Eidgenössische Technische Hochschule Zürich
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Perturbative QCD
Lecture 1

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The Standard model of particle physics

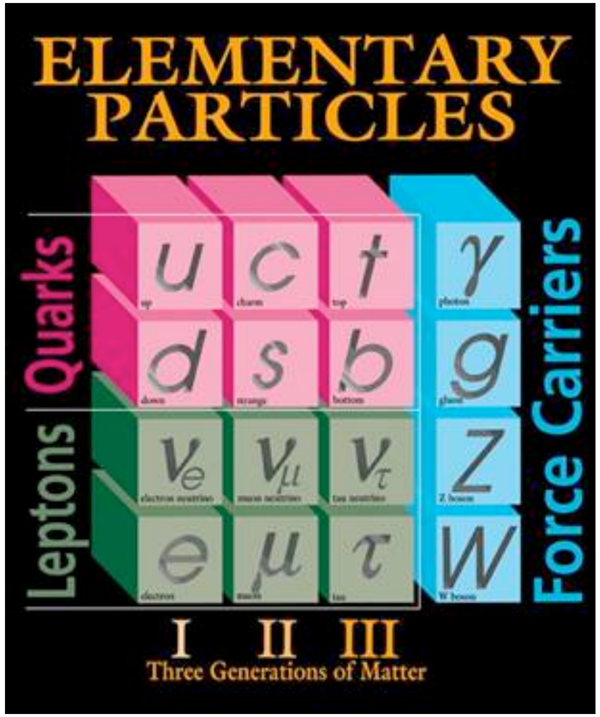
Present knowledge on nature of matter at distances of 10^{-15}-10^{-18} m

- Elementary matter particles (spin ½)
  - Charged leptons: e^-, μ^-, τ^-
  - Neutrinos: ν_e, ν_μ, ν_τ
  - Quarks: u,d,c,s,t,b
- Force carriers (spin 1)
  - Photon γ : electromagnetism
  - Gluon g: strong nuclear force
  - Z^0, $W^\pm$: weak nuclear force
- Higgs boson H (spin 0)
- Relative strength of the forces at 10^{-15}m (= proton radius):

  Strong : Electromagnetic : Weak

  1 : 1 / 100 : 1/10000

  - Note: Gravity left out here



Quantum Chromodynamics (QCD) in brief

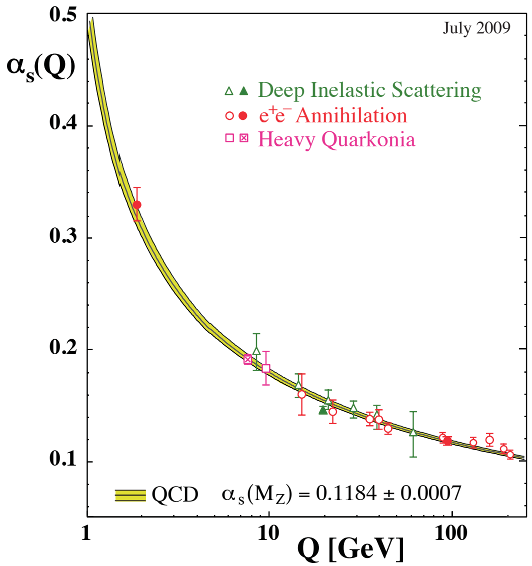
QCD: Quantum field theory of strong interactions

(C.N. Yang, R. Mills; H. Fritzsch, M. Gell-Mann, H. Leutwyler)

- interaction carried by gluons acting on quarks and gluons
- QCD-charge: colour : of three types (`colours`: red, blue, green)

QCD coupling strength α_s depends on energy

- low energy (= long distance or time)
  - $\rightarrow$ α_s is large (confinement): non-perturbative regime of QCD
- high energy (= short distance or time)
  - $\rightarrow$ α_s is small (asymptotic freedom): perturbative regime of QCD



Particle Data Group

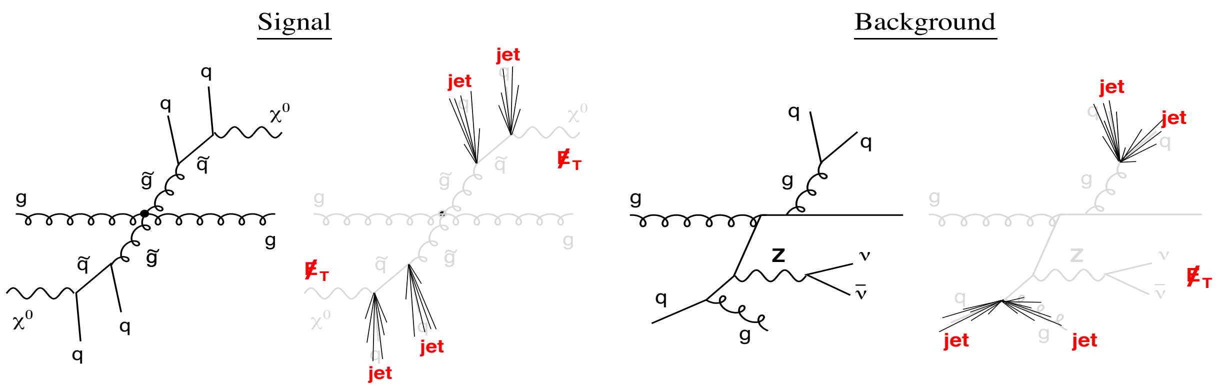
This lecture

- quantify these statements
- obtain a detailed understanding of QCD
- apply to interpretation of collider data

