

On Yangian Symmetry of Scattering Amplitudes

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Work with B. Schaub

(in progress) J. Brödel & N. Rosse

- $N=4$ SYM is conformal
 \Rightarrow Scattering amplitudes are conformally invariant. *
- Planar $N=4$ SYM is integrable
 \Rightarrow Planar amplitudes have a dual conformal invariance.
- Conformal + dual conformal close into Yangian algebra.

Integrable philosophy: Yangian constrains answers uniquely.

Unique invariant: S-Matrix. Nice!

Easy answer: $S=1$ $\xleftarrow{\text{common CFT wisdom}}$ boring

Does not agree with perturbative calculations!

What Went Wrong and Why Care?

Two important issues:

- How does the Yangian act on S-Matrix?
- What about IR divergences?

Motivation:

Does the Yangian help computing amplitudes?

Probably not: Useful & efficient methods are often developed much faster than showing their Yangian origin (e.g. original Bethe ansatz).

But: Symmetry is often essential in making the obtained methods more rigorous.

\Rightarrow Understanding Yangian could prove earlier results.

Even more: S-Matrix invariance could prove Yangian!

\Rightarrow Prove integrability of planar $N=4$ SYM.

Yangian Action

Naive action on scattering amplitudes:

Level-0: $J^a = \sum_k J_k^a$ action on local action defined
(conformal) site k for all amplitudes

Level-1: $\hat{J}^a = \sum_{k < l} f_{bc}^a J_k^b J_l^c$ non-local action
(contains dual conformal) requires single-trace!

Symmetry: $J^a A_n = \hat{J}^a A_n = 0$ unless ...

Careful about:

- specific momentum configurations,
- signature of spacetime: 3,1 vs. 2,2 | 4,
- iε's,
- signature of external particle energies.

Actually $J^a A_n \neq 0$

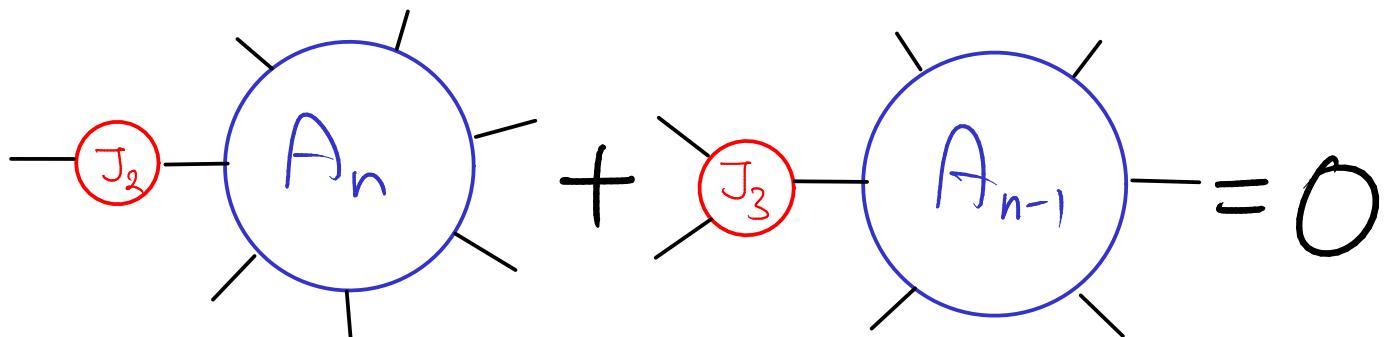
- at tree level: residual contributions on singular (collinear) momentum configurations
- at loop level: loop integrals smear momenta, symmetry broken for generic external momenta
- at loop level: Regularisation of IR divergences breaks conformal symmetry ...

all problems avoided when looking at loop integrand with generic matrix 3

Collinear Behavior and Conformal Symmetry

Can cure collinear symmetry violations at tree level [Bargheer et al '09]

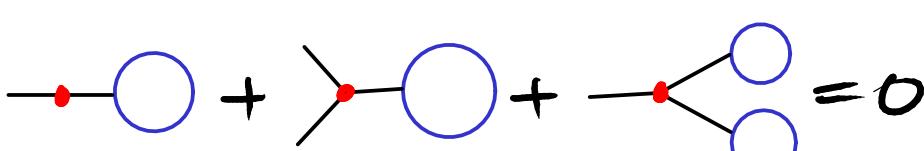
Correct action of $J = J_2$ (2-vertex) by 3-vertex J_3 :



Exciting:

- Inhomogeneous equations; lower-leg amplitudes influence invariance of higher-leg amplitudes.
- Only complete S-matrix can be invariant!

