Markov Chain Monte Carlo solution of BK equation

based on **JHEP 1307 (2013) 097** [arXiv:1305.4154]

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MPI@LHC, Antwerp 4/12/2013

Introduction

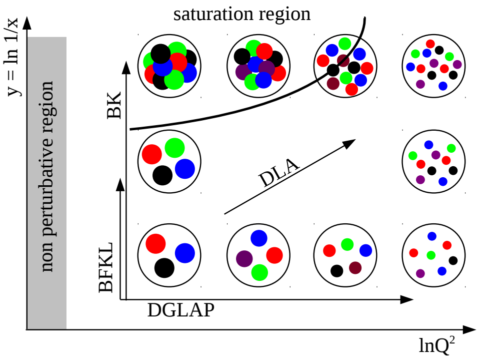
- To do any physics @LHC we need good understanding of PDFs.
- Several parton density evolution equations on the market.
- Current MC generators based on linear evolution equations (PYTHIA, Herwig++, SHERPA, CASCADE) with exception of DIPSY.

Linear evolution

- DGLAP
  Evolution in Q^2
- BFKL
  Evolution in $1/x$

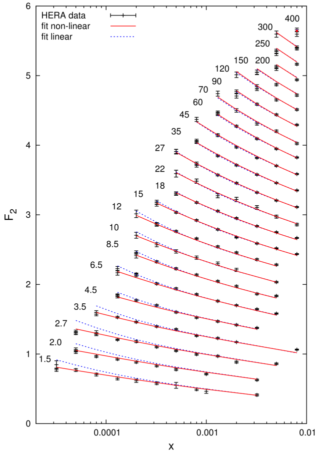
Non linear evolution

- Balitsky-Kovchegov (BK)
  BFKL + saturation



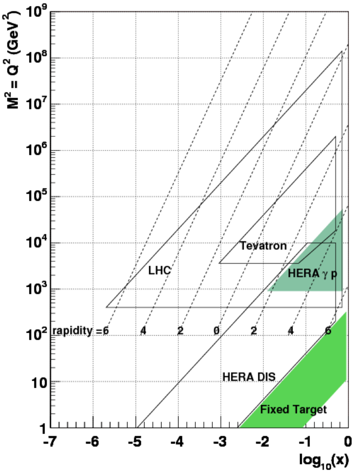
Introduction

- Hints of saturation



[Kutak, Sapeta (2012)]

- LHC opens new kinematic region



[Butterworth et al. (2004)]

BK vs BFKL

General structure of QCD equations for parton densities

$$\Phi(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\; K(y,t,\mathbb{K},\mathbb{G},\Phi(.,.)), \qquad (1)$$

where $\mathbb{K}$ and $\mathbb{G}$ are N-dim (for some N).

BFKL equation

- Kernel K linear in Φ.
- Range of numerical methods.
- Among them Monte Carlo (MC) methods as well, e.g. HEJ [Andersen, Smillie], [Schmidt (1996)].

BK equation

- Kernel K non-linear in Φ,
- Range of numerical methods BKsolver [Enberg, et al. (2005)], [Golec-Biernat, et al. (2001)]
- Convergence of Monte Carlo methods is not guaranteed,
- No known MC solutions.

Motivation

- MC method for the BK equation.
- New approach to the integral equations.

$$\Phi(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\; K(y,t,\mathbb{K},\mathbb{G},\Phi(.,.))$$

This form allows to extend the BK equation, e.g. KGBJS (nonlinear extension of CCFM) equation [Kutak, et al. (2012)], NLO BK equation [Balitsky, Chirilli (2010)].

$$\begin{aligned}\mathcal{E}(x,k^2,p) = &\ \mathcal{E}_0(x,k^2,p)\\ &+ \bar{\alpha}_s \int_x^1 dz \int \frac{d^2\bar{\mathbf{q}}}{\pi \bar{q}^2}\, \theta(p - z\bar{q}) \Delta_s(p, z\bar{q}) \left(\frac{\Delta_{ns}(z,k,q)}{z} + \frac{1}{1-z} \right) \times \\ &\left[\mathcal{E}\left(\frac{x}{z}, k'^2, \bar{q}\right) - \bar{q}^2 \delta(\bar{q}^2 - k^2)\, \mathcal{E}^2\left(\frac{x}{z}, \bar{q}^2, \bar{q}\right) \right].\end{aligned}$$

Newton–Kantorovich method

How to make the BK equation solvable by a MC method?

$$\Phi(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\; K(y,t,\mathbb{K},\mathbb{G},\Phi(.,.)),$$

Linearize non-linear kernel!