Expectations at LHC

- LHC brings new frontiers in energy and luminosity
- Production of short-lived heavy states (Higgs, top, SUSY...)
  - detected through their decay products
  - yield multi-particle final states involving: jets, leptons, γ, $\not\!\!E_T$

- Jets are the signatures of quarks and gluons:
  built with clusters of particles moving in a common direction
  → will be defined quantitatively in the second lecture

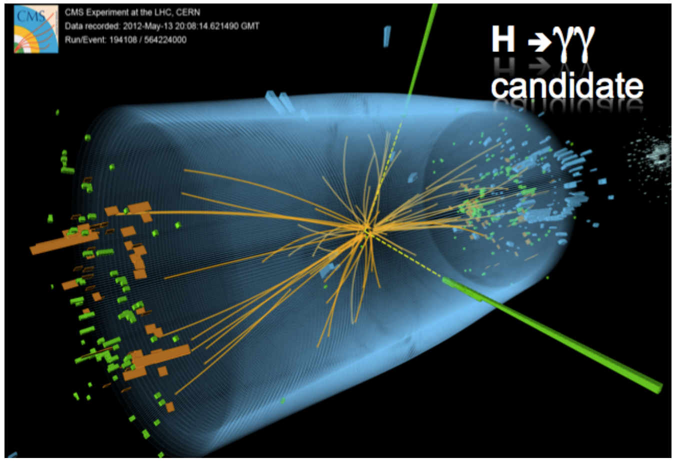


Example: SUSY signature 4j + $\not\!\!E_T$

- Mutli-particle final states produced in hard scattering processes
- QCD effects are essential here

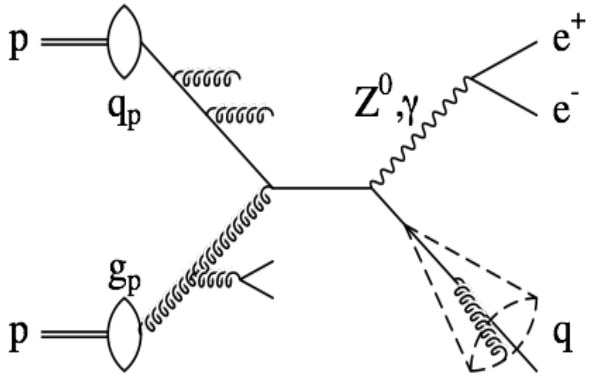
Hard scattering processes at LHC

- Hard scattering processes are rare
  - Represent only a tiny fraction of all events at a hadron collider
    - total inelastic pp cross section: σ_{pp}~70 mb
    - e.g. vector boson production: $\sigma^{W} = 150$ nb ~ $2\ 10^{-6}\ \sigma_{pp}$
- Involve a large transfer of momentum $Q \gg M_p$
    - Q: typical scale of the process
    - M_p: proton mass
- Typical hard scattering processes
  - Jet production
  - Top quark production
  - Vector boson production
  - Higgs boson production

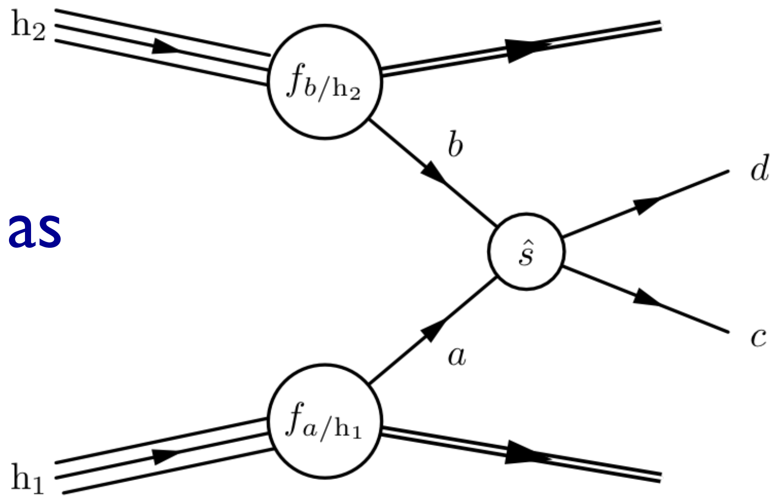


Hard scattering at hadron colliders

- Proton-proton collisions
  - two beams of partons (quarks, gluons) initiate the parton-level interaction
- Proton-proton cross sections
  - related to parton-parton cross sections through factorization of short and long distance processes
- Example: Z+jet production



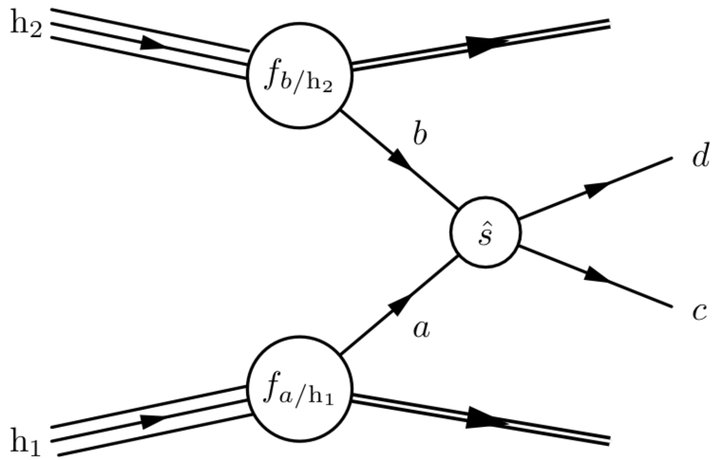
is computed as



Hard scattering cross sections

$$d\sigma^{h_1 h_2 \to cd} = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{a,b} f_{a/h_1}(x_1, \mu_F^2) f_{b/h_2}(x_2, \mu_F^2) d\hat{\sigma}^{ab \to cd}(Q^2, \mu_F^2)$$