To first approximation okay, but further corrections needed!

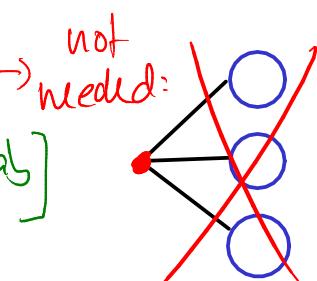


[Sever-Vieira '09]

Symmetry of S-Matrix Operator [KOD + Schwab in progress]

$$\text{energy flow } \uparrow \left[\frac{1}{J_2} + \frac{J_3^+}{J_3^-}, \frac{1}{J_3^-}, \frac{\text{S}}{\text{III}} \right] = 0$$

S-matrix operator in 3+1 spacetime



Exact Symmetry of S-Matrix

$$\left[\frac{1}{J_2} + \frac{\vee}{J_3^+} + \frac{1}{J_3^-}, \frac{|||}{S} \right] = 0$$

Features:

- + Formally exact statement at all legs and loops!
(formal manipulations of graphs, paying attention to iε's, unitarity and absence of loops in time due to time ordering.)
- ± Is intrinsically 3+1 statement. (physical but inconvenient?)
- Disregards regularisation, (see below)
- Disconnected parts of S-matrix relevant even in the planar limit. (connection via J_3^-)

How to act with Yangian on disconnected S-matrix?!

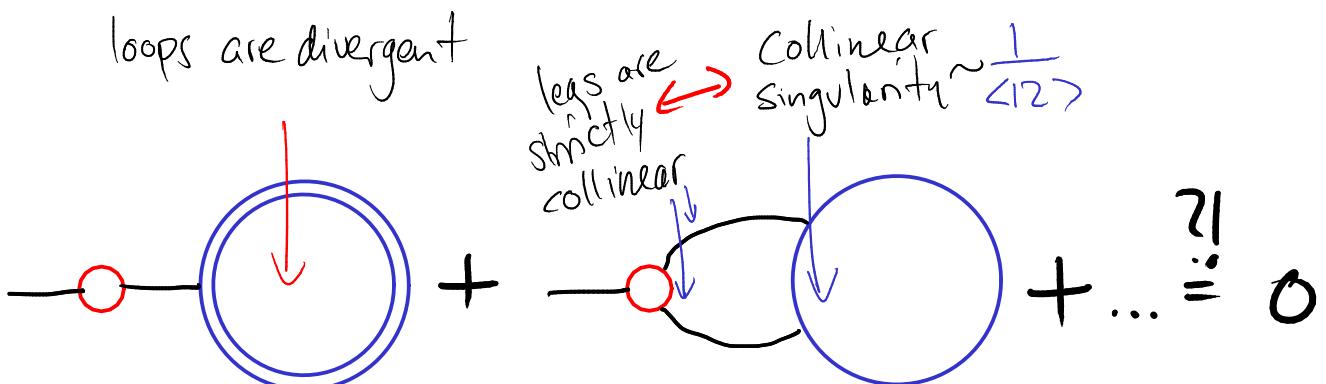
- Okay for conformal symmetries J^a as well as $\hat{P} = \tilde{k}$,
- not okay for generic \tilde{J}^a such as $\hat{Q} = \tilde{S}$,
- not okay for closure of Yangian algebra.

Idea: Act on $T = -i \log S$ operator!

- | | | |
|-------------------------|---|---|
| Features
of γ | <ul style="list-style-type: none"> • connected, Yangian can act • requires 3+1 signature • Hermitian on real momenta | <ul style="list-style-type: none"> • messy |
|-------------------------|---|---|

Yangian and IR Divergences

More serious problem: Cannot act at loop level



Divergences may cancel, perhaps also finite remainder.

How do we know in practice? **Regularise!**

- dimensional regularisation

- J_3 hard to generalise to $D=4-2\epsilon$
- do not expect conformal symmetry
- spinor-helicity does not apply

- massive regularisation

- no collinear configurations
- do not expect conformal symmetry
- spinor-helicity does not apply

In any case:

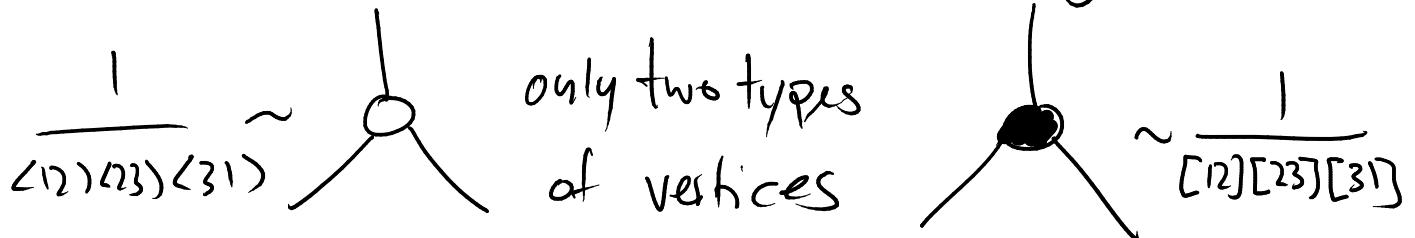
- J_3 must be deformed according to regulator
higher J_n are likely to be required
- What to expect for $J_S \approx 0$ when S is divergent?
- less divergent than S ? - limit = 0? - exactly = 0?
too weak? why? too strong!

\Rightarrow Need a finite S -matrix!

Regularisation via helicities

- Exciting proposal: \rightarrow Staudacher's talk
 [Fino, Tchouassi; Moneghetti, Plefka, Staudacher '12]
- Deform particle helicities $\eta_{\frac{1}{2}} \rightarrow \eta_{\frac{1}{2}} + c_k$ individual choice for each particle
 - Results analytic in c_k , poles at $c_k \rightarrow 0$. still not finite S-matrix, but:
 - Manifest conformal and Yangian symmetry at $c_k \neq 0$

Based on on-shell formalism for scattering amplitudes



Deformation

$$\begin{array}{ccc} \text{Diagram 1: } & \frac{1}{<12>^{1+(c_1+c_2-c_3)/2} <23>^{1+(c_2+c_3-c_1)/2} <31>^{1+(c_3+c_1-c_2)/2}} \\ \text{Diagram 2: } & \frac{1}{[12]^{1+(c_1+c_2-c_3)/2} [23]^{1+(c_2+c_3-c_1)/2} [31]^{1+(c_3+c_1-c_2)/2}} \end{array}$$

Nice: deforms amplitude integrands s.t. integrals become finite (cf. definition of Γ function)

Main example:

On-shell diagram for 1-loop 4pt

$$\sim \frac{1}{c_k}$$

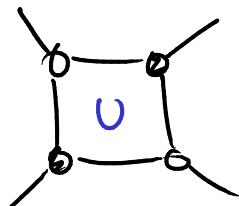
Symmetry of Deformed On-shell Amplitudes

Helicity deformation changes representation of conformal Symmetry on particles. Extend $\text{PSU}(2,2|4)$ to SU_c .
 c_n is eigenvalue of new central charge C (CPT requires $c_0=0$ always)
Want C to be a symmetry of amplitudes:

$$C = \sum_n c_n = \sum c_n = 0 \Rightarrow \text{conserved flow}$$

How about Yangian Symmetry?

Tree-level 4pt* considered by Berlin group



Amplitude is Yangian-invariant,
behaves like R-matrix $R(U)$.

\Rightarrow integrability / spectral parameter for amplitudes.

* also tree-level MHV considered

Yangian Symmetry for Deformed On-shell Amplitudes

Want to understand Yangian symmetry better towards constructing invariants.

→ systematic study with Johannes Brodbeck & Matteo Rosso

Chain of external particles is inhomogeneous (c_u 's)

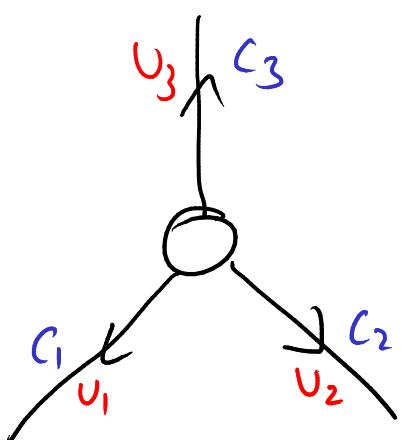
General Yangian representation on tensor product:

$$J^a = \sum_k J_k^a \quad \hat{J}^a = \sum_{k \leq c} \epsilon_{abc} J_k^b J_c^c + \sum_k \hat{J}_k$$

Ansatz for single-leg representation: evaluation rep

$$\hat{J}_k = u_k J_k \quad u_k \text{ is evaluation or spectral parameter.}$$

Apply to simplest amplitudes: 3 pt!



