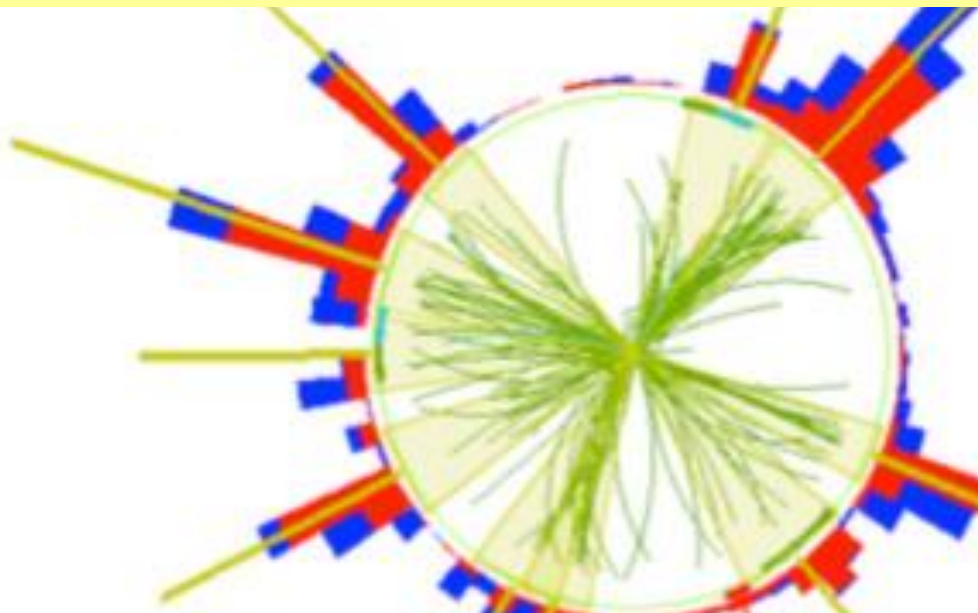




Beyond Feynman Diagrams

Lecture 3



Lance Dixon
Academic Training Lectures
CERN
April 24-26, 2013

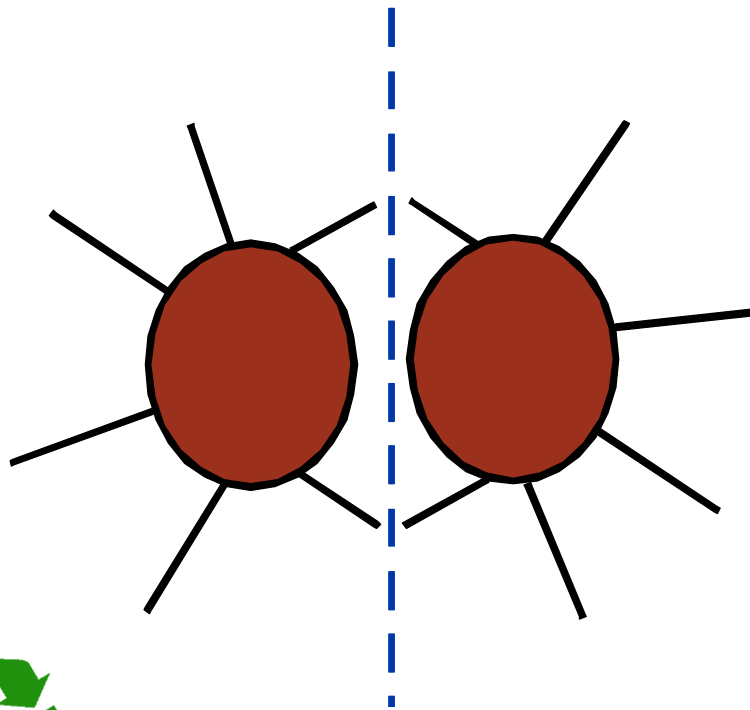


Modern methods for loops

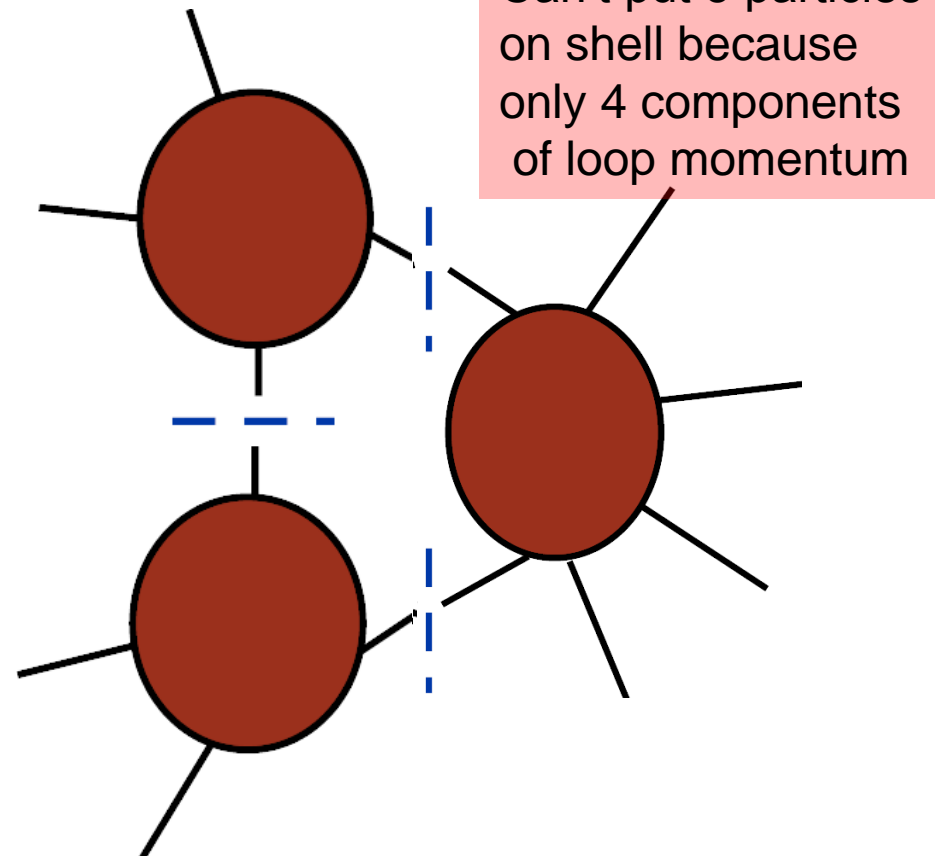
1. Generalized unitarity
2. A sample quadruple cut
3. Hierarchy of cuts
4. Triangle and bubble coefficients
5. The rational part

Branch cut information → Generalized Unitarity (One-loop fluidity)

Ordinary unitarity:
put 2 particles on shell



Generalized unitarity:
put 3 or 4 particles on shell



Can't put 5 particles
on shell because
only 4 components
of loop momentum



Trees recycled into loops!

One-loop amplitudes reduced to trees

When all external momenta are in $D = 4$, loop momenta in $D = 4 - 2\epsilon$ (dimensional regularization), one can write:

Bern, LD, Dunbar, Kosower (1994)



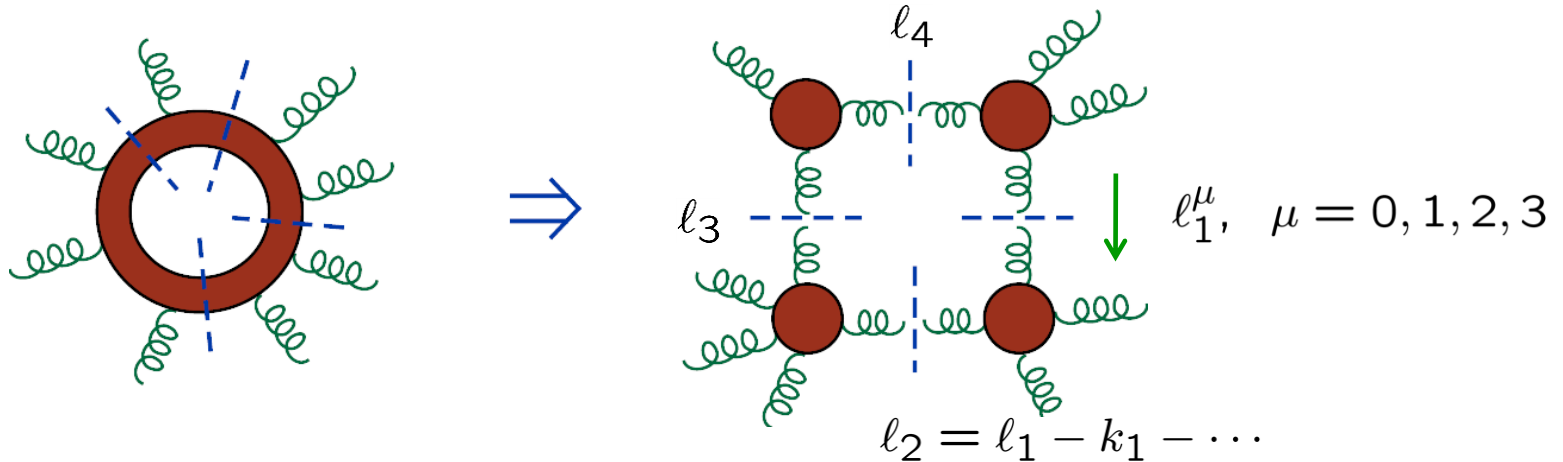