- $f_{a/hi}(x_i)$: parton distribution function: probability of finding a parton of type a with momentum fraction x_i in the hadron h_i
  - process-independent but not calculable in perturbation theory
  - needs to be determined from data
  - contains all unresolved emission below factorization scale μ_F
- $\sigma^{ab \to cd}$: parton-level hard scattering cross section
  - calculable in perturbative QCD as series expansion in α_s
  - contains only hard emissions above factorization scale μ_F



Aim of these 3 lectures

- Aim1:
  - Introduce fundamental features of QCD leading to the derivation of the hard scattering cross section formula

$$\boxed{d\sigma^{h_1h_2\to cd} = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{a,b} f_{a/h_1}(x_1,\mu_F^2) f_{b/h_2}(x_2,\mu_F^2) d\hat{\sigma}^{ab\to cd}(Q^2,\mu_F^2)}$$

- Aim2:
  - Study the phenomenology for several high energy collider processes
    - Jet production in e^+e^- annihilation
    - Deep inelastic lepton-proton scattering
    - Higgs and gauge boson production at the LHC

Outline of the first lecture

- Basics of QCD
  - SU(3) gauge invariance
  - Lagrangian of QCD and Feynman rules
  - Properties of QCD: running of α_s, asymptotic freedom, confinement, perturbative QCD
- Application of perturbative QCD to $e^+e^- \rightarrow$ hadrons
  - Infrared divergences and Kinoshita-Lee-Nauenberg theorem
  - Scale dependence

Basics of QCD

- QCD: Quantum Field Theory of strong interactions between quarks and gluons which both carry colour charge
  - three types (`colours`: red, blue, green)
- Theory is based on the symmetry group $SU(3)_C$
  - rotation in colour space
  - part of Standard Model symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$
- Quarks are in the fundamental representation of SU(3)
  - triplets of spin-1/2 spinors: $\psi = (\psi_1, \psi_2, \psi_3)$
- Gluons are in the adjoint representation of SU(3)
  - octets of spin-1 vectors: A^μ_a , a=1..8
- Lagrangian of QCD

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^{\text{class.}} + \mathcal{L}_{QCD}^{\text{gauge}} + \mathcal{L}_{QCD}^{\text{ghost}}$$

SU(3) gauge invariance

- $\mathcal{L}_{QCD}^{\text{class.}}$ can be explicitly constructed by requiring local SU(3) gauge invariance of free fermion field Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

- SU(3) transformation

$$\Psi \to \Psi' = V(x)\Psi \qquad\qquad V(x) \in SU(3)$$

  - V(x): complex 3x3 matrix with det(V) = 1 (special) and $V^+V = 1$ (unitary)

$$V(x) = e^{i\alpha_a(x)T^a}$$

  - real group parameters: $\alpha_a(x), a = 1\ldots 8$
  - T^a: SU(3) generators: hermitian and traceless 3x3 matrices

$$T^a = (T^a)^+, \qquad tr(T^a) = 0$$

    - fulfill SU(3) Lie algebra: $[T^a, T^b] = if^{abc}T^c$

The QCD Lagrangian

- Imposing invariance under

$$\Psi(x) \longrightarrow \Psi'(x) = V(x)\Psi(x) \equiv e^{i\alpha_a(x)T^a}\Psi(x).$$

- results in classical Yang-Mills Lagrangian

$$\boxed{\mathcal{L}_{\mathrm{QCD}}^{\mathrm{class.}} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}}_{\mathcal{L}_{\mathrm{YM}}} + \underbrace{\bar{\Psi}(i\not{D} - m)\Psi}_{\mathcal{L}_F}}$$

  - first term: decribes the gluon field dynamics
  - gluon field strength tensor

$$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}.$$

    - f^{abc} term yields three and four-gluon interactions (absent in QED)
  - Covariant derivative

$$\boxed{D_\mu = \partial_\mu - igA^a_\mu T^a}$$

The QCD Lagrangian

- With the covariant derivative given by:

$$\boxed{D_\mu = \partial_\mu - igA_\mu^a T^a}$$

- Transformation of the gluon field obtained by requiring:

$$D_\mu \Psi \longrightarrow D'_\mu \Psi' = e^{i\alpha^a(x)T^a} D_\mu \Psi .$$

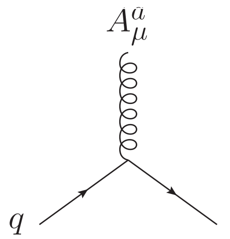
  - results in:

$$\boxed{A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c}$$

  - has an abelian and a pure non–abelian piece

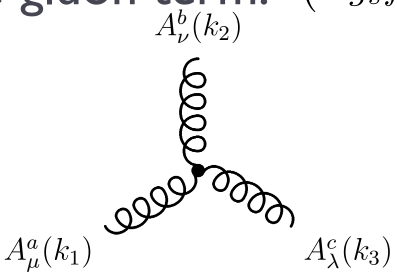
Feynman rules

- Feynman rules for the vertices: from classical Lagrangian
  - Covariant derivative term $\bar{\Psi}_i g \gamma^\mu T^a_{ij} \Psi_j A^a_\mu$ yields boson-fermion-fermion vertex



