Tsallis - Shaped Hadron Spectra in e^+e^-, pp and AA collisions

Károly Ürmössy [1]

with

T.S. Biró, G. G. Barnaföldi, G. Kalmár, P. Ván

Statistical Partical Production

April, 2013, Monbachtal, Germany

1, Wigner RCP of the HAS

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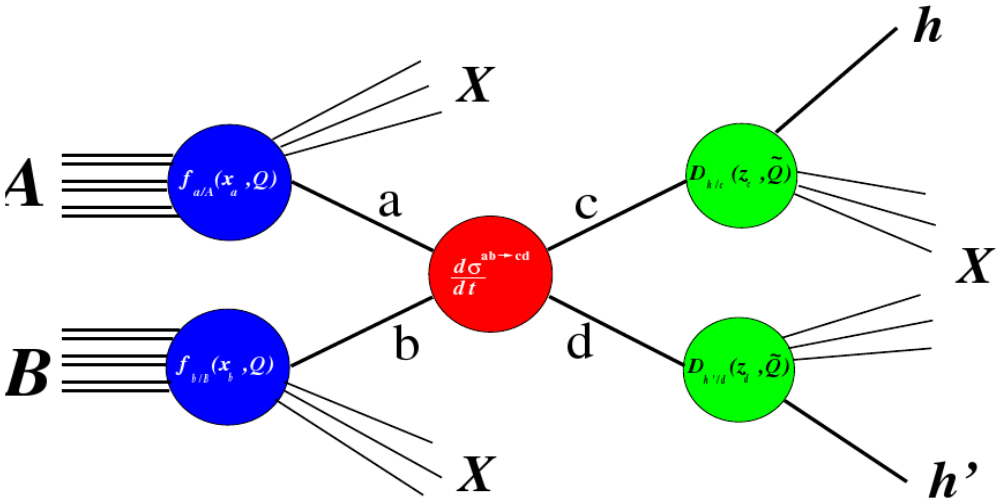
1) What Can Be Statistical?

Parton-model calculation in pp collisions

Partons *a, b* inside protons *A, B* scatter and fragment into hadrons.
Bet: *jet fragmentation might be statistical* .



Cluster hadronisation (Becattini)

***Clusters* with P_μ obtained by some monte-carlo (e.g. *HERWIG*) or from a *termal distribution* *fragment into hadrons* while P_μ, *spin, charges* are *conserved* .**

My bet:* *Let us choose a single jet as a cluster!

Content

1, From the Smallest to the Largest

- Hadron spectra in e^+e^-, *pp* and *AA* collisions
- *NEW: Multiplicity dependence* of hadron spectra in *pp* collisions

2, Statistical Jet Fragmentation @ LEP & LHC

- Model: *microcanonical jets + multiplicity fluctuations*
- Good agreement with *LEP & ATLAS* data
- *Scale evolution* of the newly obtained *fragmentation functions*

  Evolution of *q, T* with E_{jet} and Q^2

3, Hadron Spectra in pp Collisions

- *Multiplicity* and *collision energy* dependence of *hadron spectra*
- Model: *microcanonical jets + multiplicity* and *jet energy fluctuations*

4, Hadron Spectra and Transverse Flow in AA Collisions

- The adequate *scaling variable* for hadrons
- Spectra from *Quark coalescence + Collective flow + Freeze-out*
- *Non-extensive thermodynamics, resonance* production

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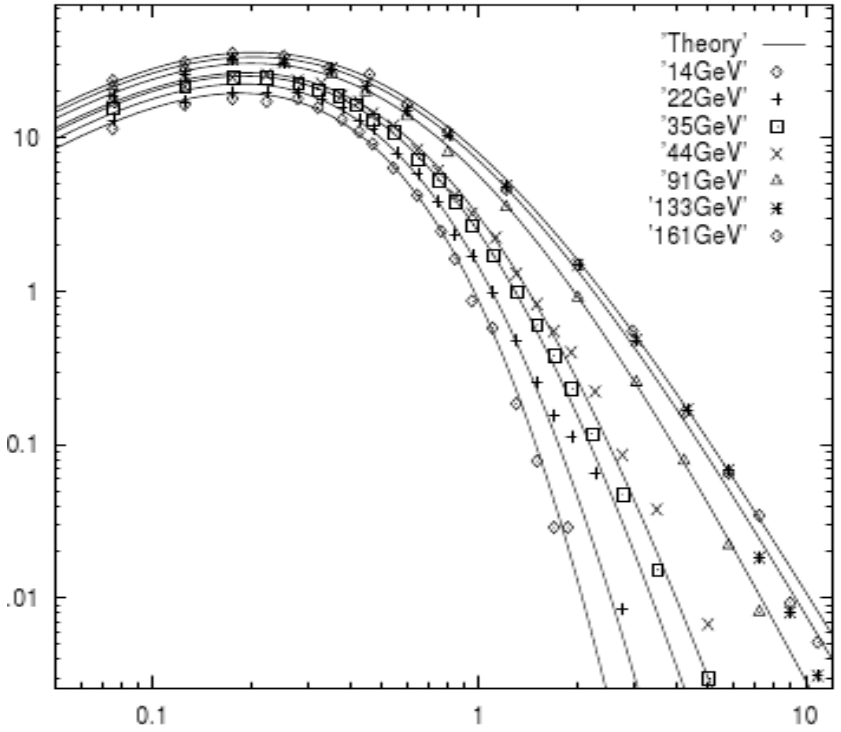
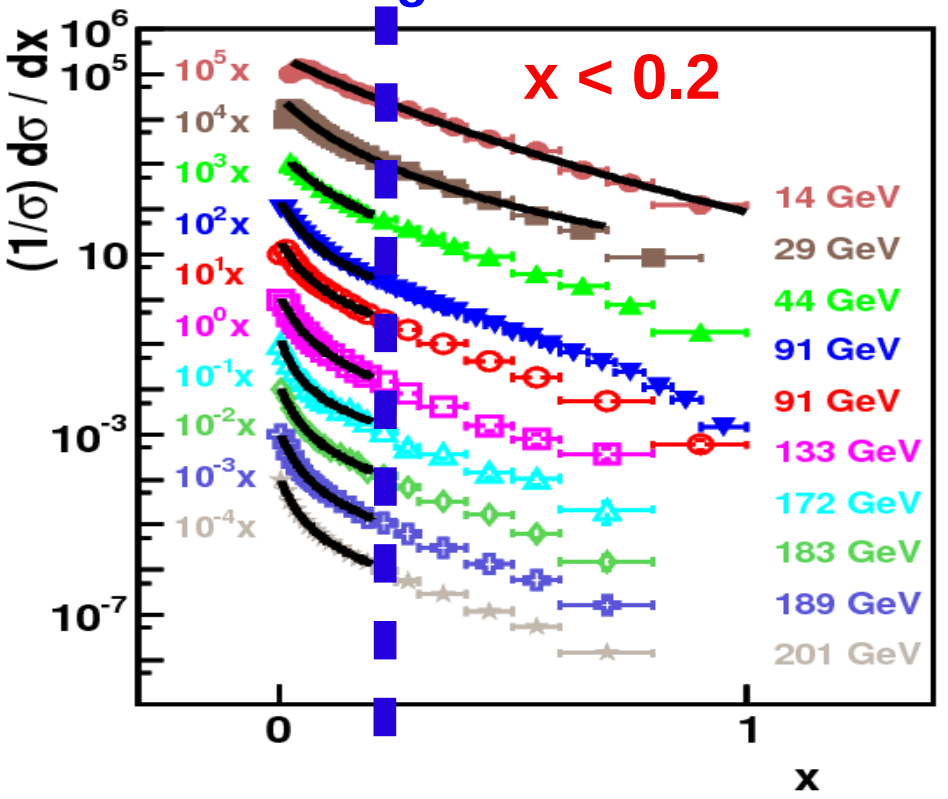
1) Tsallis distribution (e^+e^-)

Transverse momentum distribution of charged hadrons in jets



$$\frac{d\sigma}{dp_T} \propto p_T \int_0^\infty dp_L f_{TS}(E)$$

Energy-fraction distribution of charged hadrons



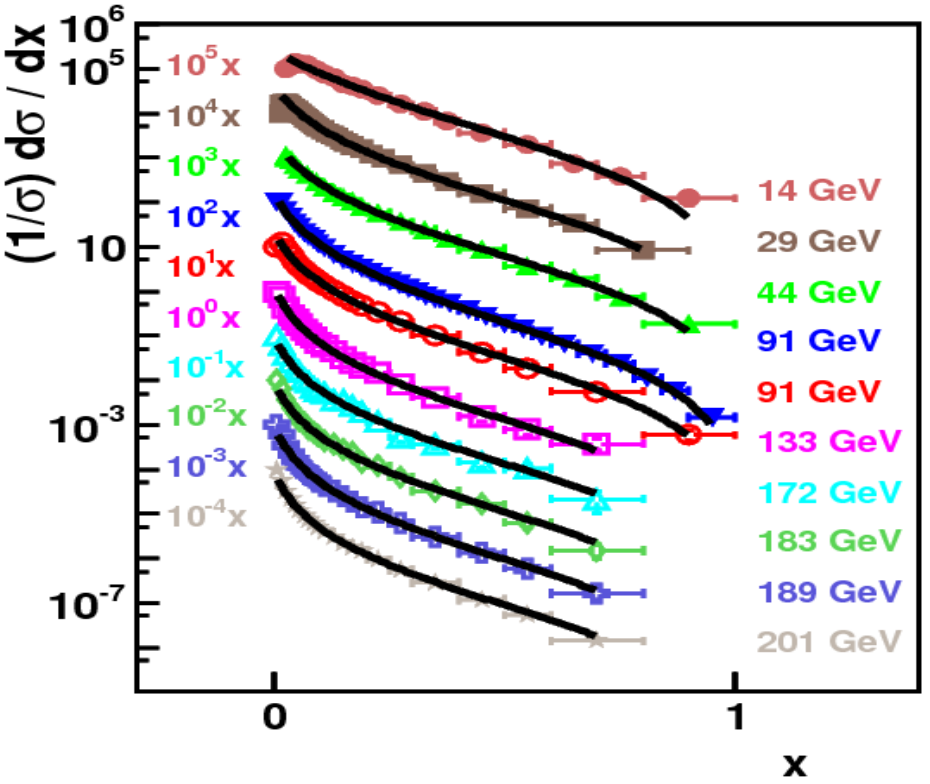
$$\frac{d\sigma}{dx} \propto \left(1+\frac{q-1}{T/(\sqrt{s}/2)}x\right)^{-1/(q-1)}$$

- C. Beck, *Physica A*, **286**, 164-180 (2000)
- I. Bediaga, et.al., *Physica A,* **286**, 156-163 (2000)

- Urmossy et.al., *Phys.Lett.B*, **701,** 111-116 (2011), arXiv:1101.3023

1) Microcanonical Tsallis distribution (e^+e^-)

Energy-fraction distribution of charged hadrons

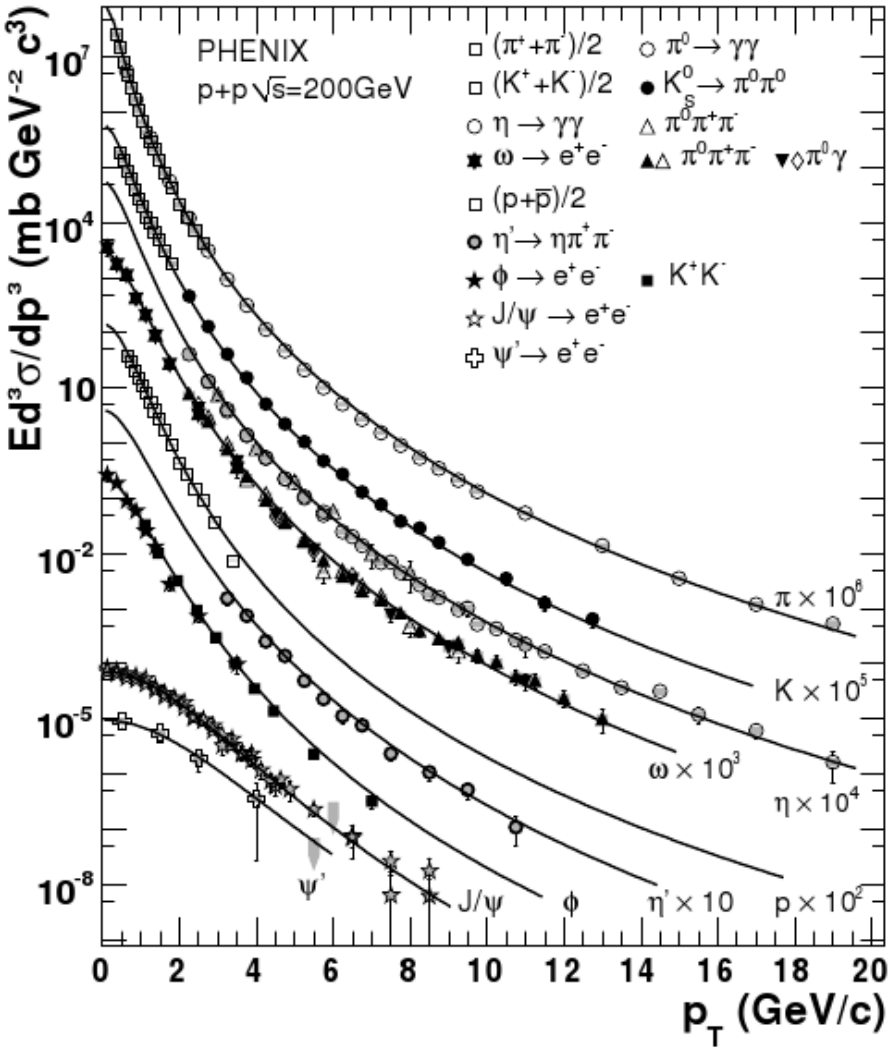


$$\frac{d\sigma}{dx} \propto \left(1-\frac{q-1}{T/(\sqrt{s}/2)}\ln(1-x)\right)^{-1/(q-1)}$$