Taylor expansion of the kernel

$$K\left(\Phi(\mathbb{X})\right) = K\left(\bar{\Phi}(\mathbb{X})\right) + K'_{\Phi}\left(\bar{\Phi}(\mathbb{X})\right)\left[\Phi(\mathbb{X}) - \bar{\Phi}(\mathbb{X})\right] + \mathcal{O}\left(\left[\Phi(\mathbb{X}) - \bar{\Phi}(\mathbb{X})\right]^2\right). \tag{2}$$

where $K'_{\Phi}\left(\bar{\Phi}(\mathbb{X})\right) = \frac{\delta K(\Phi(\mathbb{X}))}{\delta \Phi(\mathbb{X})}$ is a functional derivative.

If we neglect higher-order terms we are done.

Newton–Kantorovich method

Assuming $|\Psi(\mathbb{X})| := |\Phi(\mathbb{X}) - \bar{\Phi}(\mathbb{X})| \ll 1$

$$K\left(\Phi(\mathbb{X})\right) \approx K\left(\bar{\Phi}(\mathbb{X})\right) + K_{\Phi}^{'}\left(\bar{\Phi}(\mathbb{X})\right) \cdot \Psi(\mathbb{X}).$$

the initial equation

$$\Phi(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\; K(y,t,\mathbb{K},\mathbb{G},\Phi(.,.)),$$

transforms to the *set of equations*:

$$\Phi(y,\mathbb{K}) = \bar{\Phi}(y,\mathbb{K}) + \Psi(y,\mathbb{K}), \tag{3}$$

$$\Psi(y,\mathbb{K}) = \Lambda(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K_{\Phi}^{'}\left(y,t,\mathbb{K},\mathbb{G},\bar{\Phi}(.,.)\right)\Psi(t,\mathbb{G}), \tag{4}$$

$$\Lambda(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) - \bar{\Phi}(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K\left(y,t,\mathbb{K},\mathbb{G},\bar{\Phi}(t,\mathbb{G})\right) \tag{5}$$

Newton–Kantorovich method

Assuming $|\Psi(\mathbb{X})| := |\Phi(\mathbb{X}) - \bar{\Phi}(\mathbb{X})| \ll 1$

$$K\left(\Phi(\mathbb{X})\right) \approx K\left(\bar{\Phi}(\mathbb{X})\right) + K'_{\Phi}\left(\bar{\Phi}(\mathbb{X})\right) \cdot \Phi(\Psi(\mathbb{X})).$$

the initial equation

$$\Phi(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\; K(y,t,\mathbb{K},\mathbb{G},\Phi(.,.)),$$

transforms to the *set of equations*:

"given" $\bar{\Phi}(y,\mathbb{K})$

$$\Phi(y,\mathbb{K}) = \bar{\Phi}(y,\mathbb{K}) + \Psi(y,\mathbb{K}), \qquad (3)$$

Integral equation on Ψ, but *linear*

$$\Psi(y,\mathbb{K}) = \Lambda(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K'_{\Phi}\left(y,t,\mathbb{K},\mathbb{G},\bar{\Phi}(.,.)\right)\Psi(t,\lambda), \qquad (4)$$

$$\Lambda(y,\mathbb{G}) = \Phi^0(y,\mathbb{K}) - \bar{\Phi}(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K\left(y,t,\mathbb{K},\mathbb{G},\bar{\Phi}(t,\mathbb{G})\right) \qquad (5)$$

Simple Integration

Newton–Kantorovich method

- Unfortunately, $\bar{\Phi}(\mathbb{X})$ is not given...

Solution – Iteration

$$\Phi_n(y,\mathbb{K}) = \Phi_{n-1}(y,\mathbb{K}) + \Psi_{n-1}(y,\mathbb{K}) \quad (6)$$

$$\Psi_{n-1}(y,\mathbb{K}) = \Lambda_{n-1}(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K_{\Phi}'\left(.,\Phi_{n-1}(.)\right)\Psi_{n-1}(t,\mathbb{G}) \quad (7)$$

$$\Lambda_{n-1}(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) - \Phi_{n-1}(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K\left(.,\Phi_{n-1}(.)\right) \quad (8)$$

- To compare with, a straight forward iterative method on the initial equation reads:

Iteration method on initial equation

$$\Phi_n(y,\mathbb{K}) = \Phi_{n-1}(y,\mathbb{K}) + \Psi_{n-1}(y,\mathbb{K}) \quad (9)$$

$$\Psi_{n-1}(y,\mathbb{K}) = \Lambda_{n-1}(y,\mathbb{K}) \quad (10)$$

$$\Lambda_{n-1}(y,\mathbb{K}) = \Phi^0(y,\mathbb{K}) - \Phi_{n-1}(y,\mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K\left(.,\Phi_{n-1}(.)\right) \quad (11)$$

- The Newton–Kantorovich method more complicated, but accessible by MC methods.