coefficients are all rational functions – determine algebraically from products of **trees** using **(generalized) unitarity**

$$A^{1\text{-loop}} = \sum_i d_i \text{ (box diagram)} + \sum_i c_i \text{ (triangle diagram)} + \sum_i b_i \text{ (bubble diagram)} + R + \mathcal{O}(\epsilon)$$

rational part
known **scalar** one-loop integrals, same for all amplitudes

Generalized Unitarity for Box Coefficients d_i

Britto, Cachazo, Feng, hep-th/0412308



$$\begin{aligned}
 & \int d^4\ell \, \delta(\ell_1^2 - m_1^2) \delta(\ell_2^2 - m_2^2) \\
 & \quad \times \delta(\ell_3^2 - m_3^2) \delta(\ell_4^2 - m_4^2) \times A^{1\text{-loop}}(\ell_i) \\
 = & \sum_{\pm} A_1^{\text{tree}}(\ell_0^{\pm}) A_2^{\text{tree}}(\ell_0^{\pm}) A_3^{\text{tree}}(\ell_0^{\pm}) A_4^{\text{tree}}(\ell_0^{\pm}) \\
 = & d_i^+ + d_i^-
 \end{aligned}$$

No. of dimensions = 4 = no. of constraints \rightarrow 2 discrete solutions

Easy to code, numerically very stable

Box coefficients d_i (cont.)