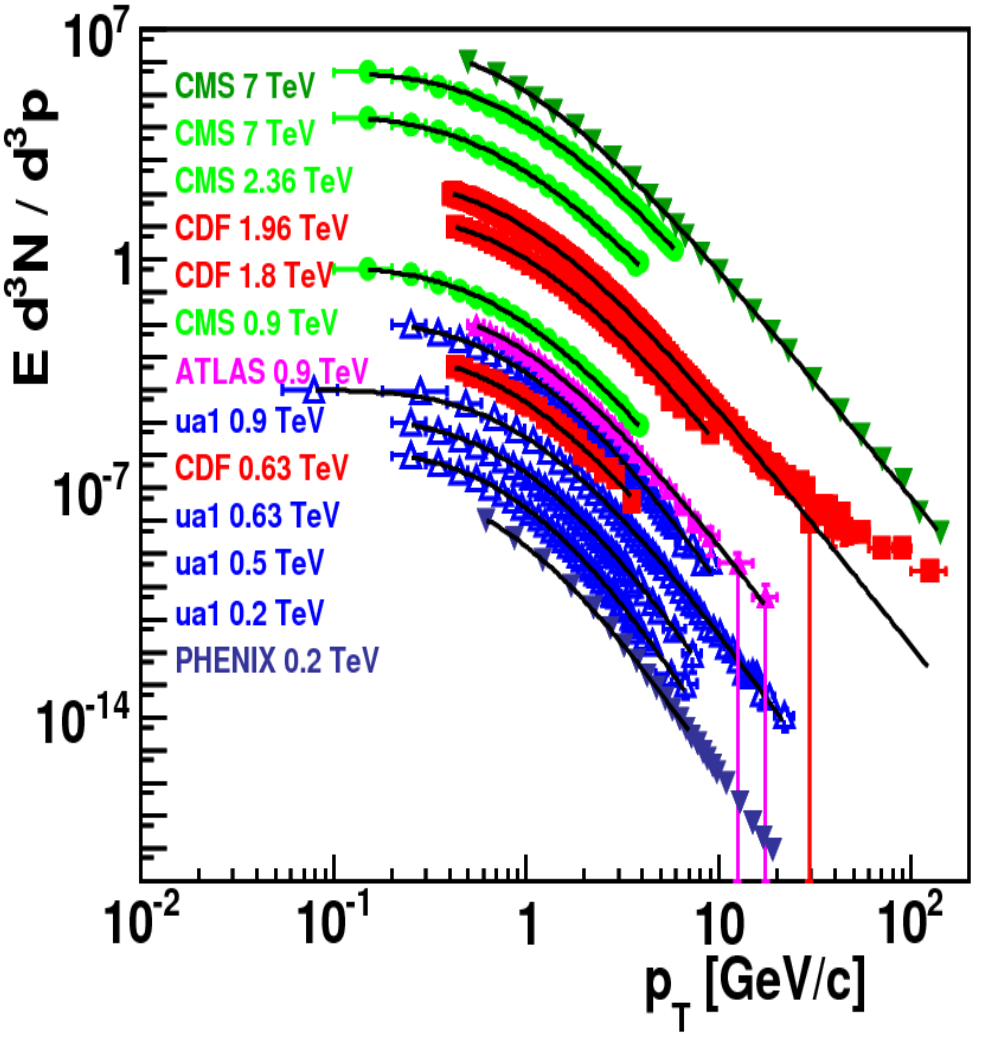
- Urmossy et.al., *Phys.Lett.B*, **701,** 111-116 (2011), arXiv:1101.3023

1) Hadron Spectra in Proton-proton Collisions

Phys. Rev. D, **83,** 052004 (2011)



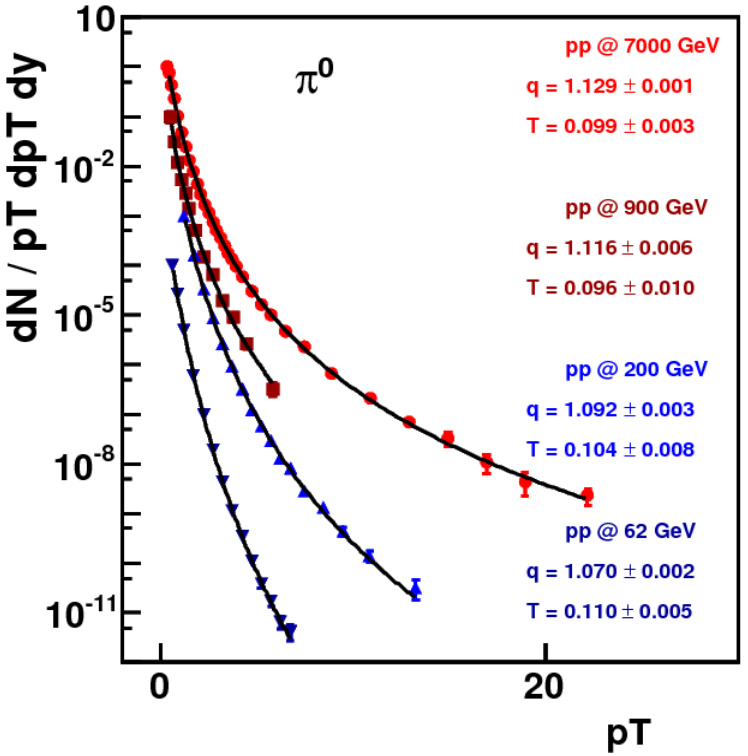
J. Phys.: Conf. Ser., **270** 012008 (2011)

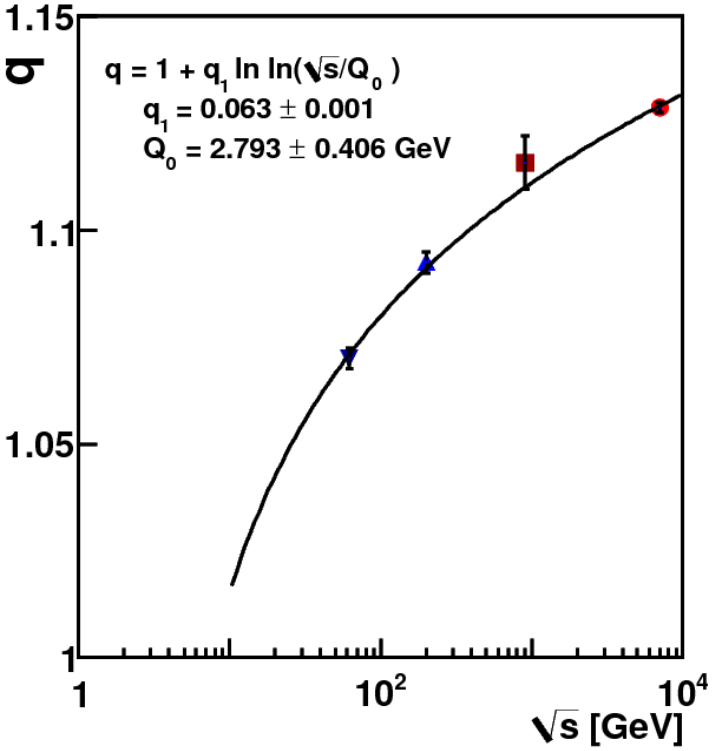


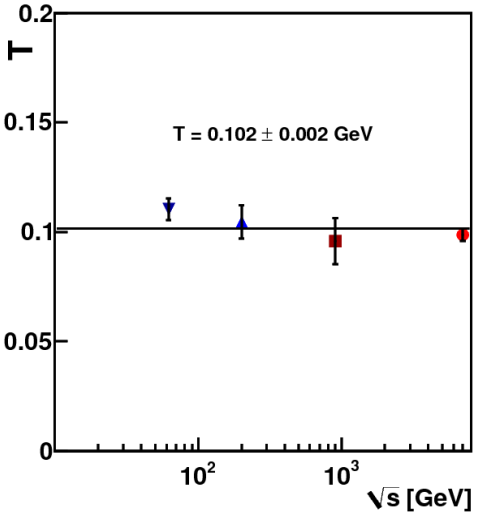
$$E\frac{d\sigma}{d^3p} \propto (1+m_T/nT)^{-n}$$

$$E\frac{d\sigma}{d^3p} \propto m_T(1+m_T/nT)^{-n}$$

1) √s Dependence of π^0-Spectra in pp Collisions



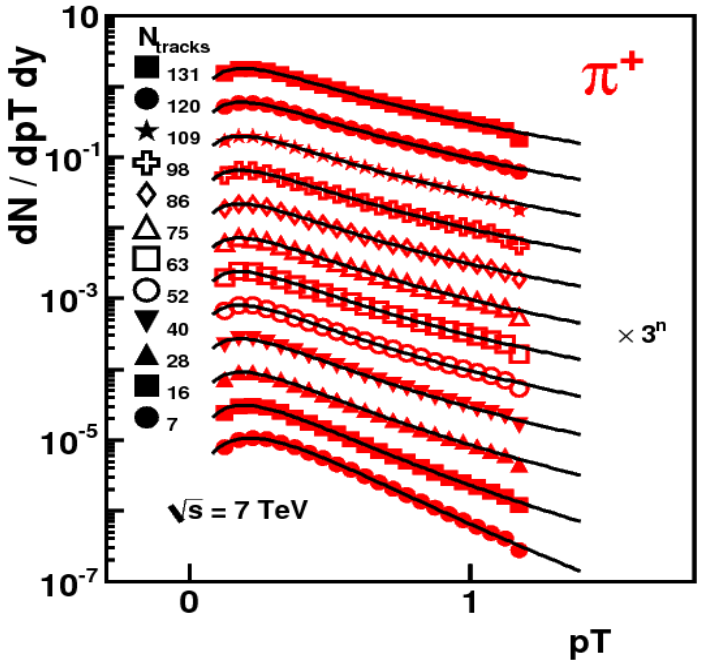


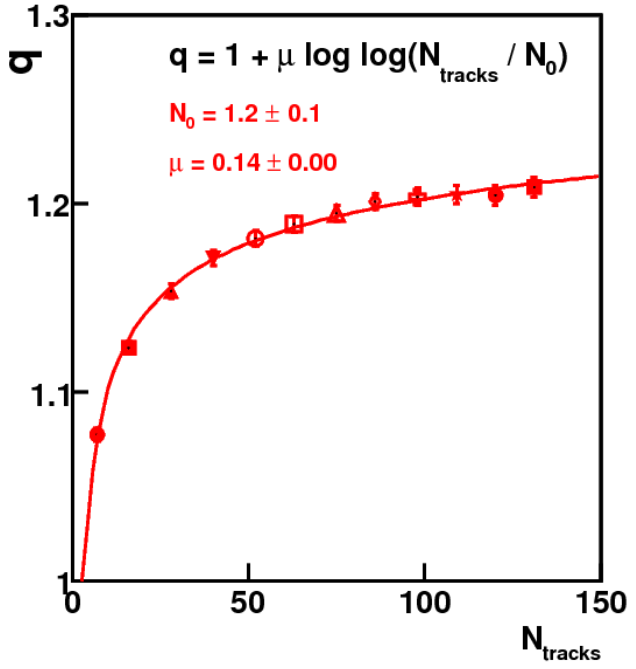


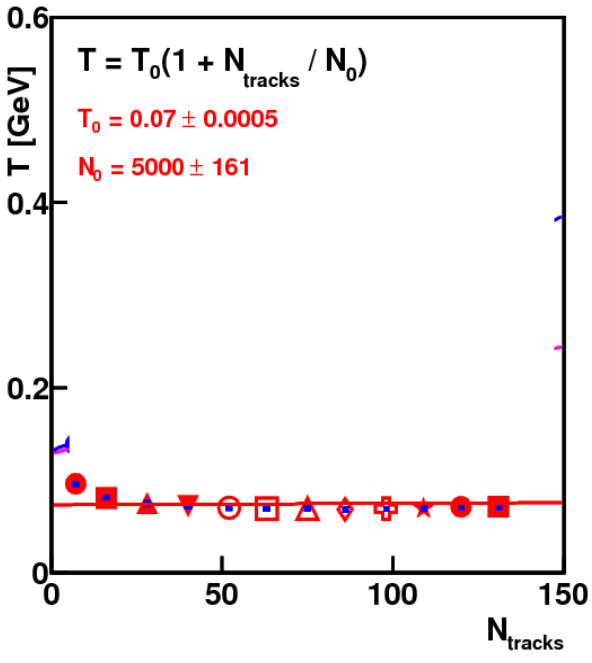
$$q = 1 + q_1 \log\log\left(\sqrt{s}/Q_0\right) \quad , \qquad T = const$$