Amplitude is Yangian-invariant

provided that: whether u_k appear more fundamental (3 vs 2), but overall shift of u_k 's always inconsequential

$$c_1 = u_2 - u_3$$

$$c_2 = u_3 - u_1$$

$$c_3 = u_1 - u_2 \quad c_u \text{ or } u_u?$$

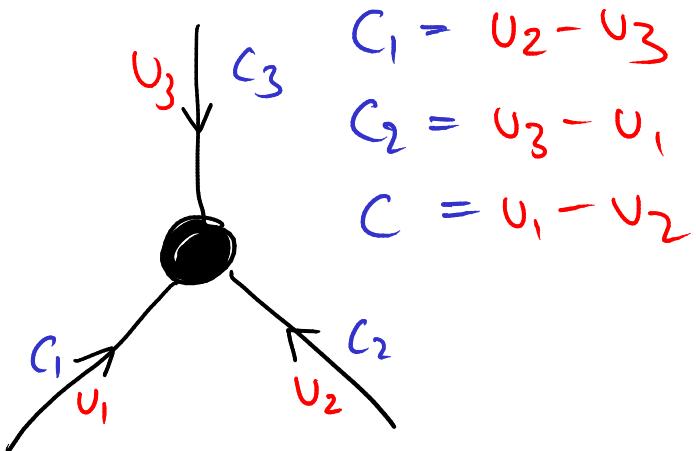
Question: Which parameters are more fundamental?

Yangian Symmetry of Composite Amplitudes

Investigate some other amplitudes.

Equivalent for
other 3pt vertex:

Note: flow of
lines inverted!



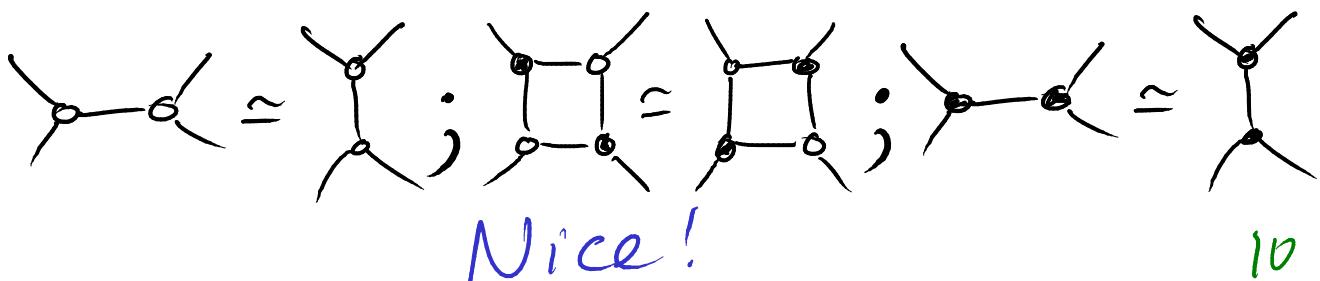
Inverting the flow
on a line ↑
assignment of direction
is arbitrary but parameters depend on it

$$\begin{array}{ccc} \xrightarrow{cu} & \text{when} & \xleftarrow{c' u'} \\ \downarrow & & \downarrow \\ \xleftarrow{c' u'} & & \end{array}$$

(consider evaluation rep)

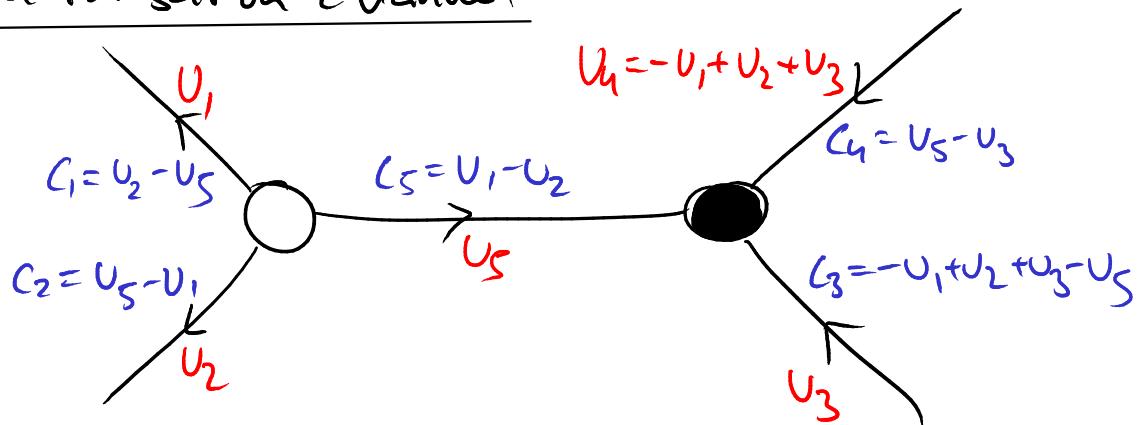
Can join amplitudes and close loops. (same argument
as for undeformed
Yangian invariance preserved by gluing amplitudes)
provided that assignment of c_k 's and u_k 's is consistent!