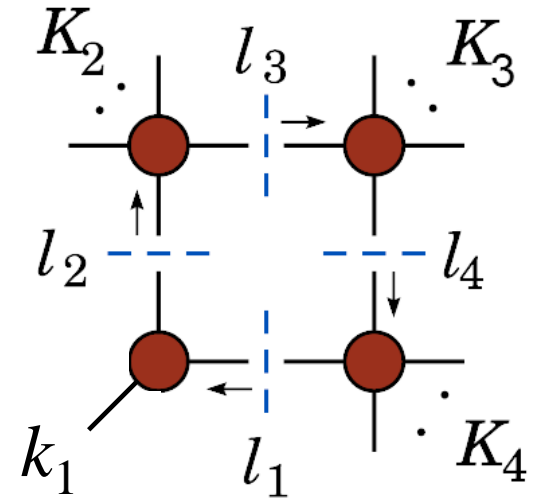
- General solution involves a quadratic formula
- Solutions simplify (and are more stable numerically) when all internal lines are **massless**, and at least one external line (k_1) is **massless**:

$$(l_1^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \not{K}_4 \gamma^\mu | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle},$$

$$(l_3^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \gamma^\mu \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle},$$

Exercise: Show

$$l_2 - l_3 = K_2, \quad l_3 - l_4 = K_3, \quad l_4 - l_1 = K_4$$



$$(l_2^{(\pm)})^\mu = -\frac{\langle 1^\mp | \gamma^\mu \not{K}_2 \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle},$$

$$(l_4^{(\pm)})^\mu = -\frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \gamma^\mu \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}.$$

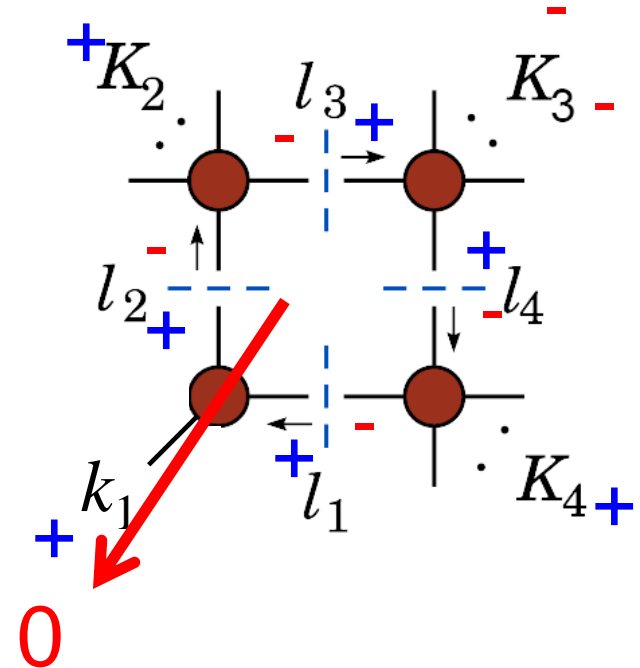
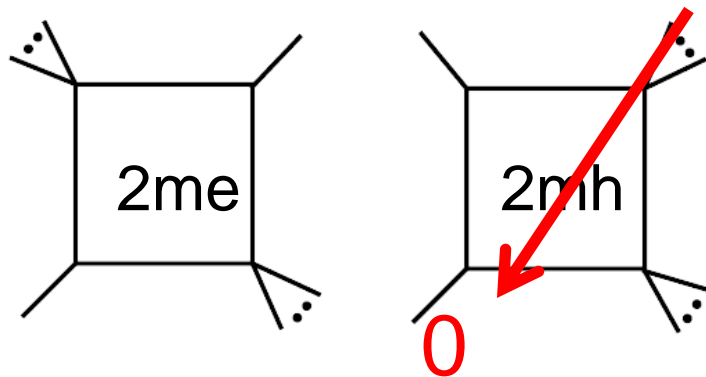
BH, 0803.4180;
Risager 0804.3310