Urmossy, arXiv: 1212.0260

1) Multiplicity Dependence of $\pi^{\pm}$ Spectra in pp@7 TeV



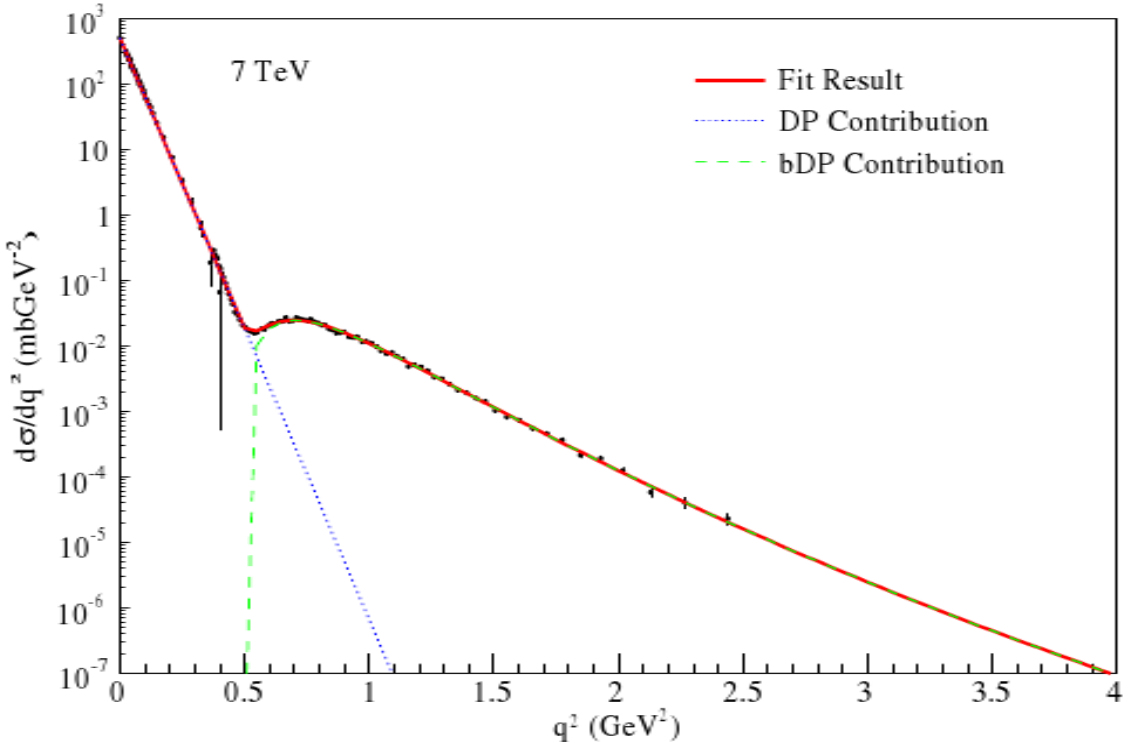




$$\pi: \; q = 1 + \mu \log\log\left(N/N_q\right) \;, \quad T = const$$

Urmossy, arXiv: 1212.0260

1) Cross-section of Elastic pp Scattering

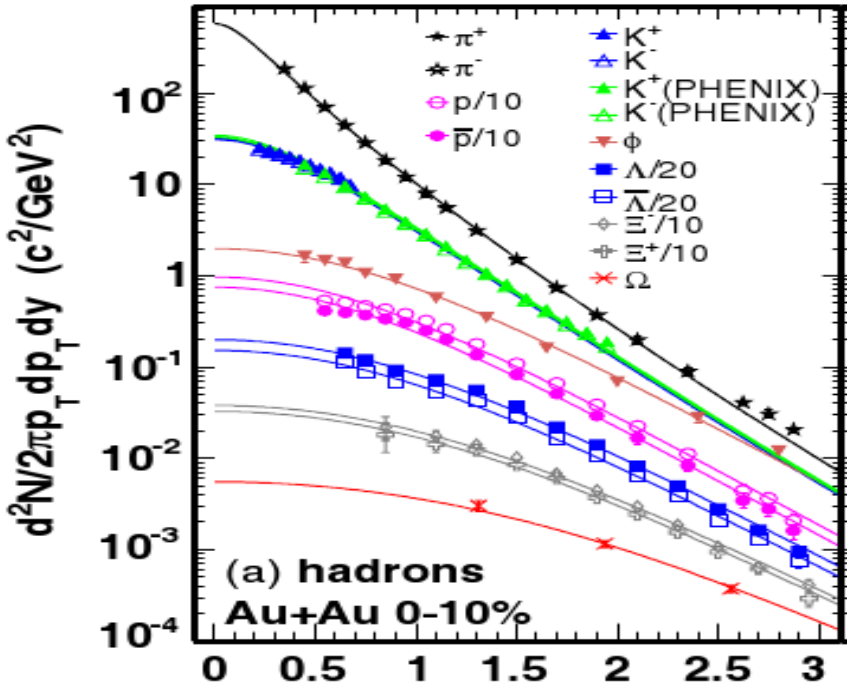


$$\frac{d\sigma}{dq^2} = a_1 e^{-b_1 q^2} + a_2 e^{-b_2 q^2} + a_3 \left(q^2 - q_0^2\right) \left[1 + \frac{\sqrt{\left(q^2 - q_0^2\right)^2 + \delta^2} - \delta}{\gamma \beta}\right]^{-\gamma}$$

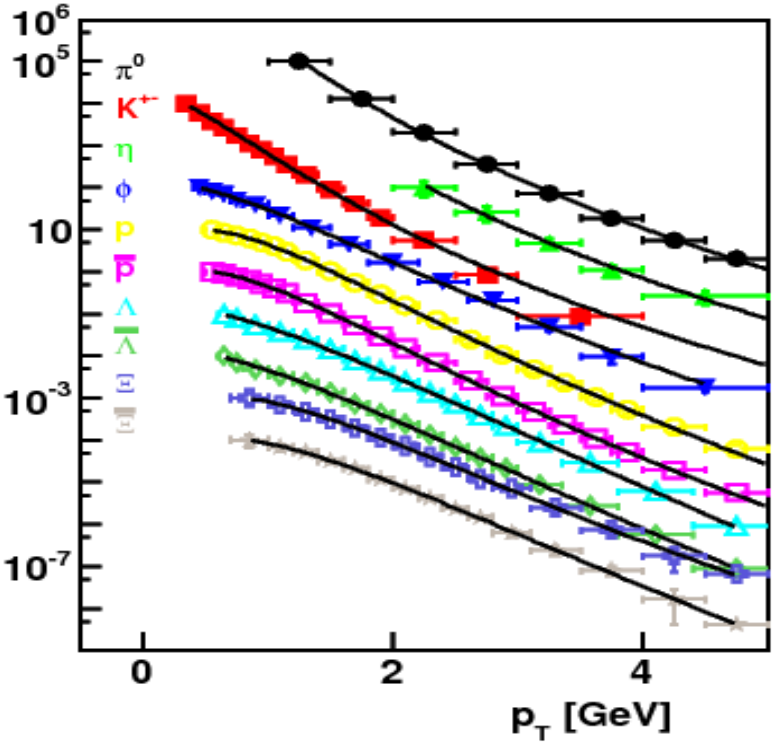
- D. A. Fagundes *etal.*, arXiv:1204.5646 (2012)

1) *Hadron Spectra in Heavy-ion Collisions (AuAu@200 AGeV)*

J. Phys. G: Nucl. Part. Phys. **37** 085104 (2010)



Phys. Lett. B, **689,** 14-17 (2010)



Theory (Right)*: *Coalescence* of *Tsallis* distributed thermal *quarks
+ *„Blast Wave"* flow profile for the expanding QGP

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- *Non-extensive thermodynamics, resonance* production

2) *Cluster hadronisation (* Bignamini, Becattini, *EPJC*, **72**, 2176 (2012) *)*

Clusters **are formed with some distribution (thermal or pQCD or MC generator) and** ***fragment into hadrons*****, while** P_μ**,** ***spin*****,** ***charges are conserved*****.**

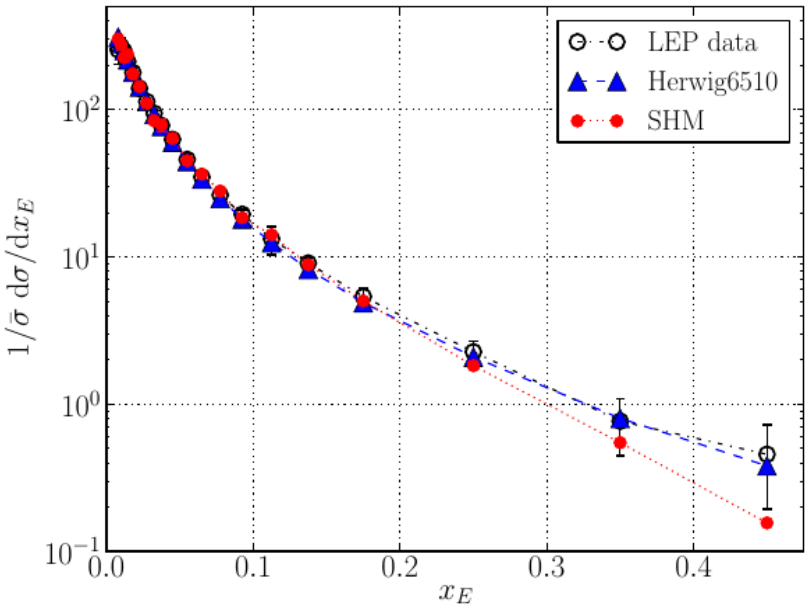


Fig. 13 π^0 scaled energy distribution: comparison among SHM predictions, `Herwig6510` predictions and OPAL data [14].

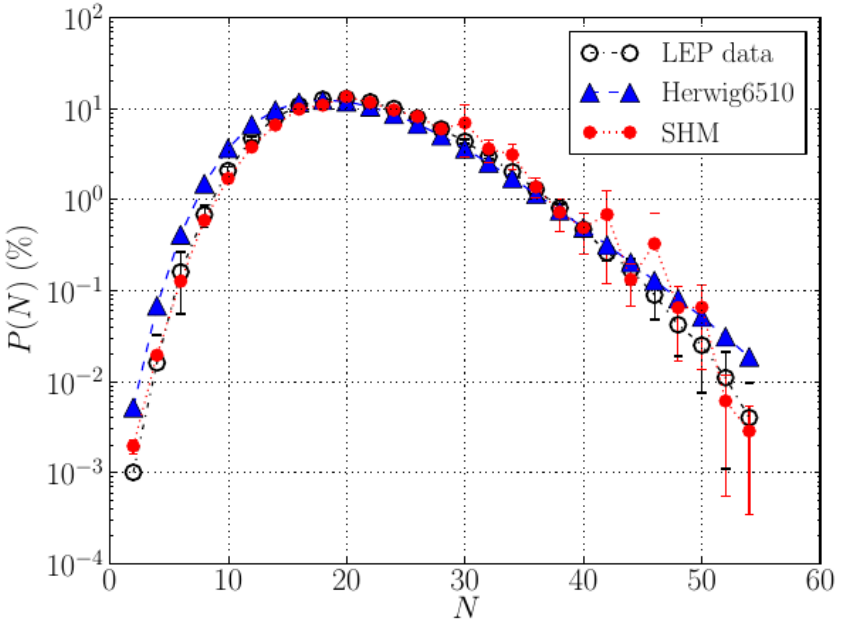


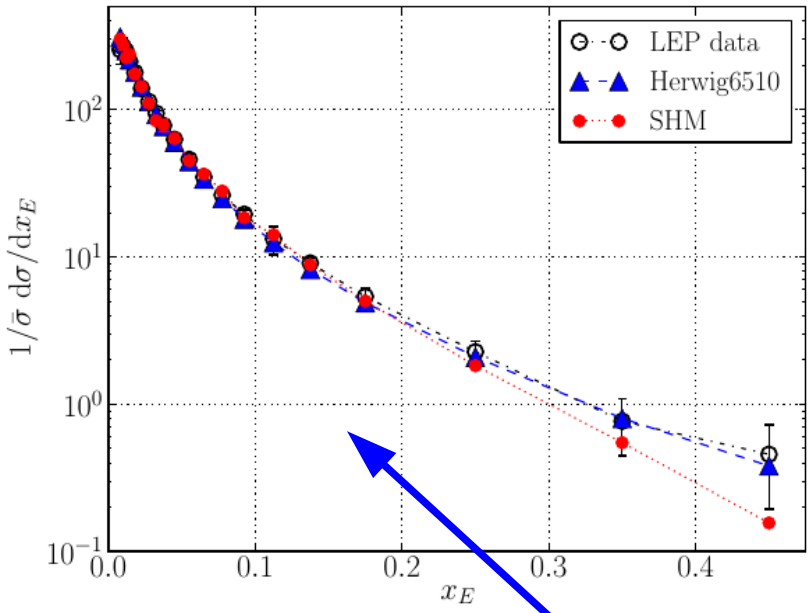
Fig. 22 Charged particle number probability distribution: comparison among SHM predictions, `Herwig6510` predictions and OPAL data [19].

Descibes LEP data**, but it is very complicated.**

Let us simplify it :)!