Also the basic moves of on-shell amplitudes
are consistent with assignment of (external) c_k, u_k



Some Examples

4pt factorisation channel



Somewhat messy! How to describe parameters best?

- one constraint among external C_n 's
- one constraint among external U_n 's New!

4pt tree level

Nice!

Somewhat symmetric

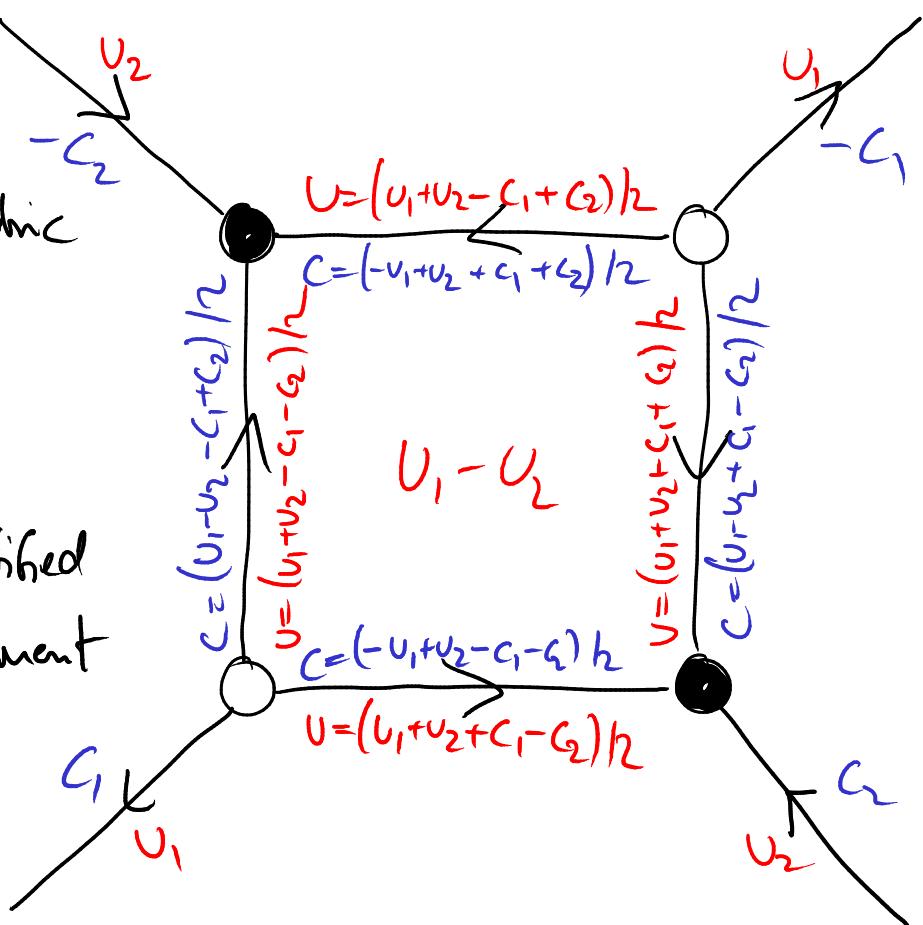
2 ext c_n 's

2 ext U_n 's

Opposite legs identified

R-matrix assignment

$$R(U_1 - U_2)$$



More Examples

Observe: Constraints between external c_u 's and v_u 's depend strongly on diagram (exterior & interior).

Dangerous: Different constraints can spoil

- comparison of graphs (tree vs. loop)
- Summing up contributions to a physical process (tree & loop)

May invalidate manifest Yangian invariance of the helicity regularisation scheme (back to dai^H)!