Example of MHV amplitude

All 3-mass boxes (and 4-mass boxes) vanish trivially – not enough (-) helicities

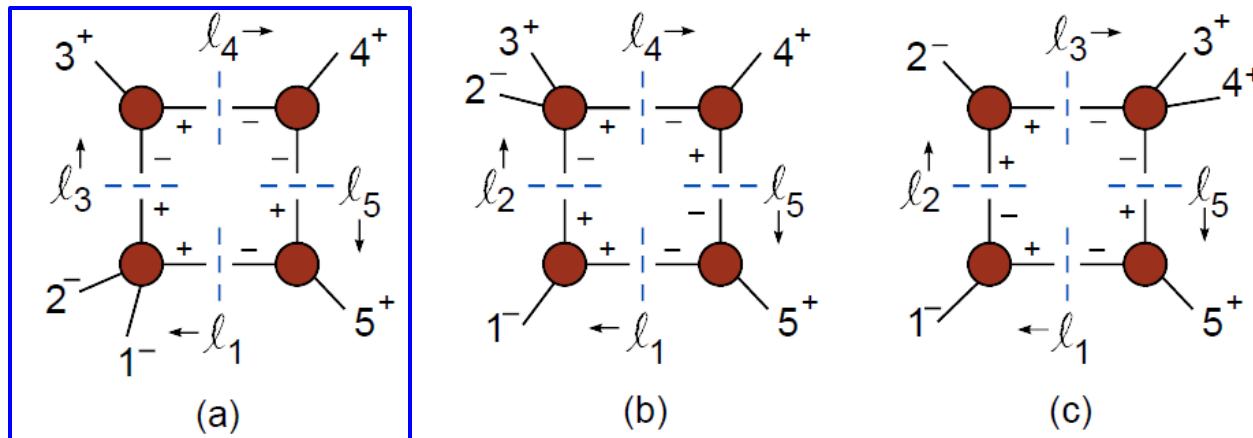
Have $2 + 4 = 6$ (-) helicities,
but need $2 + 2 + 2 + 1 = 7$

2-mass boxes come in two types:



5-point MHV Box example

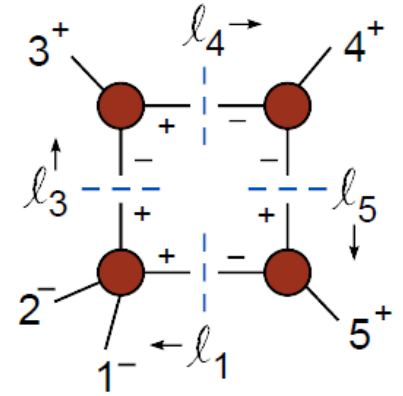
For ($--+++$), 3 inequivalent boxes to consider



Look at this one. Corresponding integral in dim. reg.:

$$\begin{aligned}
 \mathcal{I}(K_{12}) &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2 (\ell - K_{12})^2 (\ell - K_{123})^2 (\ell + k_5)^2} \\
 &= \frac{-2i c_\Gamma}{s_{34} s_{45}} \left\{ -\frac{1}{\epsilon^2} \left[\left(\frac{\mu^2}{-s_{34}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{45}} \right)^\epsilon - \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon \right] \right. \\
 &\quad \left. + \text{Li}_2 \left(1 - \frac{s_{12}}{s_{34}} \right) + \text{Li}_2 \left(1 - \frac{s_{12}}{s_{45}} \right) + \frac{1}{2} \ln^2 \left(\frac{-s_{34}}{-s_{45}} \right) + \frac{\pi^2}{6} \right\} \\
 &\quad + \mathcal{O}(\epsilon),
 \end{aligned}$$

5-point MHV Box example



$$\ell_4^\mu = \frac{1}{2}\xi_4 \langle 3^- | \gamma^\mu | 4^- \rangle .$$

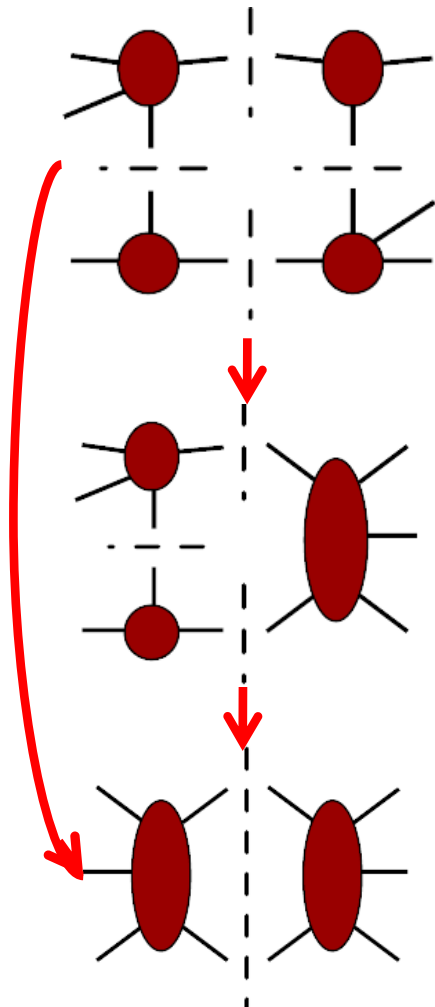
The constant ξ_4 is fixed by the last of the four on-shell equations,

$$\ell_1^2 = (\ell_4 - K_{45})^2 = -\xi_4 \langle 3^- | 5 | 4^- \rangle + s_{45} = 0 ,$$

to have the value $\xi_4 = \langle 45 \rangle / \langle 35 \rangle$.