2) *We will see that*

Euler's Gamma Distribution



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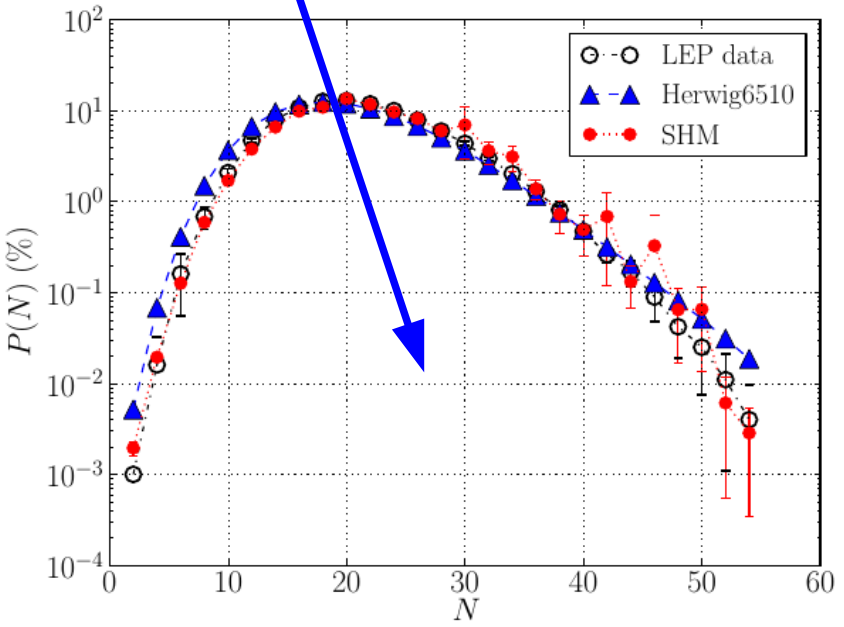


Fig. 22 Charged particle number probability distribution: comparison among SHM predictions, `Herwig6510` predictions and OPAL data [19].

Approximately the ***Microcanonical Tsallis Distribution***

2) Jet-fragmentation in e^+e^- and pp Collisions

The cross section of hadrons h_1 , … , h_N inside a jet of N hadrons is

$$d\sigma^{h_1,\ldots,h_N} = |M|^2 \delta^{(4)}\left(\sum_i p^{\mu}_{h_i} - P^{\mu}_{tot}\right) d\Omega_{h_1,\ldots,h_N}$$

- **If the *jet* is very narrow, hadrons fly *almost colinearly* (*quasi – 1 dimension!*).**
- ***If* we neglect hadron masses ($m_i = 0$), the conservation of P_μ reduces to the *conservation of E*.**
- **If $|M| \approx$ *constant*, we arrive at a *microcanonical ensemble in 1D*:**

$$d\sigma^{h_1,\ldots,h_N} \propto \delta\left(\sum_i \epsilon_{h_i} - E_{tot}\right) d\Omega_{h_1,\ldots,h_N}$$

2) Jet-fragmentation in e^+e^- and pp Collisions

Thisway, in a jet of *N* (massless) hadrons, hadrons have the *energy distribution*:

$$f_N(z) = A_N (1-z)^{N-2}, \qquad z = \frac{\epsilon_h}{E_{jet}}$$

- **The *number of hadrons* in a *jet fluctuates* as *(experimental observation)***

$$p(N) \propto (N - N_0)^{\alpha-1} e^{-\beta(N-N_0)}$$

or negative-binomial distribution

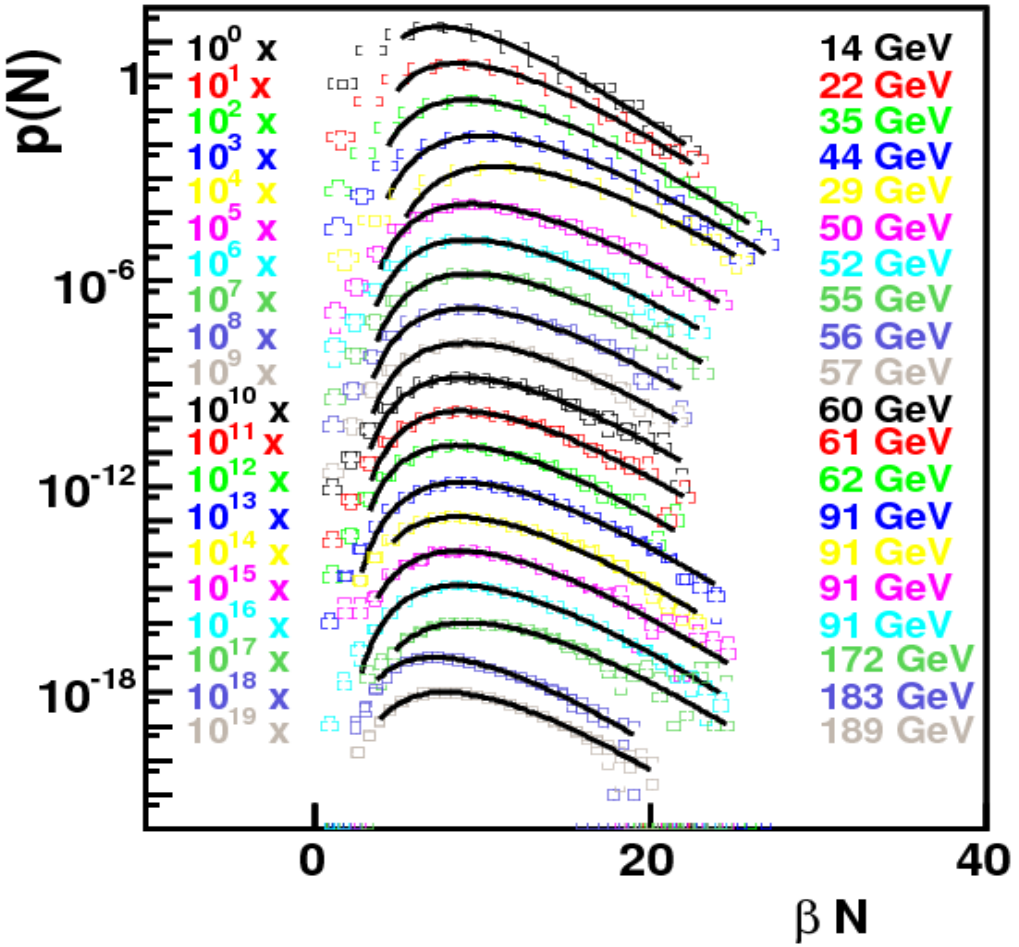
- **Thus, the *multiplicity averaged hadron distribution* (*fragmentation function*) becomes**

$$\frac{d\sigma}{dz} = \sum_{N=N_0}^{\infty} f_N(z)\, N\, p(N) \propto \frac{(1-z)^{\nu(N_0)}}{\left(1 - \frac{(q-1)}{T/E_{jet}} \ln(1-z)\right)^{1/(q-1)}}$$

$$q = 1 + 1/(\alpha+2), \quad T = E_{jet}\beta/(\alpha+2), \quad \nu = N_0 - 2$$

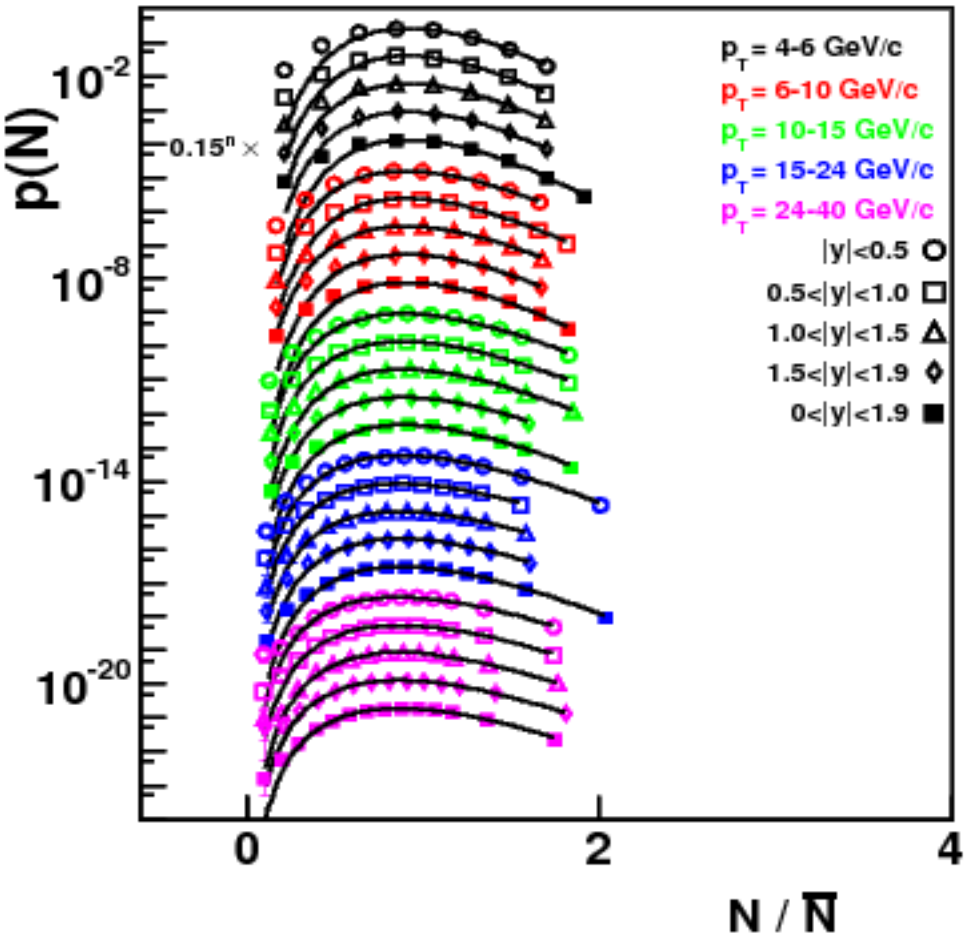
2) Multiplicity Fluctuations

Multiplicity distributions in *electron-positron* collisions



Urmossy et.al., *Phys. Lett. B*, **701**: 111-116 (2011), ArXiv:1101.3023 ***p(N) = Euler's Gamma***

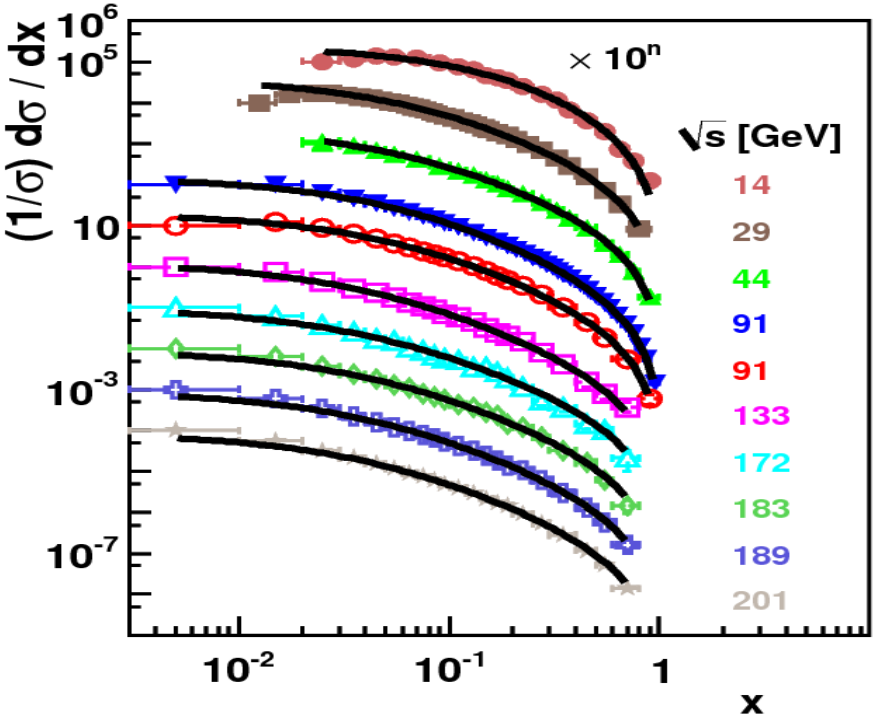
Multiplicity distributions in jets Produced in *pp* collisions @7 TeV



Urmossy et. al., *Phys. Lett. B*, 718, 125-129, (2012) arXiv:1204.1508v1

2) Confrontation with Measurements

e^+e^- annihilations @LEP ($\sqrt{s}$ = 14–200 GeV)