More insights needed. Count free variables:

$$\text{DOF} = 3V - 2I$$

$3v_u$'s $\stackrel{\nearrow}{=}$ E matching of c & v

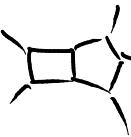
3-valent vertices: $3V = 2I + E$

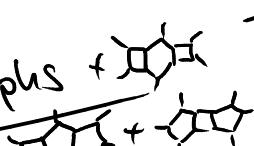
vertices \uparrow internal/external lines \uparrow

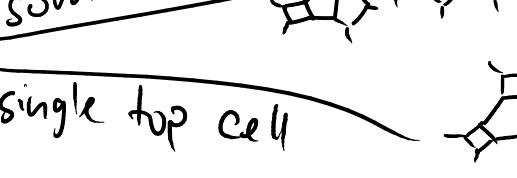
One variable per external line! But of which kind?

Further Examples:

5 spectral parameters possible

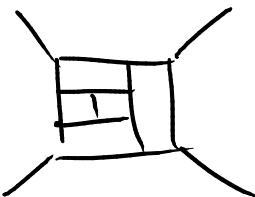
- 5pt tree  - or 4 c's and 1 v

sum of 3 graphs  - 3 incompatible sets of constraints!

- 6pt NMHV tree 

single top cell

- 3 c's & 3 v's
- opposite legs paired

- 4pt one-loop 

- all external $c_u = 0$
- 4 free v_u 's
- different from 4pt tree

Incompatible constraints

at the same or at different loop levels!

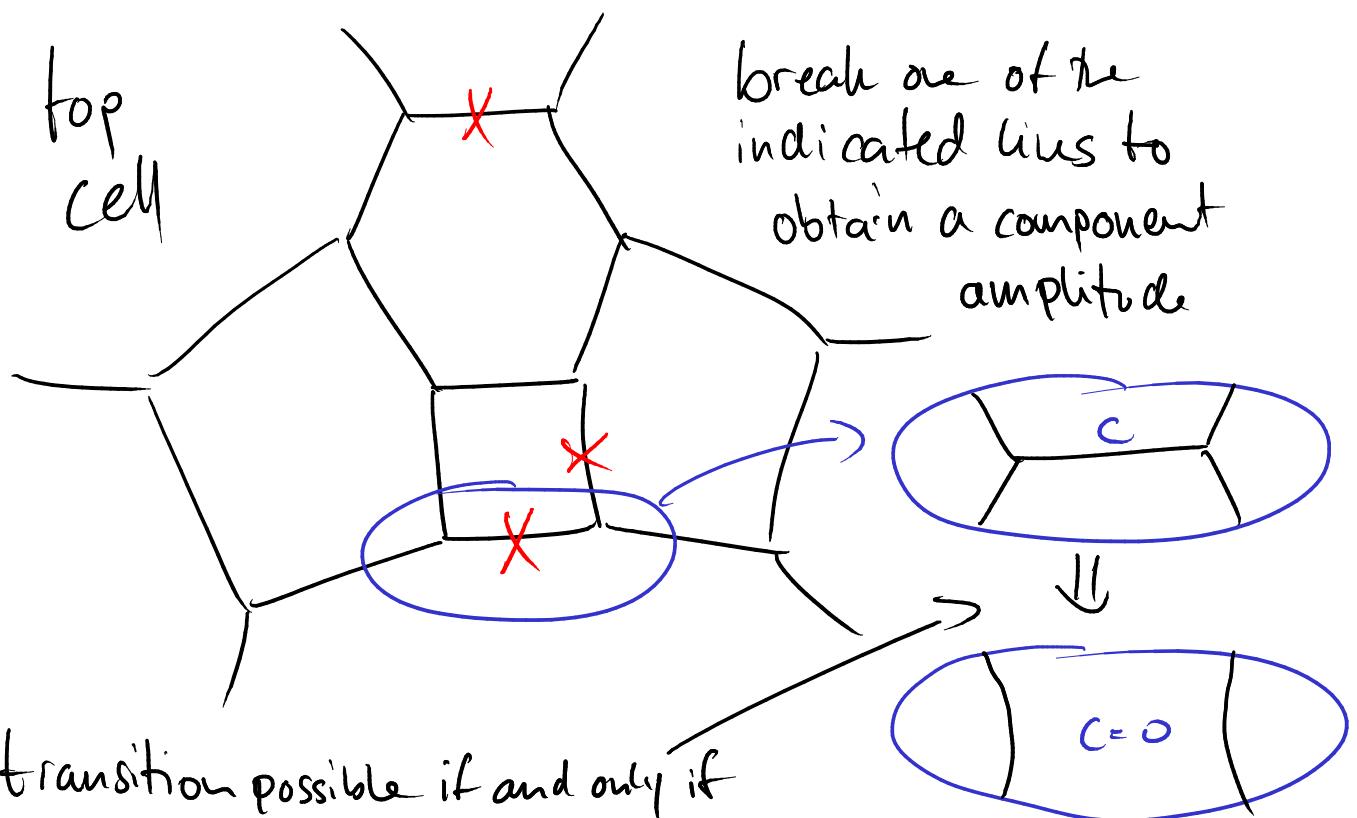
Incompatible constraints arise in some of graphs.

