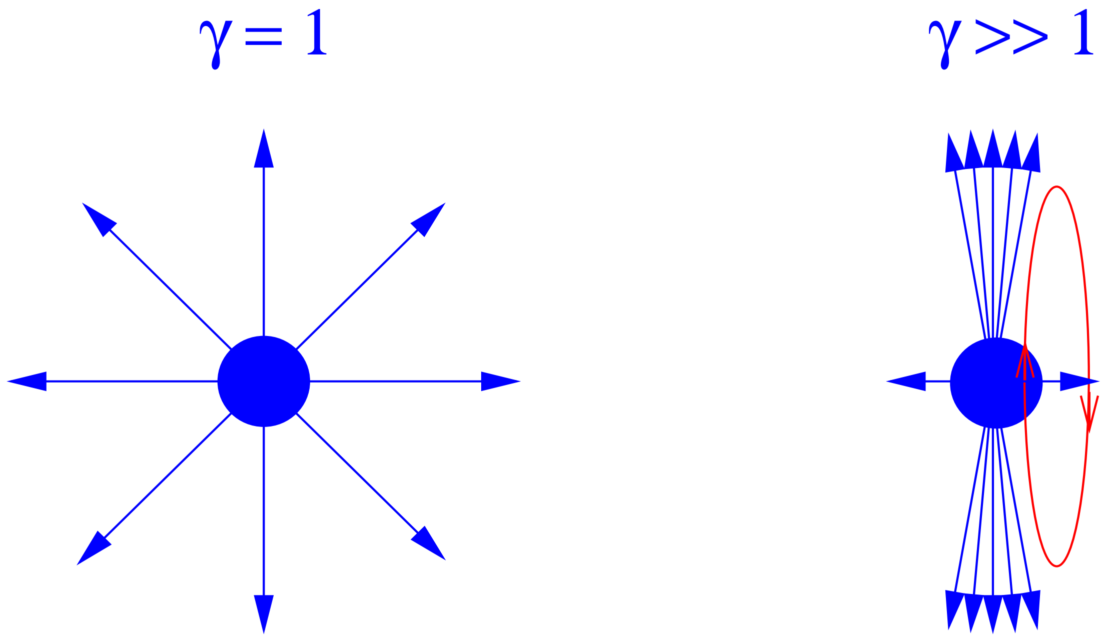
$$F^{\mu\nu} = \delta^\mu A^\nu - \delta^\nu A^\mu$$

Lorentz transformation of fields

$$
\begin{aligned}
\vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\
\vec{B}'_{\perp} &= \gamma\left(\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right) \\
\vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel}
\end{aligned}
$$

- **Field perpendicular to movement transform**

Lorentz transformation of fields



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**