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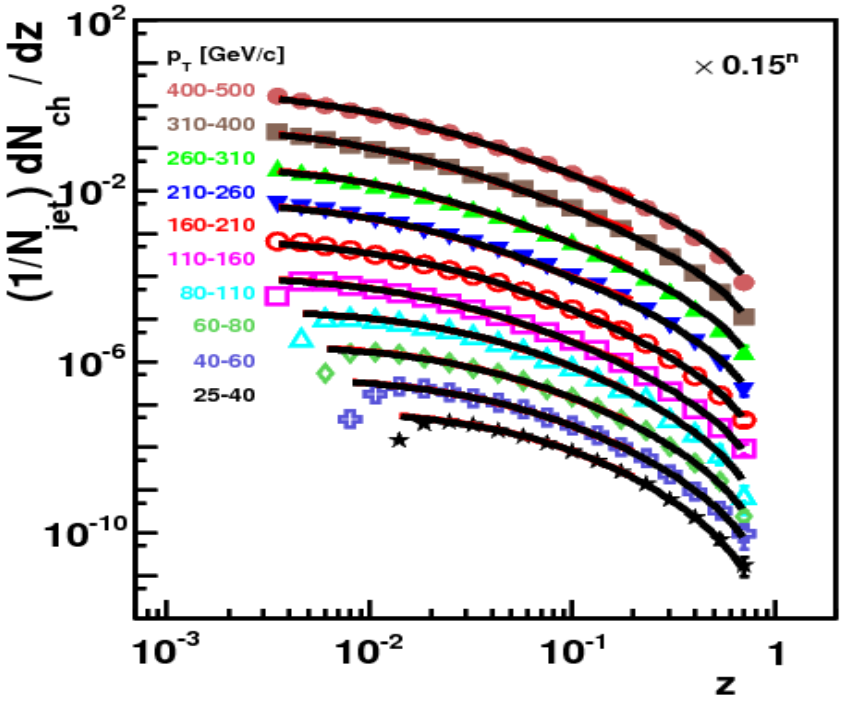
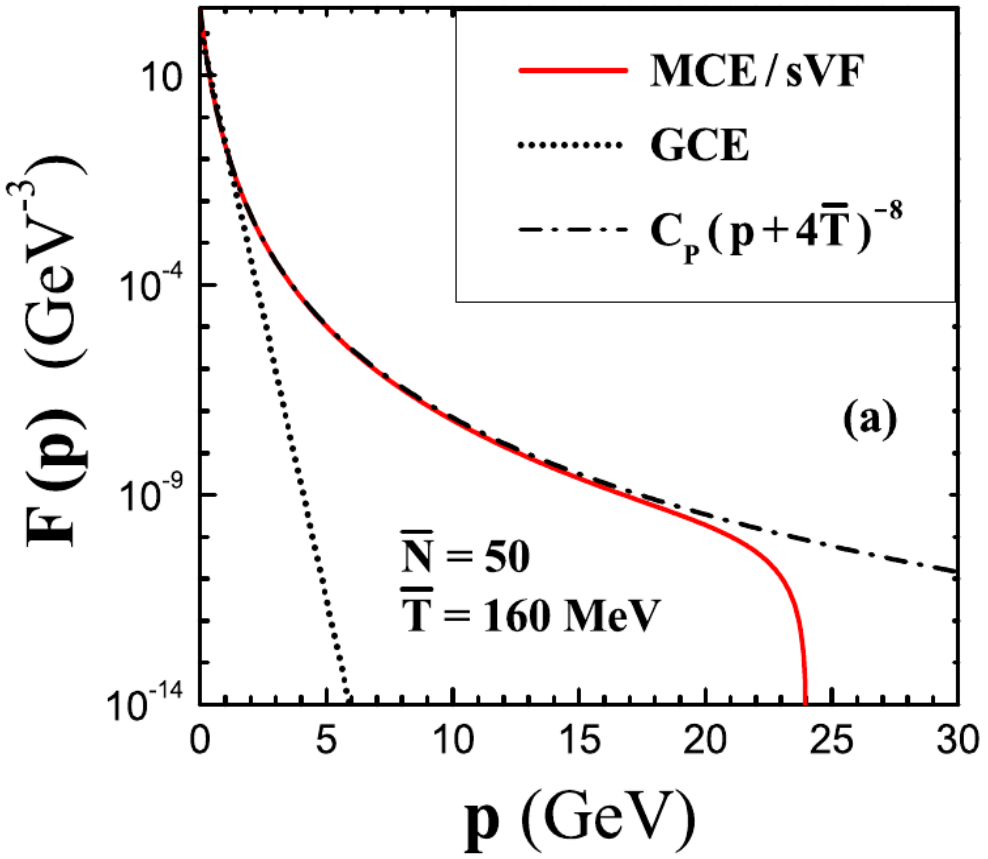
Phys. Lett. B, 701, 111-116 (2011),
arXiv:1101.3023

proton-proton collisions @LHC (pT = 25–500 GeV/c)



Urmossy et. al.,
Phys. Lett. B, 718, 125-129, (2012)
arXiv:1204.1508v1

2) *Similar Result from V Fluctuations*

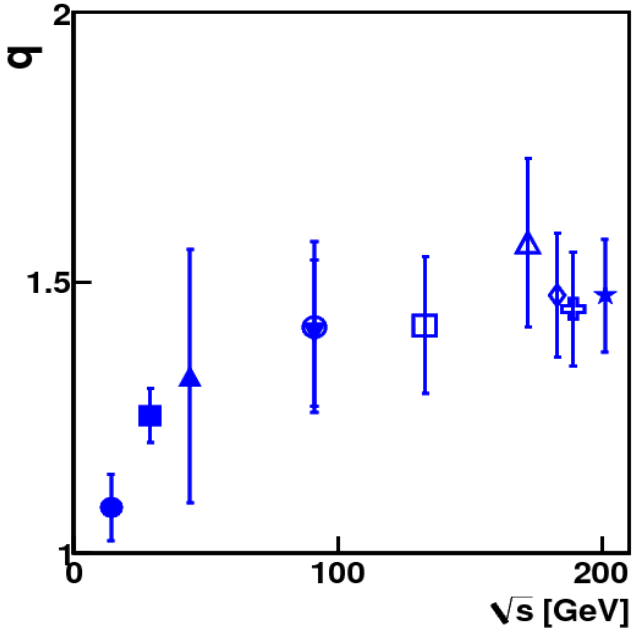


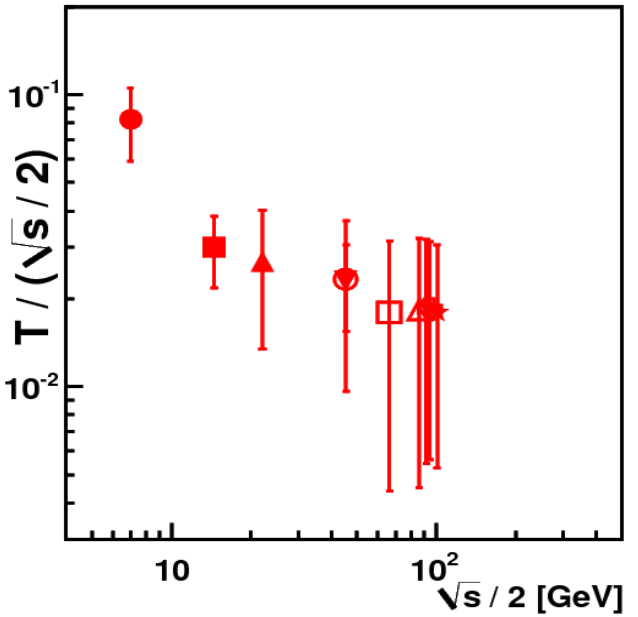
V. V. Begun, M. Gazdzicki, M. I. Gorenstein, Power Law in Micro-Canonical Ensemble with Scaling Volume Fluctuations, *Phys. Rev. C,* **78**, 024904, (2008), arXiv:0804.0075

2) $\sqrt{s}$ dependence of the parameters

e^+e^- annihilations @LEP ($\sqrt{s}$ = 14–200 GeV)

$$\frac{d\sigma}{dz} \propto (1-z)^{\nu} \left(1-\frac{(q-1)}{T/E_{jet}}\ln(1-z)\right)^{-1/(q-1)}$$



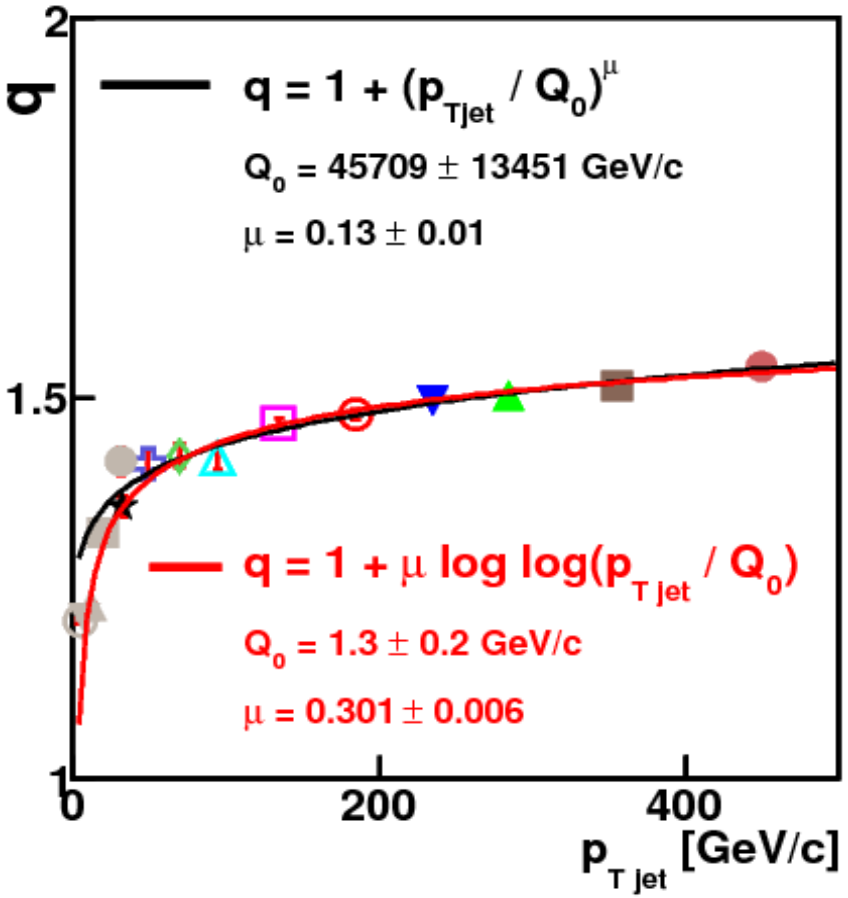


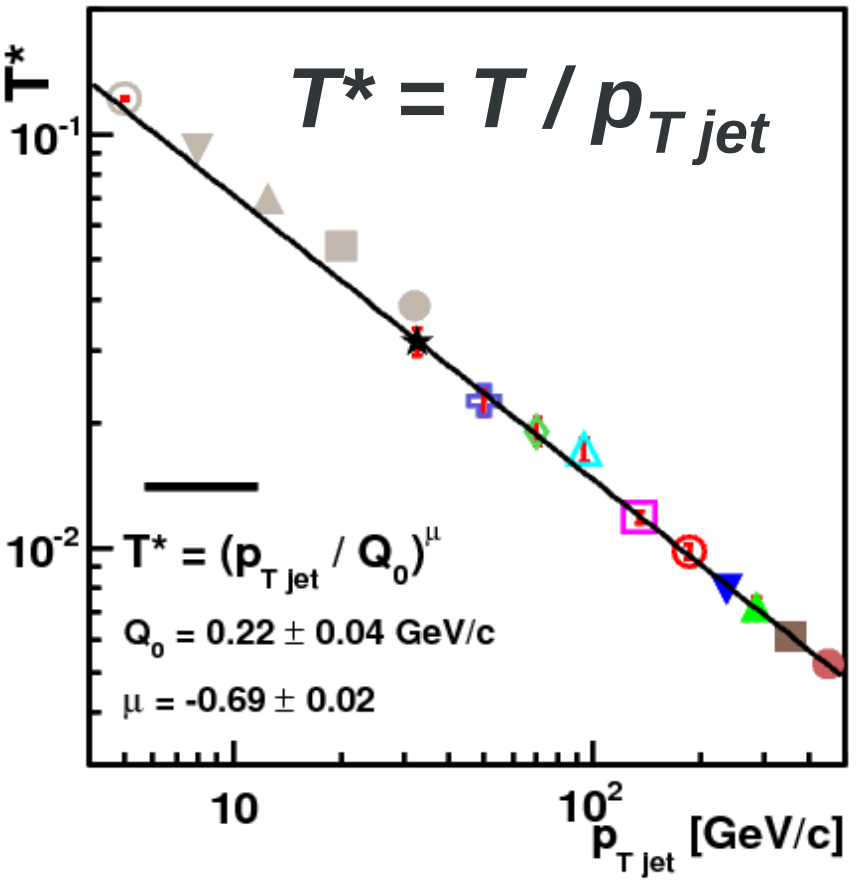
q rises, *T/√s falls* as the collision energy (√s) grows

Urmossy et. al., *Phys. Lett. B*, 701, 111-116 (2011), arXiv:1101.3023

2) Similar $p_{T\,jet}$ depencence in

proton-proton collisions @LHC (pT = 25–500 GeV/c)