Very bad for Yangian invariance

- incompatible assignments spoil Yangian invariance
- satisfying all constraints reduces $\text{DOF} < E$.
- easily end up with constant v and all $c_i \leq 0$
 \Rightarrow back to undeformed amplitudes & divergences

Smells like Yangian anomaly! Can't have that?!

Why? Consider 6pt NMHV top cell vs 3 components



\Rightarrow Better use top cell! Is there always single top cell at loops?

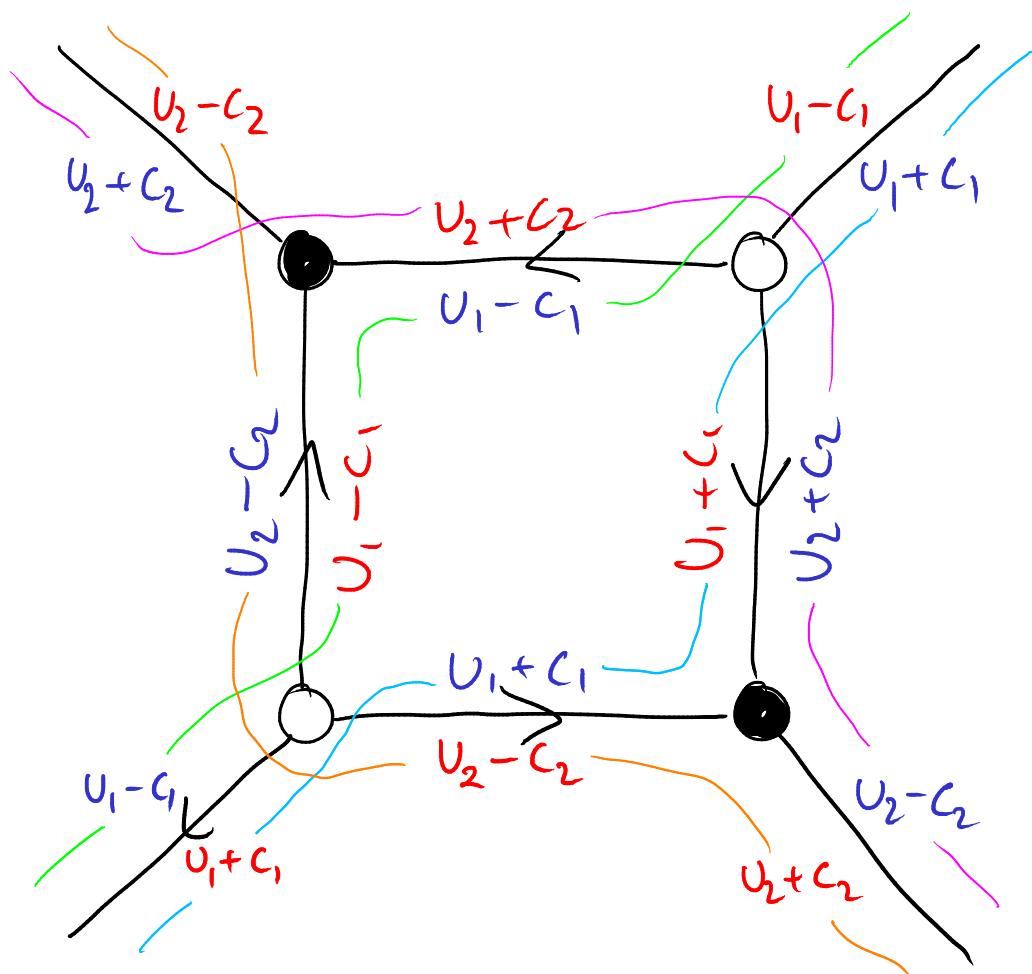
Improved Assignment

Still individual graphs are just fine!

Want to understand the assignment of c_i 's and u_i 's.

Idea: In quantum integrable systems the spectral parameters (u_i) often mix with charges (c_i).

Let's try $U^\pm = u \pm c$.



Rings a bell?