$$\begin{aligned} c_{12} &= \frac{1}{2} A_4^{\text{tree}}(-\ell_1^+, 1^-, 2^-, \ell_3^+) A_3^{\text{tree}}(-\ell_3^-, 3^+, \ell_4^+) A_3^{\text{tree}}(-\ell_4^-, 4^+, \ell_5^-) A_3^{\text{tree}}(-\ell_5^+, 5^+, \ell_1^-) \\ &= \frac{1}{2} \frac{\langle 12 \rangle^3}{\langle 2\ell_3 \rangle \langle \ell_3(-\ell_1) \rangle \langle (-\ell_1)1 \rangle} \frac{[3\ell_4]^3}{[\ell_4(-\ell_3)] [(-\ell_3)3]} \frac{\langle \ell_5(-\ell_4) \rangle^3}{\langle 4\ell_5 \rangle \langle (-\ell_4)4 \rangle} \frac{[(-\ell_5)5]^3}{[5\ell_1] [\ell_1(-\ell_5)]} \\ &= -\frac{1}{2} \frac{\langle 12 \rangle^3 \langle 3^+ | \ell_4 \ell_5 | 5^- \rangle^3}{\langle 2^- | \ell_3 | 3^- \rangle \langle 4^- | \ell_4 \ell_3 \ell_1 | 5^- \rangle \langle 1^- | \ell_1 \ell_5 | 4^+ \rangle} . \\ c_{12} &= \frac{1}{2} \frac{\langle 12 \rangle^3 \langle 4^- | \ell_4 | 3^- \rangle^2 [45]^3}{\langle 2^- | \ell_4 | 3^- \rangle \langle 34 \rangle [45] \langle 15 \rangle \langle 4^- | \ell_4 | 5^- \rangle} \\ &= -\frac{1}{2} \frac{\langle 12 \rangle^3 s_{34} s_{45}}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \\ &= \frac{i}{2} s_{34} s_{45} A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) . \end{aligned}$$

Full amplitude determined hierarchically



Each **box** coefficient comes
uniquely from 1 “quadruple cut”

Britto, Cachazo, Feng, hep-th/0412103

Ossola, Papadopolous, Pittau, hep-ph/0609007;
Mastrolia, hep-th/0611091; Forde, 0704.1835;
Ellis, Giele, Kunszt, 0708.2398; Berger et al., 0803.4180;...

Each **triangle** coefficient from 1 triple cut,
but “**contaminated**” by **boxes**

Each **bubble** coefficient from 1 double cut,
removing contamination by boxes and triangles
Rational part depends on all of above

Triangle coefficients

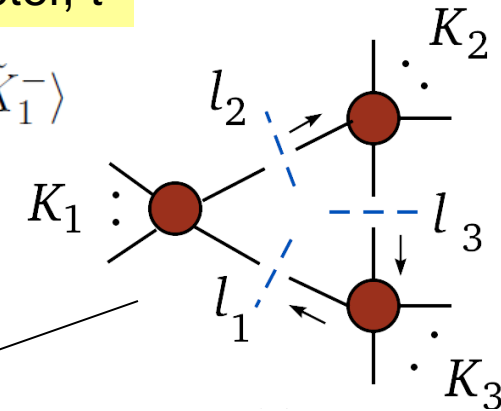
Forde, 0704.1835; BH, 0803.4180

Triple cut solution depends on one **complex** parameter, t

$$l_1^\mu(t) = \tilde{K}_1^\mu + \tilde{K}_3^\mu + \frac{t}{2} \langle \tilde{K}_1^- | \gamma^\mu | \tilde{K}_3^- \rangle + \frac{1}{2t} \langle \tilde{K}_3^- | \gamma^\mu | \tilde{K}_1^- \rangle$$

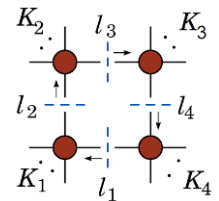
Solves $l_1^2(t) = l_2^2(t) = l_3^2(t) = 0$

for suitable definitions of (massless) $\tilde{K}_1^\mu, \tilde{K}_3^\mu$



Box-subtracted triple cut has poles only at $t = 0, \infty$

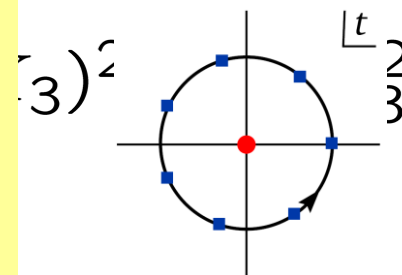
$$T_3(t) \equiv C_3(t) - \sum_{\sigma=\pm} \sum_i \frac{d_i^\sigma}{\xi_i^\sigma(t - t_i^\sigma)}$$



$$T_3(t) = \sum_{j=-p}^p c_j t^j$$

Bubble coeff's similar

Triangle coefficient c_0 plus all other coefficients c_j obtained by **discrete Fourier projection**, sampling at $(2p+1)^{\text{th}}$ roots of unity