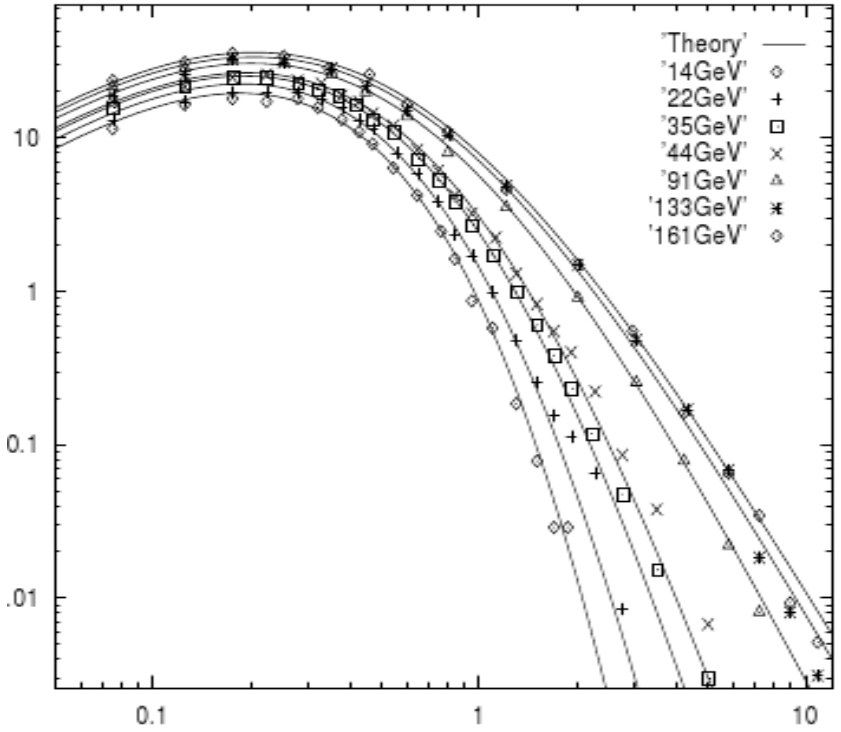




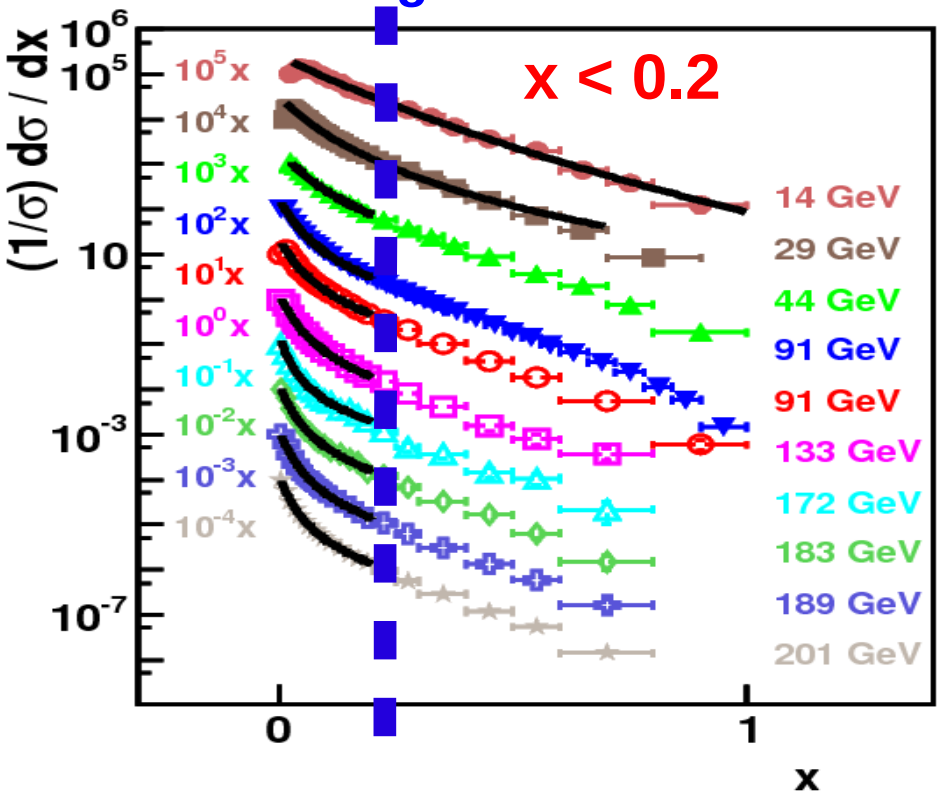
Urmossy et. al., *Phys. Lett. B*, 718, 125-129, (2012), arXiv:1204.1508v1

2) Low x limit: Tsallis distribution (e^+e^-)

Transverse momentum distribution of charged hadrons in jets



Energy-fraction distribution of charged hadrons



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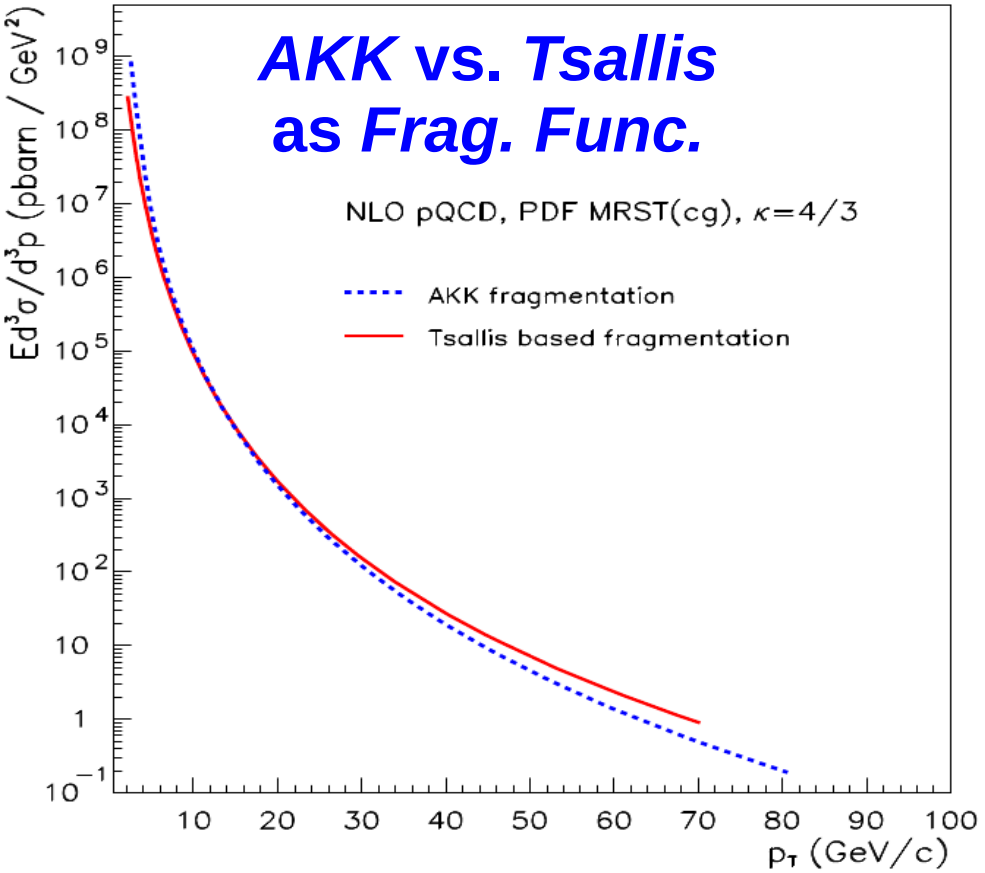
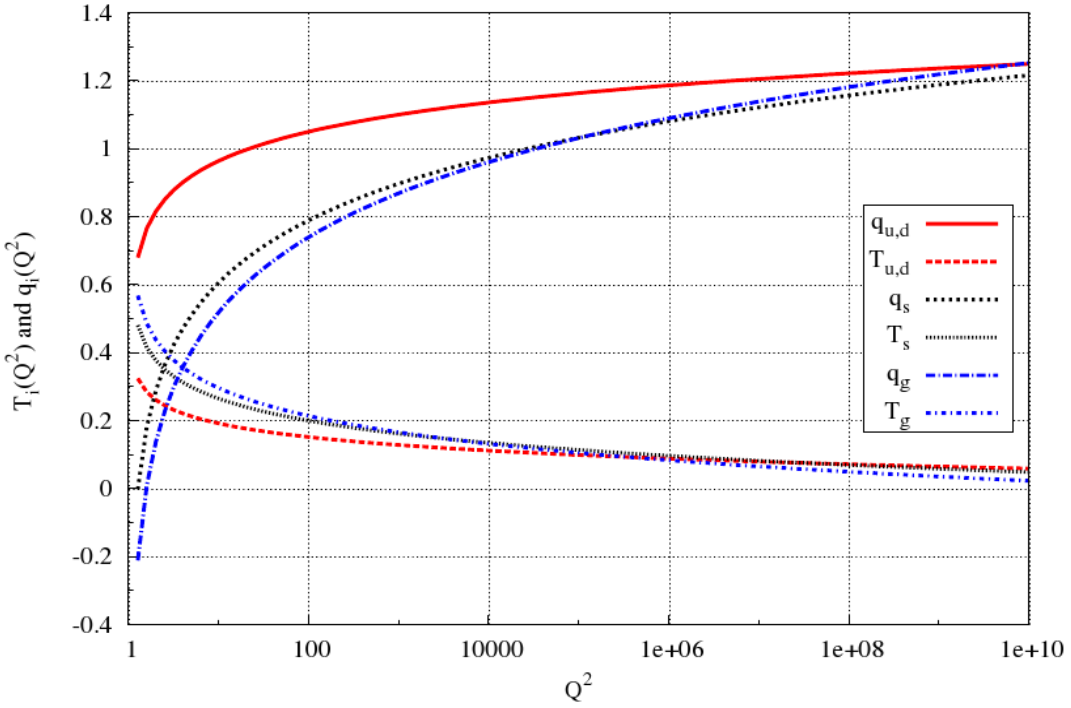
2) DGLAP Evolution of the parameters

$\pi^{\pm}$ spectrum in pp --> $\pi^{\pm}$X @ √s=7 TeV (NLO pQCD)



Scale dependence of *q* and *T*

From fits to AKK type FragFuncts



$$D_{p_i}^{\pi^+}(z) \sim (1+(q_i-1)z/T_i)^{-1/(q_i-1)}$$

$$q_i = q_{0i} + q_{1i}\ln(\ln(Q^2))$$

$$T_i = T_{0i} + T_{1i}\ln(\ln(Q^2))$$

Barnaföldi et. al., *Proceedings of the Workshop Gribov '80* (2010)

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3, Hadron Spectra in pp Collisions

- ***Multiplicity*** **and** ***collision energy*** **dependence of** ***hadron spectra***
- **Model:** ***microcanonical jets*** **+** ***multiplicity*** **and** ***jet energy fluctuations***

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- The adequate *scaling variable* for hadrons
- Spectra from *Quark coalescence + Collective flow + Freeze-out*
- *Non-extensive thermodynamics, resonance* production

3) *Hadron Spectra in pp Collisions*

- **Spectra can be described by the Tsallis distribution**

$$\frac{dN}{d^3p} \propto \left(1+\frac{q-1}{T}\left(m_T-m\right)\right)^{-1/(q-1)}$$

- **Dependence of q on √s is**

$$q = 1+q_1 \log\log\left(\sqrt{s}/Q_0\right)$$

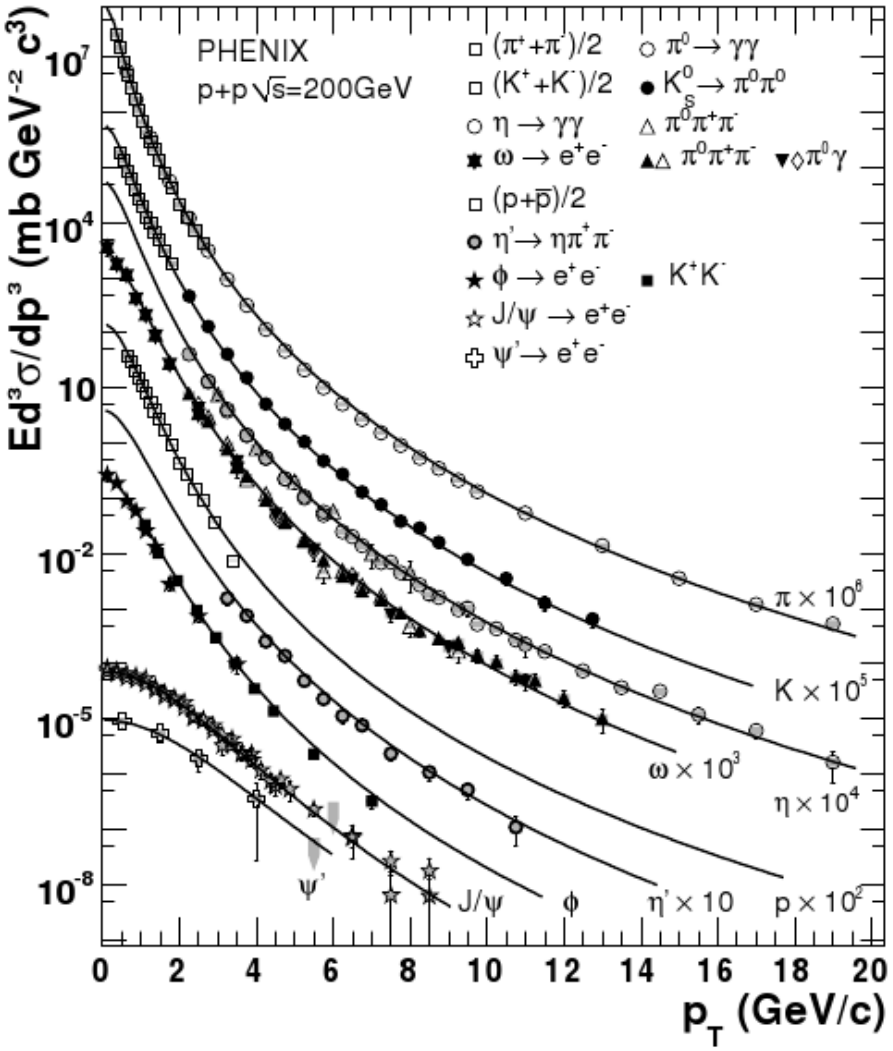
- **Dependence of q on the multiplicity is**