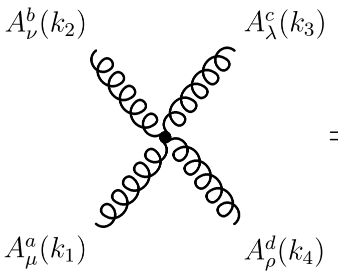
$$= -ig\gamma^\mu T^a_{ij}$$

  - three-gluon term: $(-g_s f_{abc} A^b_\mu A^c_\nu)\,(\partial_\mu A^\nu_a - \partial^\nu A^\mu_a)$ yields three-gluon vertex



$$= g_s f_{abc}\left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu\right].$$

  - four-gluon term: $(-g_s f_{abc} A^b_\mu A^c_\nu)\,(-g_s f_{ade} A^\mu_d A^\nu_e)$ yields four-gluon vertex



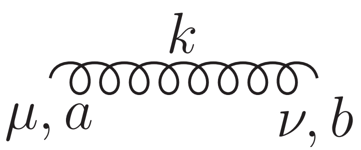
$$= -ig_s^2\left[f_{abe}f_{cde}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) + f_{ade}f_{bce}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}) + f_{ace}f_{bde}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\rho\lambda})\right]$$

Gauges and Ghosts

- Classical Lagrangian not sufficient to define the gluon propagator (kinetic term not invertible)
- need to specify a choice of gauge
  - add gauge-fixing term to the Lagrangian

$$\mathcal{L}_{QCD}^{\text{gauge}} = -\frac{1}{2\xi}(\partial_\mu A^{\mu a})^2$$

  - defines the class of covariant gauges with parameter ξ
  - yields gluon propagator



$$\frac{i}{k^2}\delta^{ab}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Gauges and Ghosts

- In non-abelian gauge theories (like QCD) this term must be supplemented (not discussed in this lecture) by a ghost term

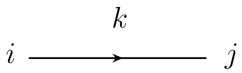
$$\mathcal{L}_{QCD}^{\text{ghost}} = (\partial_\mu \bar{c})(D^\mu c)$$

- Ghost fields *c* are unphysical (scalars with fermionic statistics)
  - cancel the unphysical polarizations of gluons in covariant gauges
  - For practical purposes: it is enough to know that there are also Feynman rules for ghosts and that every Feynman diagram with a closed loop of internal gluons needs to be supplemented by a corresponding diagram with gluons replaced by ghosts (to obtain a meaningful result)

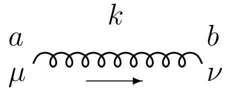
Feynman rules (Summary)

Vertices:

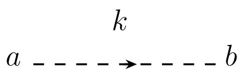
Propagators:



$$i\,\delta_{ij}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}$$

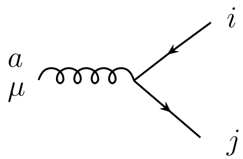


$$\frac{-i\,\delta_{ab}}{k^2+i\epsilon}\left[g_{\mu\nu}-(1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

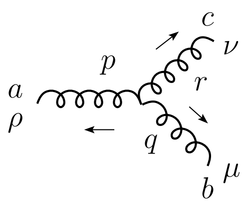


$$\frac{-i\,\delta_{ab}}{k^2+i\epsilon}\,,$$

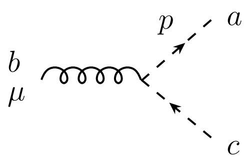
$$\eta \text{ fixes the gauge: } \eta = \begin{cases} 1, & \text{Feynman gauge} \\ 0, & \text{Landau gauge} \end{cases}$$



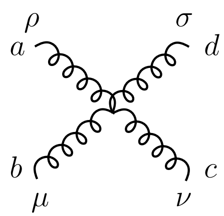
$$ig_s\gamma_\mu T^a_{ji}$$



$$-g_s f^{abc}\left[(p-q)_\nu\, g_{\rho\mu}+(q-r)_\rho\, g_{\mu\nu}+(r-p)_\mu\, g_{\nu\rho}\right]$$



$$g_s\, f^{abc} p_\mu \ (p_\mu \text{ outgoing})$$

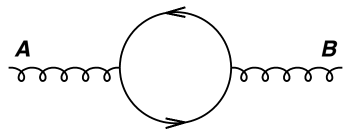


$$\begin{gathered} -ig_s^2 f^{abe} f^{cde}\,(g_{\rho\nu}g_{\mu\sigma}-g_{\rho\sigma}g_{\mu\nu}) \\ -ig_s^2 f^{ace} f^{bde}\,(g_{\rho\mu}g_{\nu\sigma}-g_{\rho\sigma}g_{\mu\nu}) \\ -ig_s^2 f^{ade} f^{cbe}\,(g_{\rho\nu}g_{\mu\sigma}-g_{\rho\mu}g_{\sigma\nu}) \end{gathered}$$

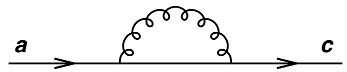
Colour factors

- SU(3) generators T^a and SU(3) structure constants f^{abc} present in Feynman rules
- In cross section calculations, appear in particular combinations:

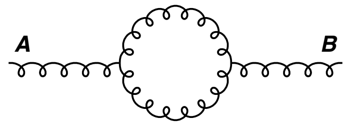
$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$$



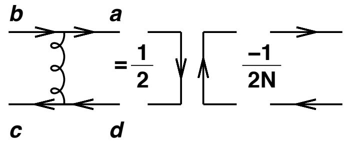
$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}$$



$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_C = 3$$



$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_C}\delta_{ab}\delta_{cd} \ \text{(Fierz)}$$



- Each colour factor corresponds to a specific splitting
  - T_R: $g \to q\bar{q}$; C_F: $q \to qg$; C_A: $g \to gg$

Perturbative QCD

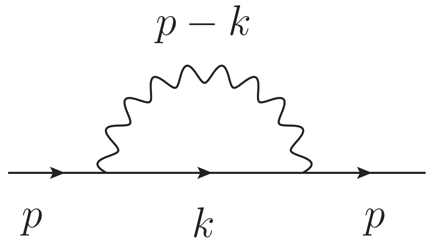
- Parameters in QCD Lagrangian
  - Quark masses
  - Strong coupling constant: $\alpha_s = g_s^2/(4\pi)$
- Perturbative QCD: expansion in powers of $\alpha_s \ll 1$
  - Expansion of observable

$$f = f_0 + \alpha_s f_1 + \alpha_s^2 f_2 + \alpha_s^3 f_3 + \ldots$$

  - often compute only the first one (leading order, LO) or two (next-to-leading order, NLO) terms
- Technique for calculations : using Feynman diagrams
- Energy dependence of α_s : first appears at NLO

Ultraviolet Divergencies

- Closed loop contributions at higher orders
  - yield ultraviolet divergences if loop momentum $k \rightarrow \infty$
- E.g.:



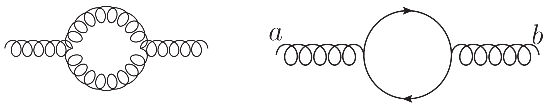
$$= (-ie)^2 \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2 + i\varepsilon}$$

- Renormalization of QCD Lagrangian:
  - Redefine parameters (coupling, masses, field strengths) to absorb ultraviolet divergences
  - Renormalization performed at mass scale μ_r
  - Parameters become renormalization scale dependent: $\alpha_s(\mu_r), m_q(\mu_r)$

Running of the QCD coupling

- Scale dependence (running) of α_s is expressed through the renormalization group equation (RGE)

$$\frac{d\alpha_s(\mu^2)}{d\ln\mu^2} = \beta(\alpha_s(\mu^2))\,, \qquad \beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + \ldots)$$



$$b_0 = \frac{11C_A - 2n_f}{12\pi}\,, \qquad b_1 = \frac{17C_A^2 - 5C_An_f - 3C_Fn_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2}\,,$$

n_f: number of light quarks; C_A, C_F: QCD colour factors

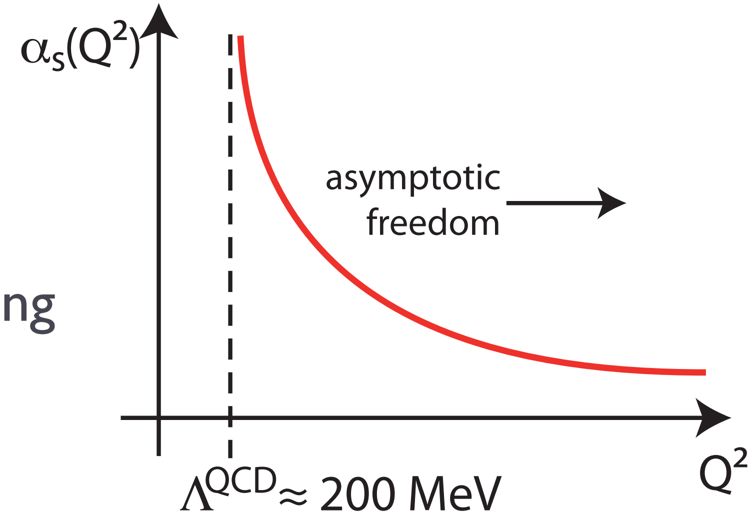
- Negative sign of beta function: asymptotic freedom
  - coupling becomes weaker at high momentum scales

Asymptotic freedom

- Keeping only the b_0 term in $\beta(\alpha_s)$, can solve the RGE exactly:

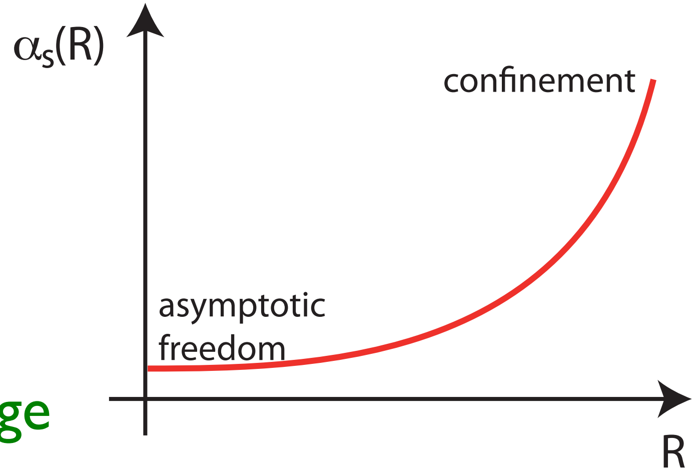
$$\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + b_0\alpha_s(\mu_0^2)\ln\frac{\mu^2}{\mu_0^2}} = \frac{1}{b_0 \ln\frac{\mu^2}{\Lambda^2}}$$