Rational parts

- These cannot be detected from unitarity cuts with loop momenta in $D=4$. They come from extra-dimensional components of the loop momentum (in dim. reg.)
- Three ways have been found to compute them:
 1. One-loop on-shell recursion (BBDFK, BH)
 2. D-dimensional unitarity (EGKMZ, BH, NGluon, ...) involving also quintuple cuts
 3. Specialized effective vertices (OPP R_2 terms)

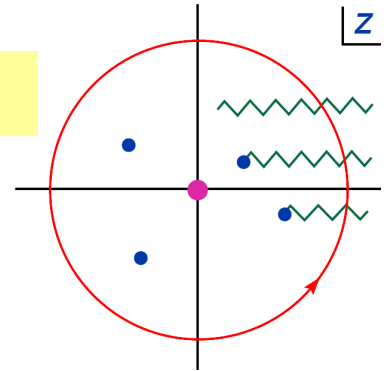
1. One-loop on-shell recursion

Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005;
Berger, et al., hep-ph/0604195, hep-ph/0607014, 0803.4180

- **Same BCFW approach** works for rational parts of **one-loop QCD** amplitudes:

Inject **complex momentum** at (say) leg 1, remove it at leg n.

$$\begin{aligned} k_1(z) + k_n(z) &= k_1 + k_n &\Rightarrow A(0) &\rightarrow A(z) \\ k_1^2(z) &= k_n^2(z) = 0 \end{aligned}$$



- **Full amplitude has branch cuts, from**

e.g. $\ln(s_{23}) \Rightarrow \ln[(\langle 23 \rangle + z\langle 13 \rangle)[32]]$

- **However, cut terms already determined using generalized unitarity**

Subtract cut parts

Generic analytic properties of shifted 1-loop amplitude, $A_n(z)$

Cuts and poles in z -plane:

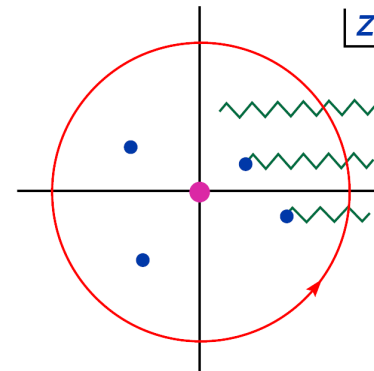
$$\ln(s_{23}) \Rightarrow \ln[(\langle 23 \rangle + z\langle 13 \rangle)[32]]$$

But if we know the cuts (via unitarity in $D=4$), we can subtract them: $R_n \equiv A_n - C_n$

rational part

full amplitude

cut-containing part

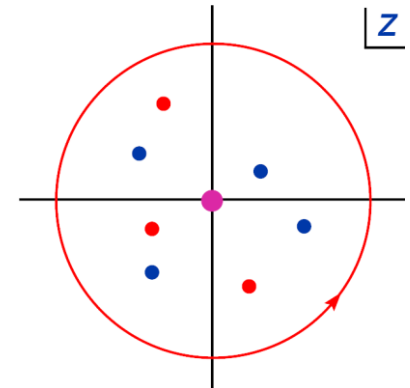


Shifted rational function

$$R_n(z) = A_n(z) - C_n(z)$$

has no cuts, but has spurious poles in z because of C_n :

$$C_n \rightarrow \frac{\ln(r) + 1 - r}{(1 - r)^2} \leftarrow R_n$$

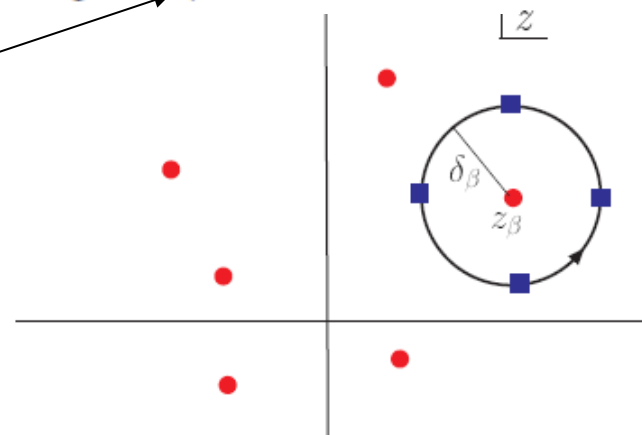


Computation of spurious pole terms

- More generally, spurious poles originate from vanishing of integral Gram determinants: $\Delta_n(z_\beta) = 0$
- Locations z_β all are known.
- And, spurious pole residues **cancel** between C_n and R_n
 \rightarrow Compute them from known C_n

$$R_n^S(0) = - \sum_{\text{spur. poles } \beta} \text{Res}_{z=z_\beta} \frac{R_n(z)}{z} = \sum_{\text{spur. poles } \beta} \text{Res}_{z=z_\beta} \frac{C_n(z)}{z}$$