$$q = 1+\mu \log\log\left(N/N_q\right)$$

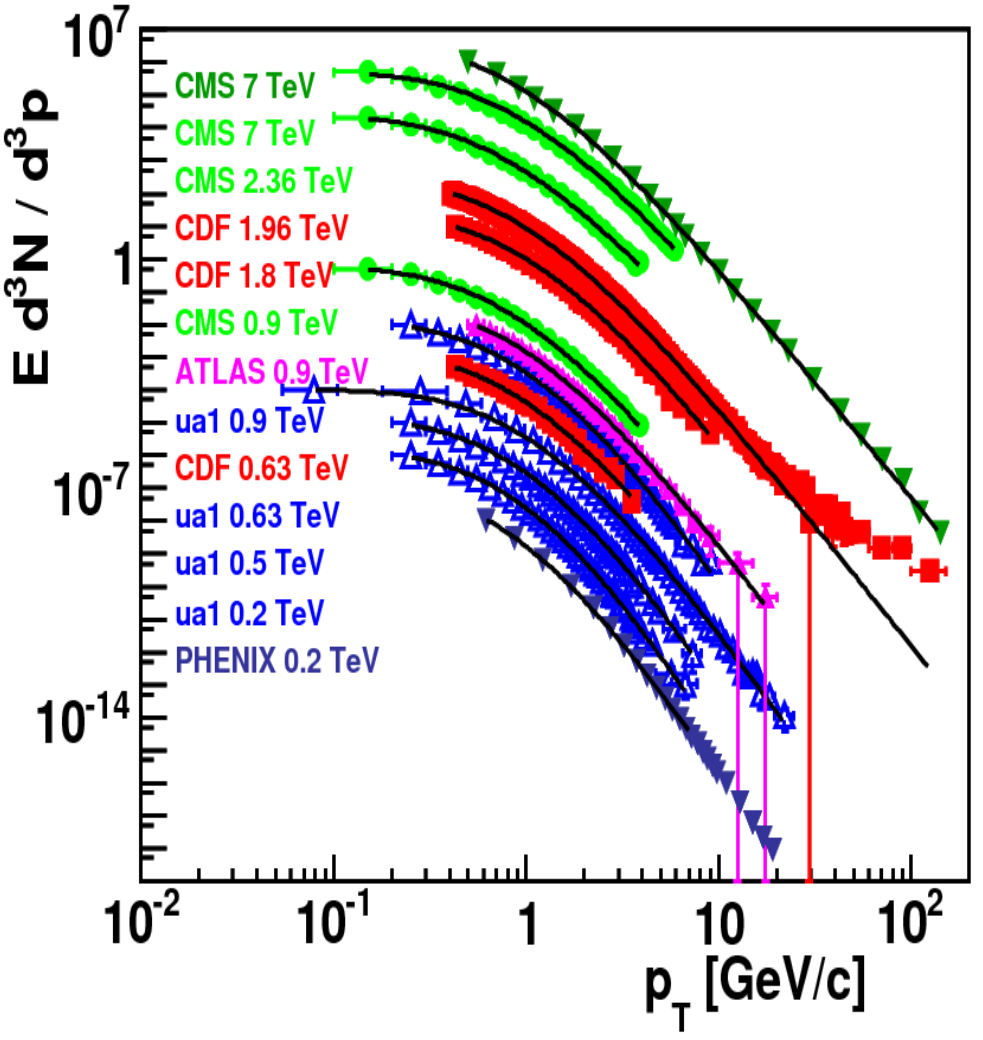
- **Multiplicity distribution is NBD / Euler's Gamma distribution**

3) Hadron Spectra in Proton-proton Collisions

Phys. Rev. D, **83,** 052004 (2011)



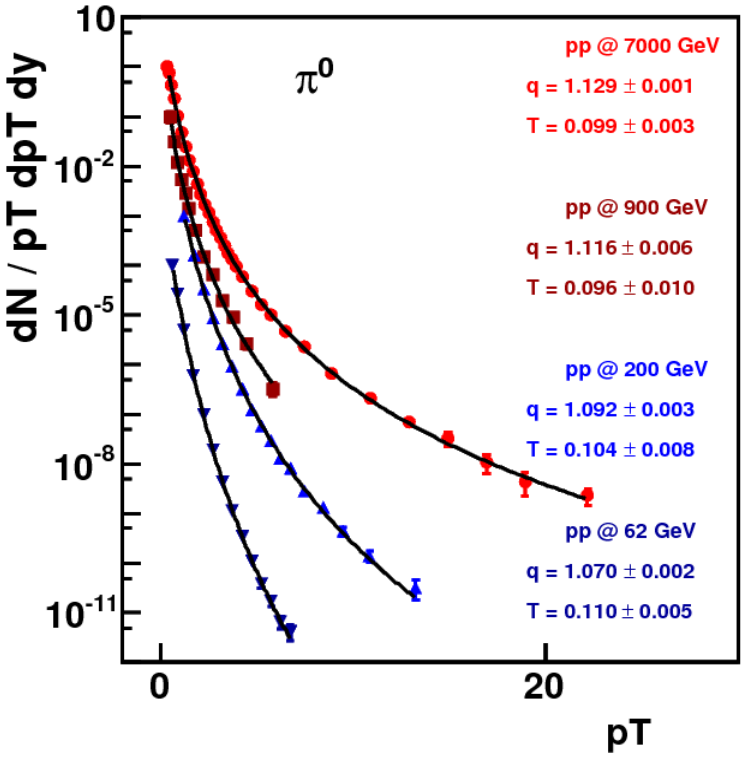
J. Phys.: Conf. Ser., **270** 012008 (2011)

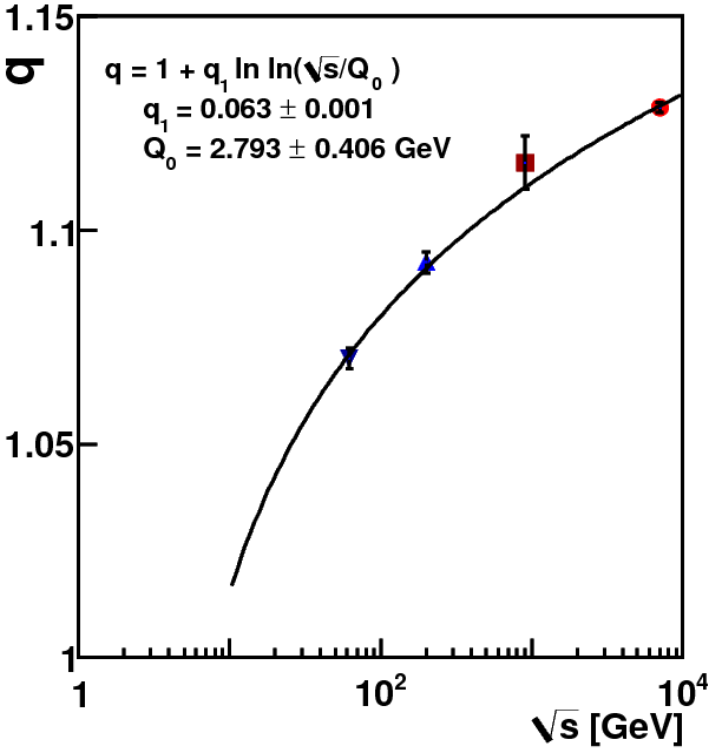


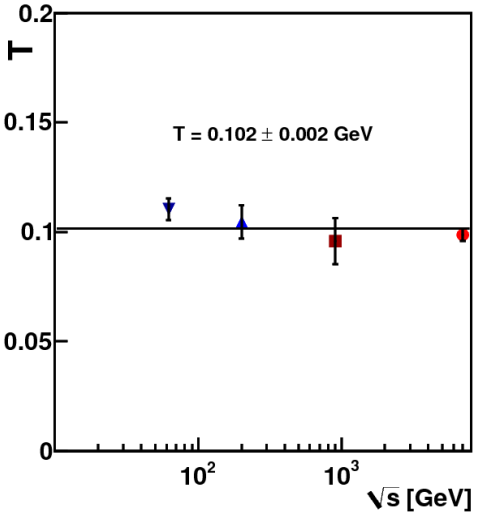
$$E\frac{d\sigma}{d^3p} \propto \left(1+m_T/nT\right)^{-n}$$

$$E\frac{d\sigma}{d^3p} \propto m_T\left(1+m_T/nT\right)^{-n}$$

3) √s Dependence of π^0 Spectra in pp Collisions





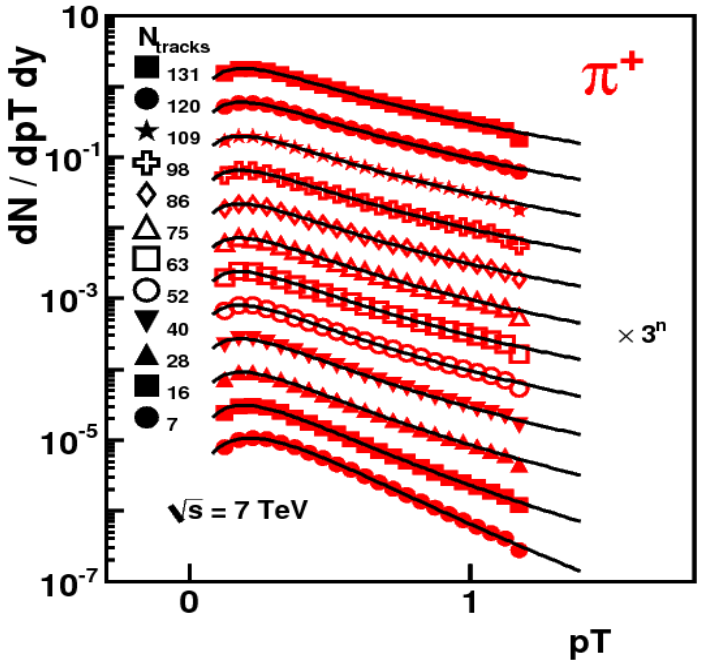


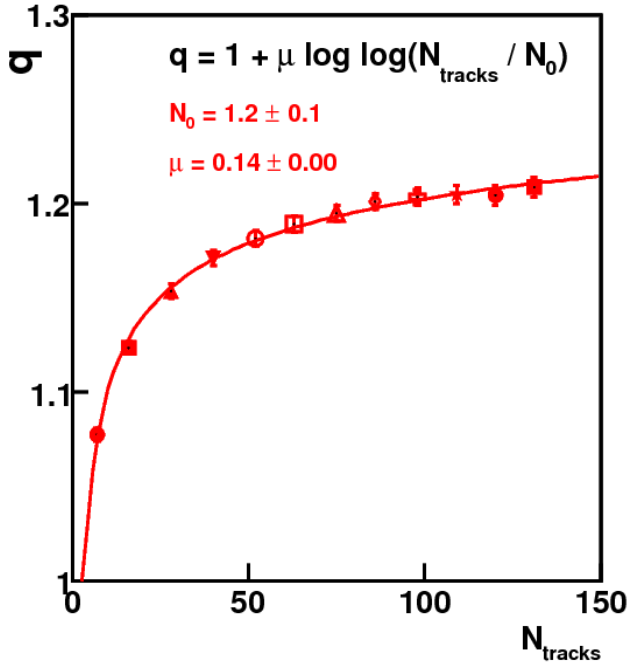
$$q = 1 + q_1 \log\log\left(\sqrt{s}/Q_0\right) \quad , \qquad T = const$$

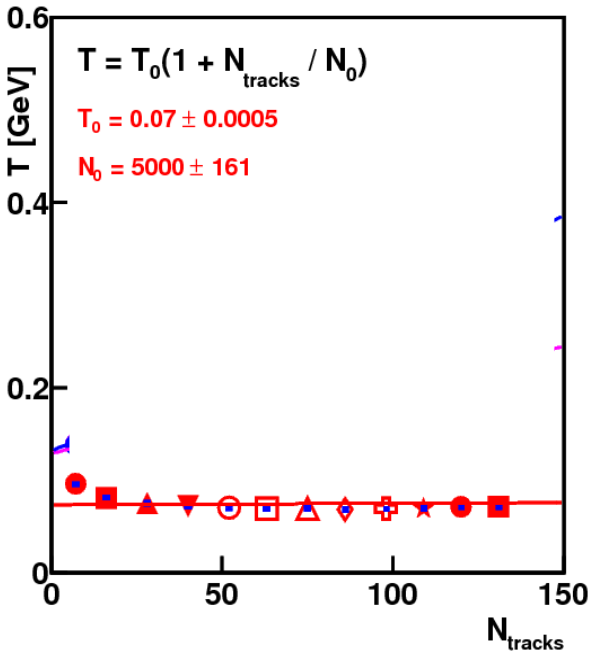
Urmossy, arXiv: 1212.0260

3) *Multiplicity Dependence of π^+ Spectra in pp@7 TeV*

(Ferenc Siklér's Talk)