Markov Chain Monte Carlo (MCMC)

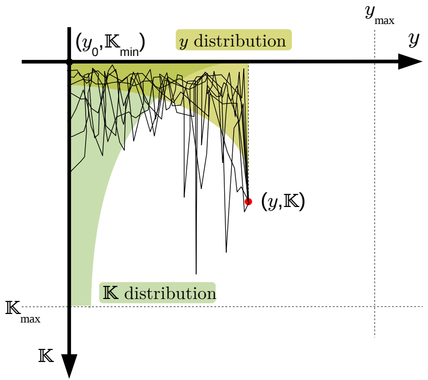
We can solve the linear equation that emerges in the Newton–Kantorovich method using a Monte Carlo method.

$$\Psi_{n-1}(y, \mathbb{K}) = \Lambda_{n-1}(y, \mathbb{K}) + \int_{y_0}^{y} dt \int d\mathbb{G}\, K'_{\Phi}\left(., \Phi_{n-1}(.)\right) \Psi_{n-1}(t, \mathbb{G})$$

For example by a random walk (Markov chain) ordered in y.

For each point $(y, \mathbb{K})$:

- generate a trajectory $(y, \mathbb{K}) \longrightarrow (y_0, \mathbb{X})$ with points generated according to some distributions.
- average over a number M of trajectories.



Example

To test numerical feasibility of our method we solve the leading-order BK equation for the Weizsäcker–Williams gluon density $\Phi(y, k^2)$:

$$\Phi(y, k^2) = \Phi^0(y, k^2) + \int_{y_0}^{y} dt \int_0^{\infty} dl^2\, K\left(y, t, k^2, l^2, \Phi(t, l^2)\right), \qquad (12)$$

where the non-linear kernel K reads:

$$K = \frac{\bar{\alpha}_s}{l^2}\left[\frac{l^2\Phi(t, l^2) - k^2\Phi(t, k^2)}{|k^2 - l^2|} + \frac{k^2\Phi(t, k^2)}{\sqrt{4l^4 + k^4}}\right] - \bar{\alpha}_s \delta\left(l^2 - k^2\right)\Phi^2(t, l^2). \qquad (13)$$

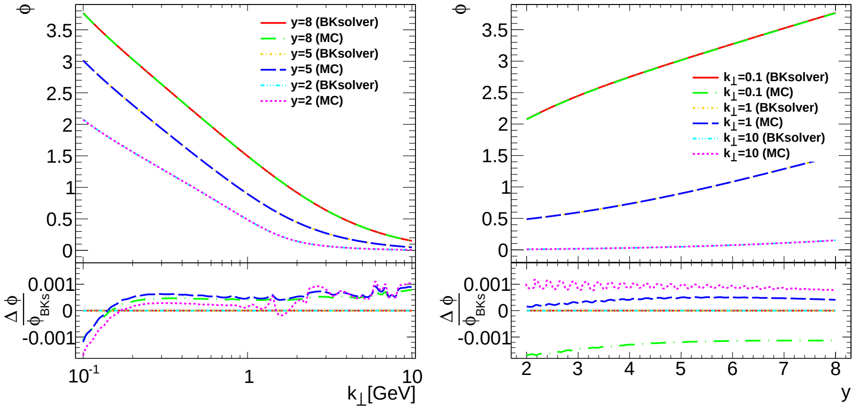
We took $\Phi^0(y, k^2) = \exp(-\frac{k^2}{\mathrm{GeV}})$, and generated the MC trajectories according to the following distributions:

$$\eta(l_i^2) = \frac{\mu^2}{(l_i^2)^2} \implies \text{for } \mathrm{l}^2 \qquad (14)$$

$$\rho(\Delta t_i) = e^{\Delta t_i} \implies \text{for t, } \Delta t_i = t_i - t_{i-1} \leq 0 \qquad (15)$$

Example

- Solutions of the BK equation obtained by our MCMC method are compared with results from the external package BKSolver.



- The agreement is at the level of 0.1%!

Final words

Conclusion

- We proposed a MC method to solve the non-linear BK equation in integral form which is the most general form of the PDF evolution equations.
- The result from the new method agree at the level of 0.1% with the solution from the independent program BKSolver.
- The proposed method is feasible to handle efficiently multi-dimentional and complex problems.

Outlook

- A code to solve an arbitrary high-dimensional integral equation – ongoing.
- Solving other BK-like equations, e.g. the exclusive version of the BK equation, KGBJS, NLO BK equation.
- Possibility to construct a Monte Carlo event generator based on non-linear integral equations?