  - $\alpha_s(\mu^2)$ can be expressed
    - in terms of coupling $\alpha_s(\mu_0^2)$ at reference scale μ_0
    - or by introducing non-perturbative constant $\Lambda \approx 200$ MeV, corresponding to the divergence (Landau pole) of $\alpha_s(\mu^2)$

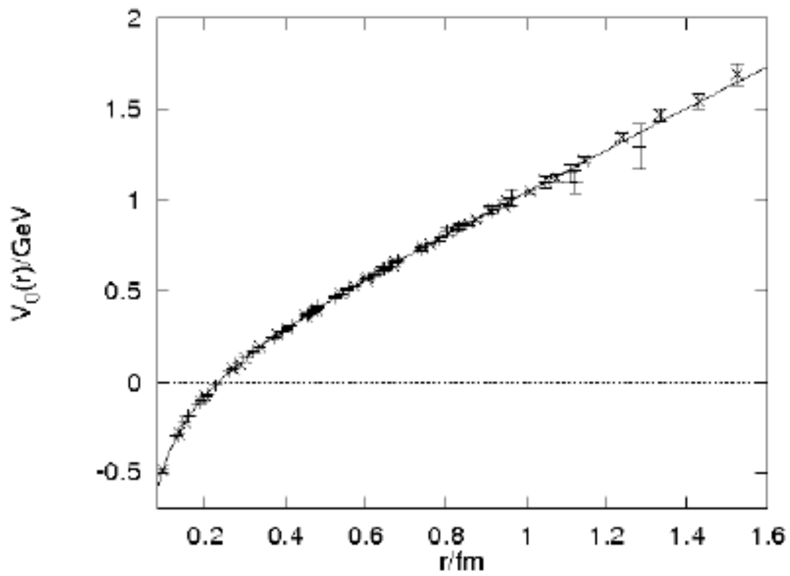


Asymptotic freedom and confinement

- Behaviour of $\alpha_s(\mu^2)$ as a function of energy Q (=inverse distance 1/R) determines the properties of QCD and the dynamics of quarks and gluons
  - Large Q (small distance R): α_s small
    - QCD weakly interacting
    - quarks and gluons asymptotically free
    - regime of perturbative QCD
  - Small Q $\approx \Lambda$ (large distance R): α_s large
    - QCD strongly interacting
    - quarks and gluons form colour-less bound states: baryons (qqq) and mesons ($q\bar{q}$) and do not exist as free particles
    - confinement regime of QCD



Asymptotic freedom and confinement



JLQCD Collab.

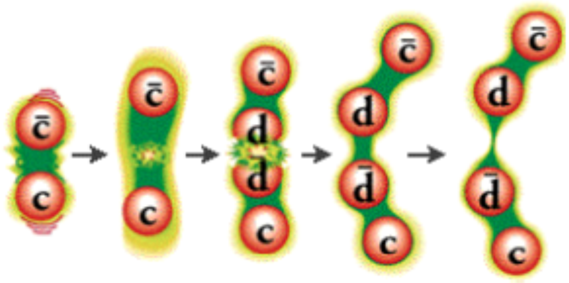
Effective QCD potential

- V ~ 1/r at small distance

  (like electromagnetism)

- V ~ r at large distance

  (like rubber band)



Separating quark-antiquark bound state:

- potential energy becomes large enough to create second pair
- form new pair of bound states

→ quarks can not exist as free particles

Renormalisation scale dependence of physical observables

- Let R be a dimensionless physical observable
  - function of a single scale Q
- After UV-renormalisation:

$$R = R(Q^2/\mu^2, \alpha_s(\mu^2))$$

- If calculated to all orders in α_s, R does not depend on μ
- If only the first terms in the perturbative series are calculated, i.e for R computed to order α_s^n:
  - uncompensated $\log(Q/\mu)$ dependence appears at order α_s^{n+1}

- Example: $e^+e^- \rightarrow$ hadrons

$e^+e^- \rightarrow$ Hadrons

- The hadronic R-ratio

$$R(Q^2) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

  - R-ratio: Classical precision observable (e.g. at LEP)
  - Depends only on e^+e^- center-of-mass energy Q
  - Can be computed in perturbation theory from

$$\sigma(e^+e^- \to q\bar{q}/q\bar{q}g/\ldots)$$

  - Perturbative series for R-ratio known to order α_s^4

    (Baikov, Chetyrkin, Kühn)

Renormalisation scale dependence: $e^+e^- \to$ hadrons

- Cross section at NLO: (at order α_s)

$$\sigma^{\mathrm{NLO}} = \sigma_{q\bar{q}}(1 + c_1 \alpha_{\mathrm{s}}(\mu_{\mathrm{R}}))$$

- Using an expansion of the running coupling

$$\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}) = \alpha_{\mathrm{s}}(Q) - 2b_0 \alpha_{\mathrm{s}}^2(Q) \ln \frac{\mu_{\mathrm{R}}}{Q} + \mathcal{O}\left(\alpha_{\mathrm{s}}^3\right)$$