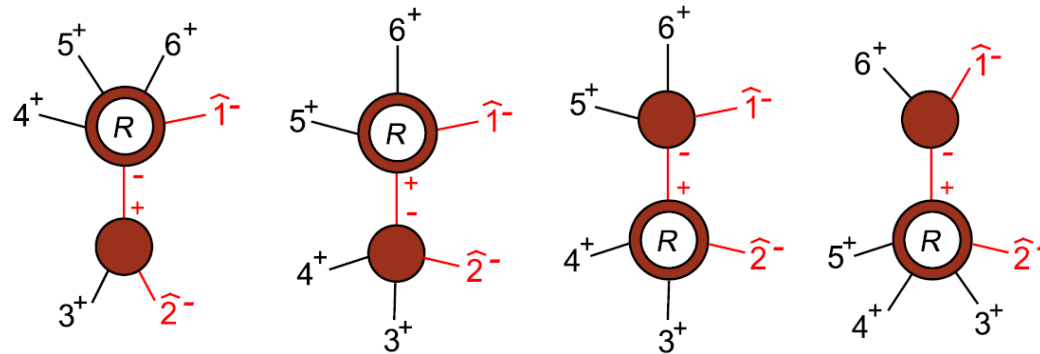
Extract these residues numerically



Physical poles, as in BCFW → recursive diagrams (simple)

For rational part of $A_6^{1\text{-loop}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$

recursive:



Summary of on-shell recursion:

- Loops recycled into loops with more legs (very fast)
- No ghosts, no extra-dimensional loop momenta
- Have to choose shift carefully, depending on the helicity, because of issues with $z \rightarrow \infty$ behavior, and a few bad factorization channels (double poles in z plane).
- Numerical evaluation of spurious poles is a bit tricky.

2. D-dimensional unitarity

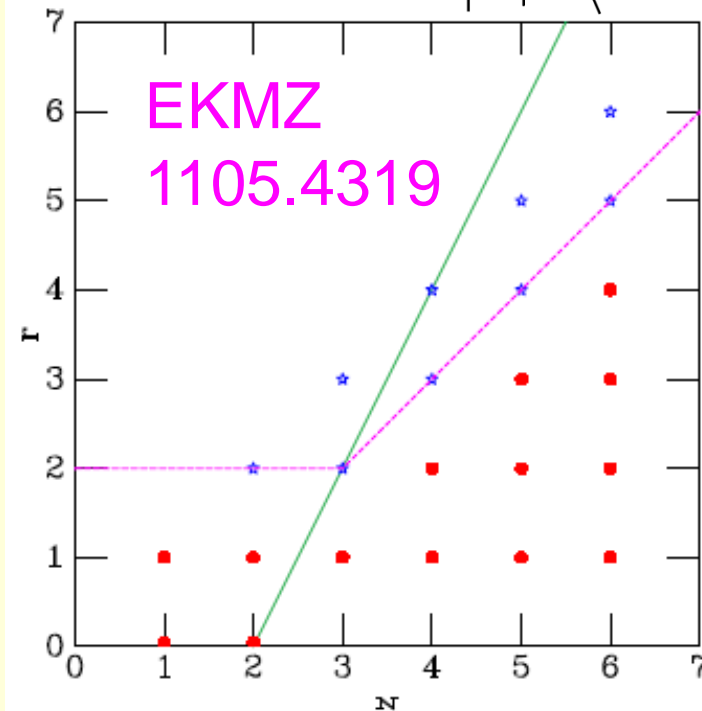
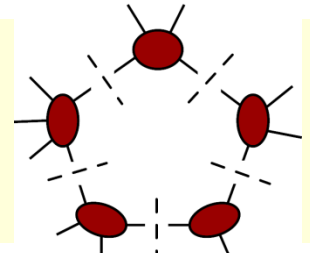
- In $D=4-2\epsilon$, loop amplitudes have **fractional dimension** $\sim (-s)^{2\epsilon}$, due to loop integration measure $d^{4-2\epsilon}l$.
- So a rational function in $D=4$ is really:
$$R(s_{ij}) (-s)^{2\epsilon} = R(s_{ij}) [1 + 2\epsilon \ln(-s) + \dots]$$
- It has a branch cut at $\mathcal{O}(\epsilon)$
- Rational parts can be determined if unitarity cuts are computed including $[-2\epsilon]$ components of the cut loop momenta.

Bern, Morgan, hep-ph/9511336; BDDK, hep-th/9611127;
Anastasiou et al., hep-ph/0609191; ...

Numerical D-dimensional unitarity

Giele, Kunszt, Melnikov, 0801.2237; Ellis, GKM, 0806.3467; EGKMZ, 0810.2762; Badger, 0806.4600; BlackHat; ...

- Extra-dimensional component $\vec{\mu}$ of loop momentum effectively lives in a 5th dimension.
- To determine μ^2 and $(\mu^2)^2$ terms in integrand, need quintuple cuts as well as quadruple, triple, ...
- Because volume of $d^{-2\epsilon}l$ is $O(\epsilon)$, only need particular “UV div” parts: $(\mu^2)^2$ boxes, μ^2 triangles and bubbles
- **Red dots are “cut constructible”:** μ terms in that range $\rightarrow O(\epsilon)$ only



D-dimensional unitarity summary

- Systematic method for arbitrary helicity, arbitrary masses
- Only requires tree amplitude input (manifestly gauge invariant, no need for ghosts)
- Trees contain 2 particles with momenta in extra dimensions (massless particles become similar to massive particles)
- Need to evaluate quintuple cuts as well as quad, triple, ...