Yangian Permutation Flow

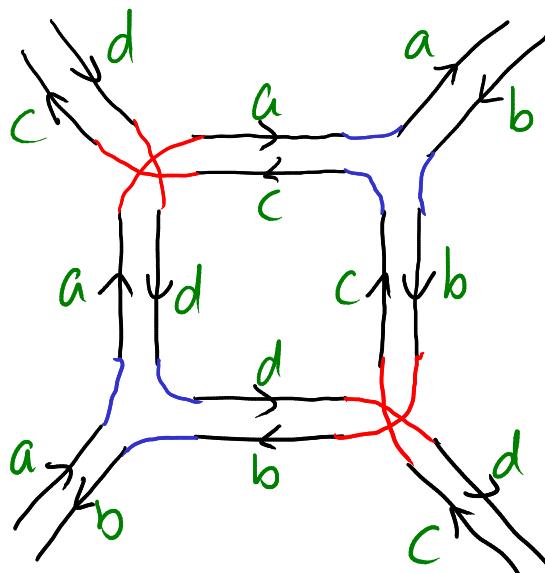
let's relabel

$$v_1 - c_1 = a$$

$$v_1 + c_1 = b$$

$$v_2 + c_2 = c$$

$$v_2 - c_2 = d$$



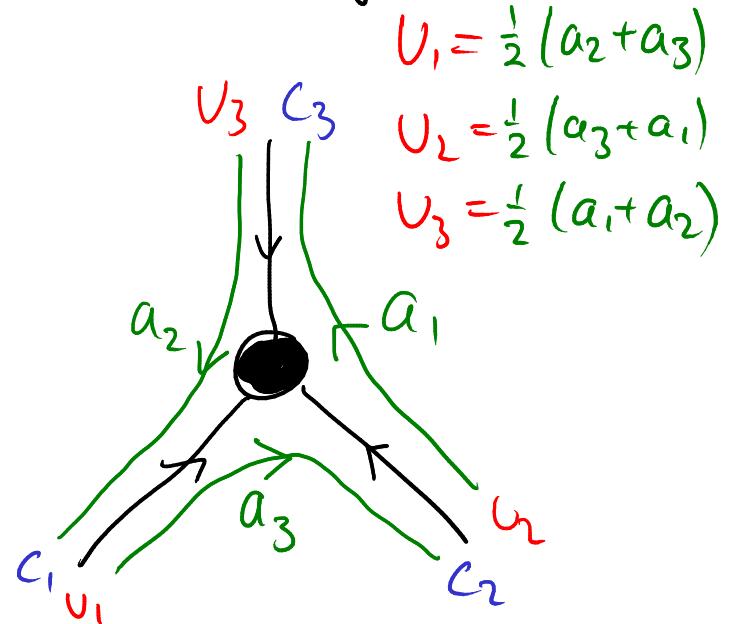
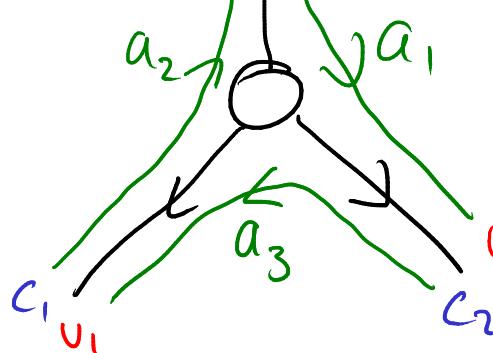
Plotted $v+c$
where c is taken
along the arrow
(two choices/hu)

This pattern is evident from revisiting Bet functions:

$$c_1 = \frac{1}{2}(a_3 - a_2)$$

$$c_2 = \frac{1}{2}(a_1 - a_3) \quad v_3 \quad c_3$$

$$c_3 = \frac{1}{2}(a_2 - a_1)$$



Very ate: a particle line splits into two lines

of the permutation flow associated to the diagram
with two independent spectral parameters!

(common behaviour in quantum integrable systems)

Final Remarks

- Found a connection between the permutation flow associated to a diagram and its central charges (c_h) and spectral parameters (v_h).
- Consistently assigning c'_h 's and v'_h 's is now trivial
- Possible assignment of c'_h 's and v'_h 's follows easily from decomposition of permutation into cycles
(eg. 1 v , $(n-1)$ c 's per n -cycle or as many as $\frac{n-1}{n}$ v' 's for n odd)
- Can construct lots of Yangian invariants, but relationship to physics (rather $N=4$ SYM) unclear.
- Does it mean that two graphs with equal permutation are actually the same by Yangian symmetry?
 - if they are connected by moves: Yes!
 - otherwise: No!

The graphs are actually not Yangian invariants!
They are almost invariant and the inhomogeneous term depends on the structure of the graph...
(see first half of the talk)

Thanks! 16