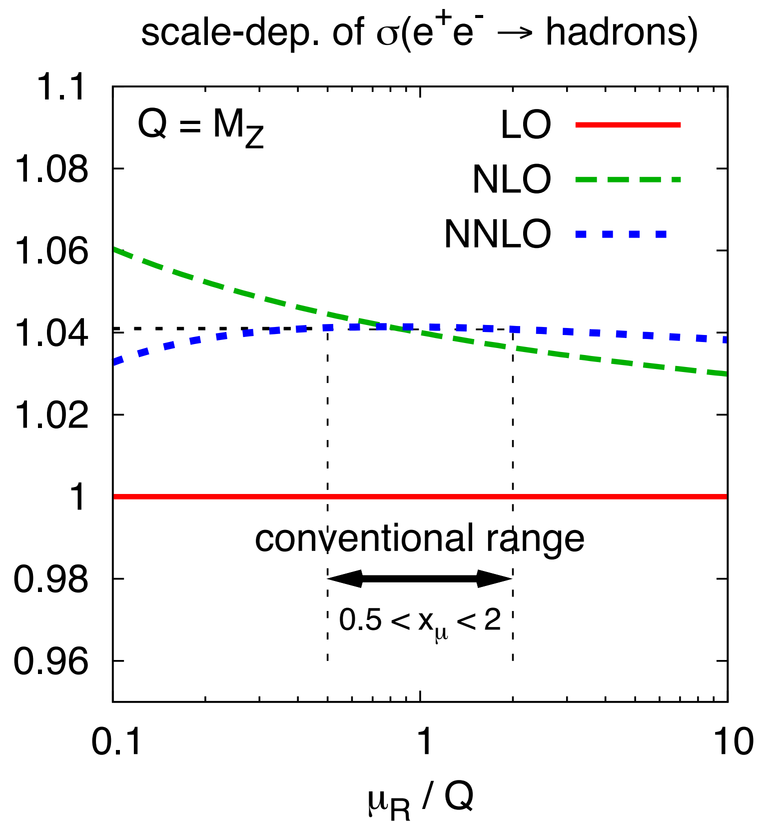
- Rewrite the NLO cross section as

$$\sigma^{\mathrm{NLO}}(\mu_{\mathrm{R}}) = \sigma_{q\bar{q}} \left(1 + c_1 \alpha_{\mathrm{s}}(Q) - 2c_1 b_0 \alpha_{\mathrm{s}}^2(Q) \ln \frac{\mu_{\mathrm{R}}}{Q} + \mathcal{O}\left(\alpha_{\mathrm{s}}^3\right)\right)$$

- Observe: scale dependent logarithms enter only one order higher: at order α_s^2

Renormalisation scale uncertainty

- Variation of renormalization scale μ_r in NLO prediction introduces uncompensated NNLO terms
  - Can use these to estimate the uncertainty from uncalculated NNLO contribution
- Same behavior at higher orders
  - scale variation probes impact of the next uncalculated order
- Scale dependence of observables is reduced order-by-order
- Hadron collider observables also depend on factorization scale μ_f of parton distributions (see lecture 2)



Infrared singularities

- After UV renormalization: higher order perturbative QCD contributions still contain divergences from infrared configurations arising in:
  - real emission of a soft or collinear parton
  - soft or collinear configurations of momenta in a virtual loop

- Infrared divergences cancel order-by-order in perturbation theory when adding real and virtual corrections as stated in:
  - Kinoshita-Lee-Nauenberg (KLN) theorem
  - Factorization theorem (for hadron collider processes)

The KLN theorem and its applicability

- Consider sufficiently inclusive observables
  - Satisfying infrared safety: definition of observable is unchanged by soft emission or collinear splitting (See lecture 2)
    - e.g. jet cross sections, event shapes, but not: final state multiplicities
  - Hard final states from virtual and real corrections are experimentally indistinguishable
- Theorem: Infrared divergences cancel among different subprocesses yielding the same hard final states:
  - From real emission of a soft gluon or real collinear splitting

  and virtual corrections
  - Theorem valid at any order in perturbation theory
- Consider $e^+e^- \to$ hadrons at NLO as an example

$e^+e^- \rightarrow$ hadrons at leading order

- Leading order cross section: from quark - anti-quark production (parton-hadron duality)

$$\sigma_0^{e^+e^-\to q\bar{q}} = \frac{4\pi\alpha_{em}}{3s} e_q^2 N_c$$

- Normalized to muon pair production

$$\sigma_0^{e^+e^-\to \mu^+\mu^-} = \frac{4\pi\alpha_{em}}{3s}$$

- yields hadronic R-Ratio

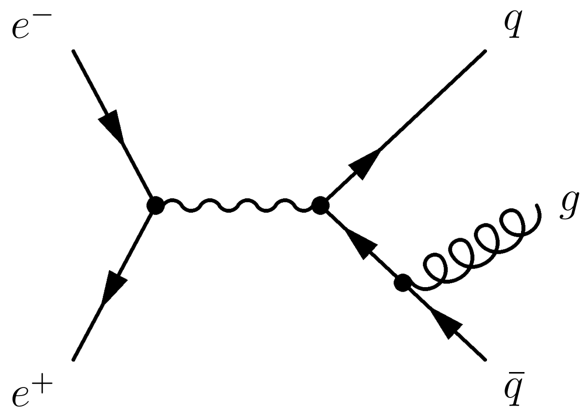
$$R = \frac{\sigma^{e^+e^-\to \mathrm{hadrons}}}{\sigma^{e^+e^-\to \mu^+\mu^-}} = N_c \sum_q e_q^2$$