3. OPP method

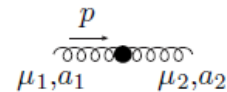
Ossola, Papadopoulos, Pittau, hep-ph/0609007

- Four-dimensional integrand decomposition of OPP corresponds to quad, triple, double cut hierarchy for “cut part”.
- OPP also give a prescription for obtaining part of the rational part, R_1 from the same 4-d data, by taking into account μ^2 dependence in integral **denominators**.
OPP, 0802.1876
- The rest, R_2 , comes from μ^2 terms in the **numerator**. Because there are a limited set of “UV divergent” terms, R_2 can be computed for all processes using a set of effective 2-, 3-, and 4-point vertices

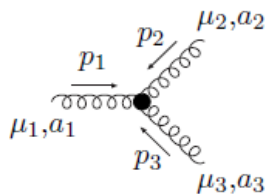
Some OPP R_2 vertices

For 't Hooft-Feynman gauge, $\xi = 1$

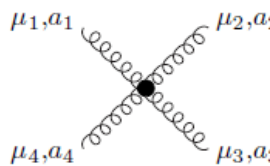
Draggiotis, Garzelli,
Papadopoulos, Pittau,
0903.0356



$$= \frac{ig^2 N_{col}}{48\pi^2} \delta_{a_1 a_2} \left[\frac{p^2}{2} g_{\mu_1 \mu_2} + \lambda_{HV} (g_{\mu_1 \mu_2} p^2 - p_{\mu_1} p_{\mu_2}) + \frac{N_f}{N_{col}} (p^2 - 6 m_q^2) g_{\mu_1 \mu_2} \right]$$



$$= -\frac{g^3 N_{col}}{48\pi^2} \left(\frac{7}{4} + \lambda_{HV} + 2 \frac{N_f}{N_{col}} \right) f^{a_1 a_2 a_3} V_{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$



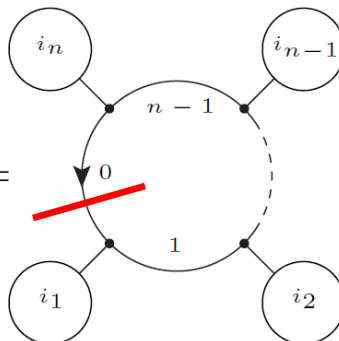
$$= -\frac{ig^4 N_{col}}{96\pi^2} \sum_{P(234)} \left\{ \left[\frac{\delta_{a_1 a_2} \delta_{a_3 a_4} + \delta_{a_1 a_3} \delta_{a_4 a_2} + \delta_{a_1 a_4} \delta_{a_2 a_3}}{N_{col}} + 4 \text{Tr}(t^{a_1} t^{a_3} t^{a_2} t^{a_4} + t^{a_1} t^{a_4} t^{a_2} t^{a_3}) (3 + \lambda_{HV}) - \text{Tr}(\{t^{a_1} t^{a_2}\} \{t^{a_3} t^{a_4}\}) (5 + 2\lambda_{HV}) \right] g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} + 12 \frac{N_f}{N_{col}} \text{Tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \left(\frac{5}{3} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} - g_{\mu_2 \mu_3} g_{\mu_1 \mu_4} \right) \right\}$$

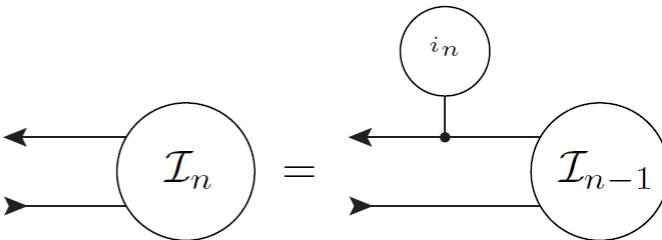
- Evaluation of R_2 very fast (tree like)
- Split into R_1 and R_2 gauge dependent
- Cannot use products of tree amplitudes to compute R_1 .

Open Loops and Unitarity

Cascioli, Maierhöfer, Pozzorini, 1111.5206; Fabio C.'s talk

- OPP method requires one-loop Feynman diagrams in a particular gauge to generate numerators. This can be slow.
- However, it is possible to use a recursive organization of the Feynman diagrams to speed up their evaluation → **Open Loops**

$$\delta\mathcal{A}^{(d)} = \int \frac{d^D q \mathcal{N}(\mathcal{I}_n; q)}{D_0 D_1 \dots D_{n-1}} =$$


$$\mathcal{N}_\alpha^\beta(\mathcal{I}_n; q) =$$


One example of numerical stability

Some one-loop helicity amplitudes contributing to NLO QCD corrections to the processes $pp \rightarrow (W,Z) + 3 \text{ jets}$, computed using unitarity-based method. Scan over 100,000 phase space points, plot distribution in $\log(\text{fractional error})$:

$$0 \rightarrow e^+ e^- q \bar{q} g^\pm g^\pm g^\pm$$