  - In good agreement with experimental observation
  - Sufficient to compute decay of a virtual photon

NLO corrections to $e^+e^- \to$ hadrons

- Real emission corrections: from $e^+e^- \to q\bar{q}g$
- 3-particle phase space

$$\mathrm{d}\phi_3 = \frac{1}{2\sqrt{s}}\frac{1}{(2\pi)^5}\int \mathrm{d}^4p_1\,\mathrm{d}^4p_2\,\mathrm{d}^4p_g,\;\delta(p_1^2)\delta(p_2^2)\delta(p_g^2)\delta^{(4)}(p_\gamma - p_1 - p_2 - p_g)$$



  - define $x_i = 2E_i/\sqrt{s}$, phase space factorizes as

$$d\phi_3 = d\phi_2\,\frac{s}{16\pi^2}\int \mathrm{d}x_1\mathrm{d}x_2$$

- 3-particle matrix element

$$|M_{q\bar{q}g}|^2 = 2C_F g_s^2\,\frac{1}{s}\,|M_{q\bar{q}}|^2\,\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

NLO corrections to R-ratio

- Three-parton contribution to R-ratio

$$R_1^{q\bar{q}g} = R_0^{q\bar{q}} \frac{2C_F g_s^2}{16\pi^2} \int_0^1 \mathrm{d}x_1 \int_{1-x_1}^1 \mathrm{d}x_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

  - Is singular for $x_1 \to 1$ or $x_2 \to 1$ (correspond to $s_{ig} \to 0$)
- Infrared singularities
  - Soft gluon: $x_g = 2 - x_1 - x_2$: $x_g \to 0$ for $x_1 \to 1$ and $x_2 \to 1$
  - Collinear quark-gluon: $s_{qg} = 2E_q E_g(1-\cos\Theta_{qg}) = s(1-x_2)$
    $s_{qg} \to 0$ for $x_2 \to 1$
  - Collinear antiquark-gluon: $s_{\bar{q}g} = 2E_{\bar{q}} E_g(1-\cos\Theta_{\bar{q}g}) = s(1-x_1)$
    $s_{\bar{q}g} \to 0$ for $x_1 \to 1$
- Divergent three-parton contributions are compensated by divergent virtual gluon corrections to two-parton contribution
  - Need a method to quantify divergences

Dimensional regularization

- Extend Lorentz and Dirac algebra to d = 4 – 2 ε space-time dimensions
  - Regulates both ultraviolet and infrared divergences
  - Preserves Lorentz and gauge invariance of the Lagrangian
  - By dimensional analysis: coupling obtains a mass dimension

$$g_s \to g_s \mu^{\epsilon}$$

  - Need to extend loop and phase space integrals to d dimensions
  - Poles from ultraviolet and infrared divergent contributions appear as inverse powers of ε

Renormalisation in dimensional regularization

- Most QCD calculations are performed in dimensional regularization
  - Dimensionally regularized divergences appear as poles in ε
- Renormalization of coupling, masses and fields not unambiguous
  - Different schemes: differ in finite parts that are absorbed in redefinitions of parameters:
- Commonly used: modified minimal subtraction ($\overline{\text{MS}}$) scheme
  - Absorb only poles, multiplied with universal factor

$$\frac{1}{\epsilon}\, e^{-\epsilon\gamma_E}\, (4\pi)^{\epsilon}$$

Real corrections to R-ratio in d-dimensions

- Phase space recomputed in d dimensions

$$d\phi_3 = d\phi_2 \, \frac{s}{16\pi^2} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \int \mathrm{d}x_1 \mathrm{d}x_2 \, [(1-x_g)(1-x_1)(1-x_2)]^{-\epsilon}$$

- Matrix element recomputed in d dimensions

$$|M_{q\bar{q}g}|^2 = 2C_F g_s^2 \, \frac{1}{s} \, |M_{q\bar{q}}|^2 \frac{(1-\epsilon)(x_1^2+x_2^2)+2\epsilon(1-x_g)}{(1-x_1)(1-x_2)} - 2\epsilon$$

- Contribution to R-ratio

$$\begin{aligned} R_1^{q\bar{q}g} &= R_0^{q\bar{q}} \, \frac{2C_F g_s^2}{16\pi^2 \Gamma(1-\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \int_0^1 \mathrm{d}x_1 \int_{1-x_1}^1 \mathrm{d}x_2 \\ &\quad \times \left(\frac{(1-\epsilon)(x_1^2+x_2^2)+2\epsilon(1-x_g)}{(1-x_1)(1-x_2)} - 2\epsilon \right) [(1-x_g)(1-x_1)(1-x_2)]^{-\epsilon} \\ &= R_0^{q\bar{q}} \, \frac{C_F \alpha_s}{2\pi\Gamma(1-\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2\right] \end{aligned}$$

  - Double pole: associated with soft gluon radiation
  - Single pole: with collinear gluon

NLO corrections to R-ratio

- One-loop correction to quark-antiquark final state yields

$$R_1^{q\bar{q}} = R_0^{q\bar{q}} \frac{C_F \alpha_s}{2\pi\Gamma(1-\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - \pi^2\right]$$

- Adding real and virtual corrections:

$$R = R_0 + R_1^{q\bar{q}} + R_1^{q\bar{q}g} = R_0 \left(1 + \frac{\alpha_s(\mu)}{\pi}\right)$$

  - Cancellation of infrared singularities explicit (KLN)
  - Obtain finite NLO corrections to R-ratio

- End of Lecture 1