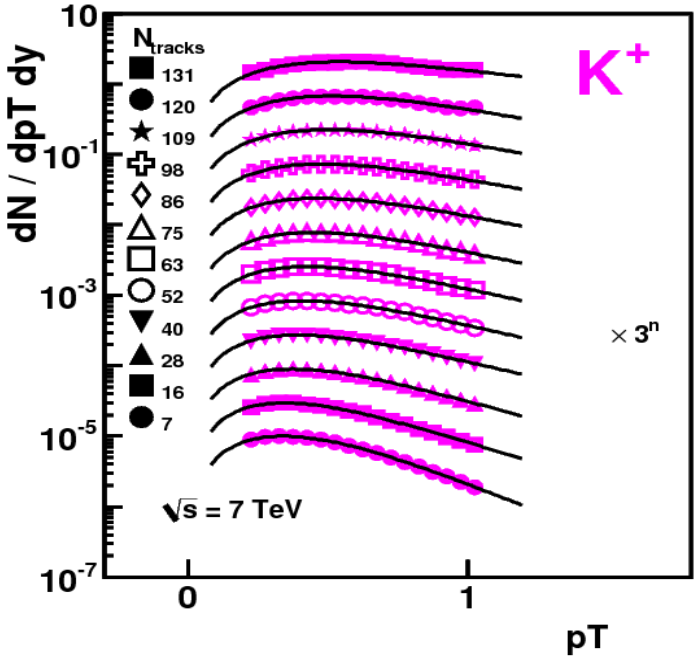


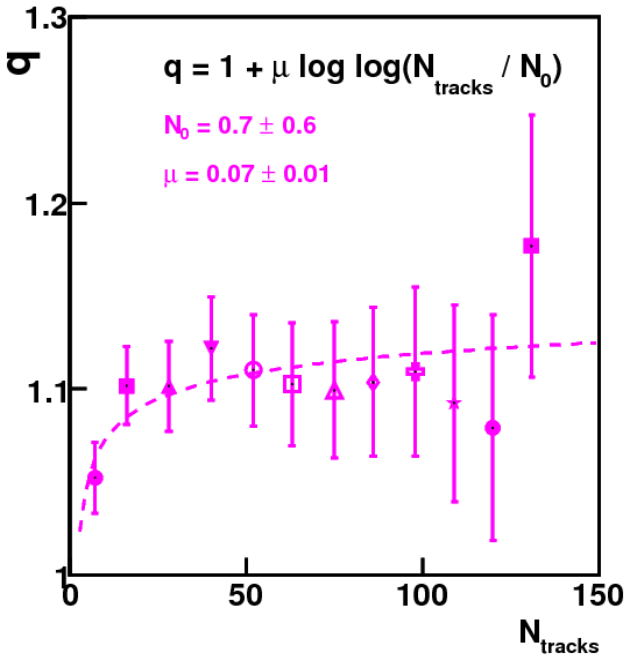


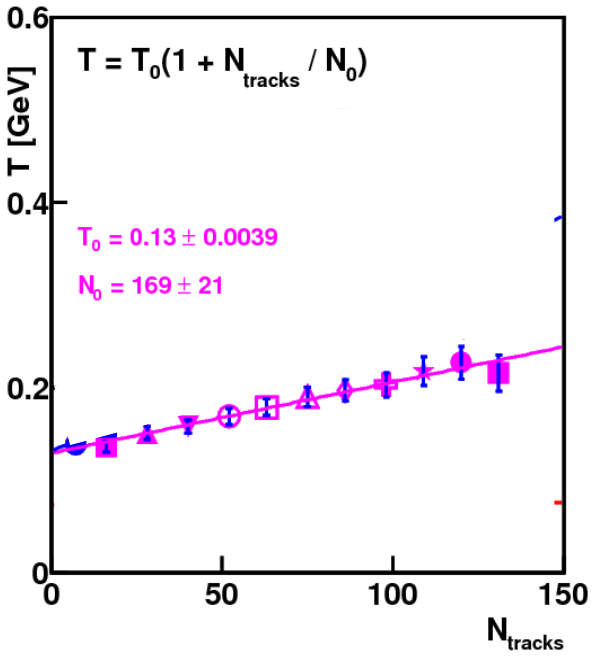
$$\pi^+ : \; q \;=\; 1+\mu \log\log\left(N/N_q\right) \;, \quad T \;=\; T_0$$

Urmossy, arXiv: 1212.0260

3) *Multiplicity Dependence of K^+ Spectra in pp@7 TeV*



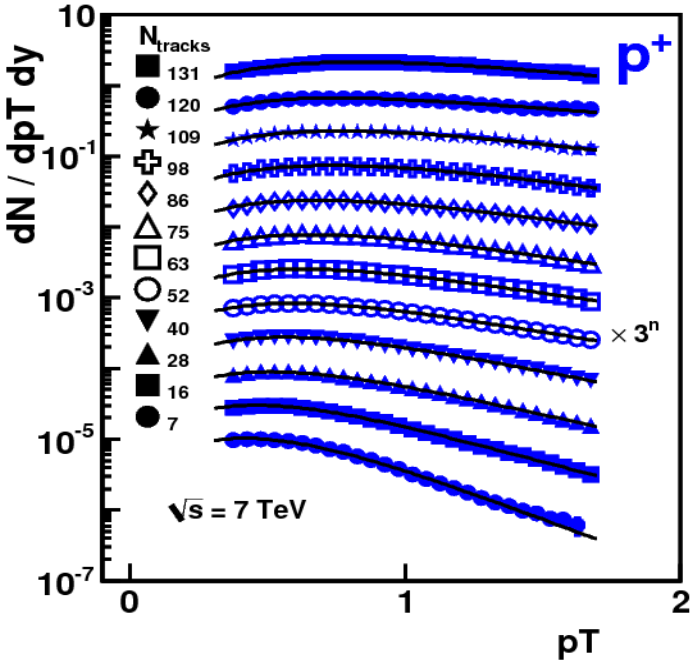


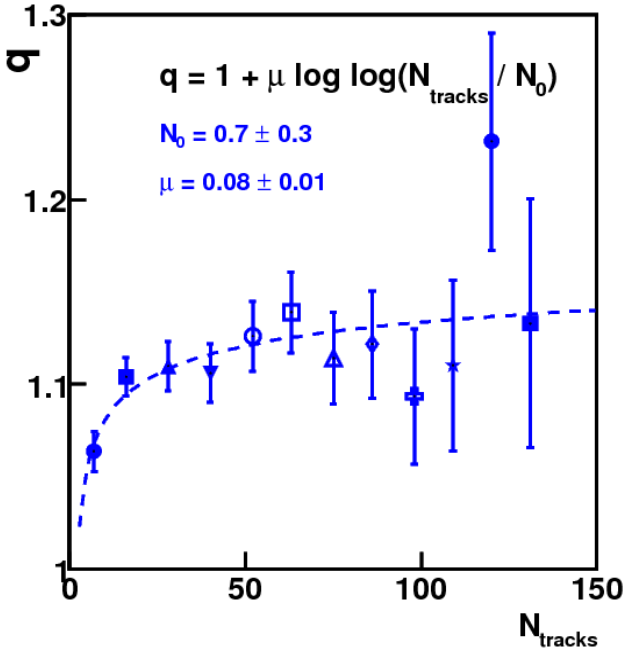


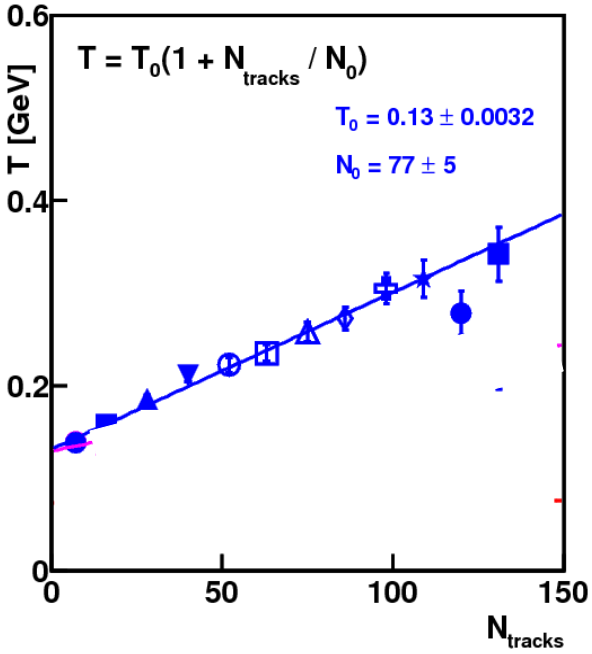
$$K^+ : \; q \;=\; 1+\mu \log\log\left(N/N_q\right) \;,\; T=T_0\left(1+N/N_T\right)$$

Urmossy, arXiv: 1212.0260

3) *Multiplicity Dependence of p^+ Spectra in pp@7 TeV*







$$p^+ : \; q \;=\; 1+\mu \log\log\left(N/N_q\right) \;,\; T=T_0\left(1+N/N_T\right)$$

Urmossy, arXiv: 1212.0260

3) *Simple Model*

- **We may use the obtained fragmentation function in a parton model calculation.**

  Problem: the low pT regime is not perturbative.

- **From the analysis of jet data, we have seen that hadron spectra in jets of fix multiplicity *N* may be a simple *microcanonical* (for low p_T *canonical*) distribution.**

$$D(z) \rightarrow D_{E_{jet},N_{jet}}(p_T) \sim (1-z)^{N-2} \quad , \quad \exp(-DN\epsilon/E)$$

- **Hadron spectra in pp events may be obtained (apart from the underlying event) by averaging over multiplicity and jet energy fluctuations.**

$$\frac{dN}{d^3p}^{N=fix} \sim \langle D(p_T)\rangle_{E_{jet}} \quad , \quad \frac{dN}{d^3p} \sim \langle D(p_T)\rangle_{E_{jet},N_{jet}}$$

What is the distribution of E_{jet}?

3) *Simple Model*

Hadron distribution in a jet **of multiplicity** ***N*** **and energy** E_{jet}

$$D_{E_{jet},N}(\epsilon) \;=\; A \;\; \exp\{-\beta(m_T - m)\} \;\;,\;\;\; \beta = N/E_{jet}$$

Conjecture for the joint distribution of ***N*** **and** E_{jet} **:**

$$p(N, E_{jet}) \;=\; h(N) \;\; g_N(E_{jet})$$

with independent multiplicity fluctuations

$$h(N) \;\sim\; N^{a-1} e^{-aN/N_0}$$

but multiplicity dependent energy distribution

$$g_N(E_{jet}) \;\sim\; E_{jet}^{-(\alpha+2)} e^{-\alpha E_0/E_{jet}}$$

… and we will recover the measured marginal distributions:

$$\frac{dN}{d^3p}^{N=fix} \;\sim\; \langle D(p_T)\rangle_{E_{jet}} \;\;,\;\;\; \frac{dN}{d^3p} \;\sim\; \langle D(p_T)\rangle_{E_{jet},N_{jet}}$$

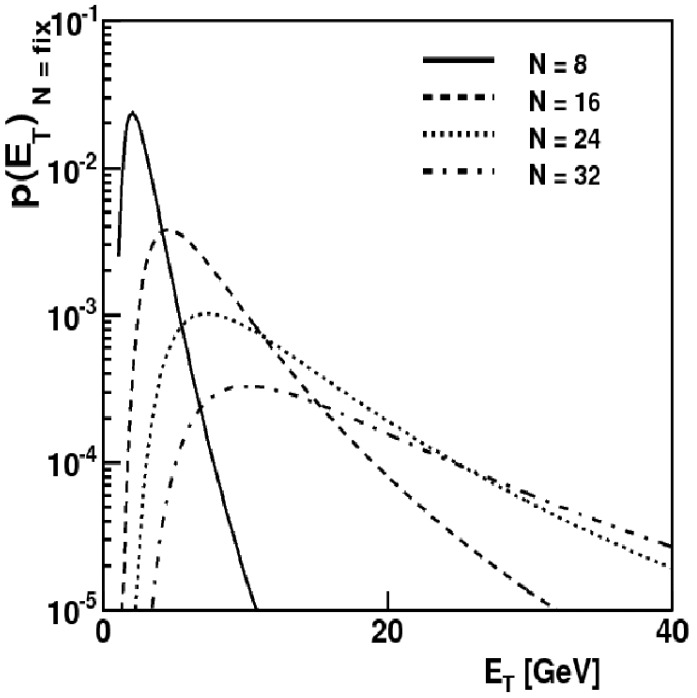
3) *Predictions of the Model*

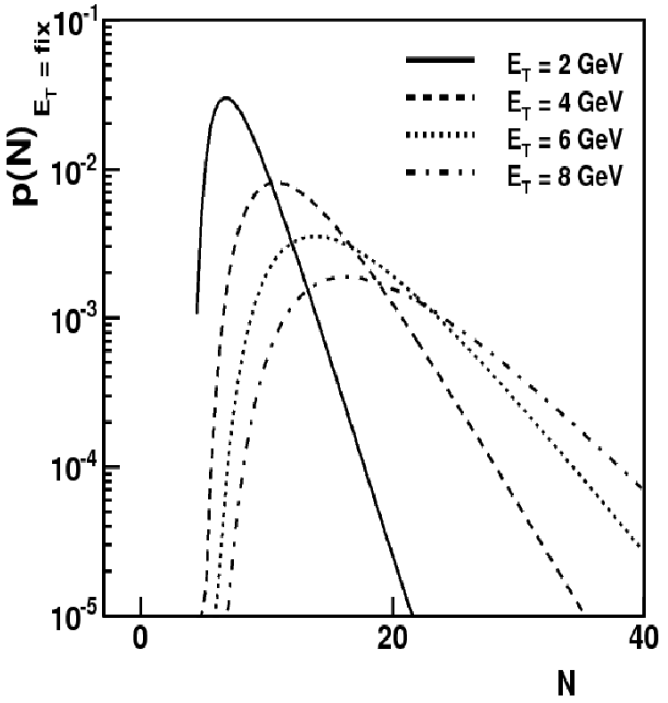
N **and** ***E_{jet}*** **are** ***correlated!*** **Their joint distribution:**

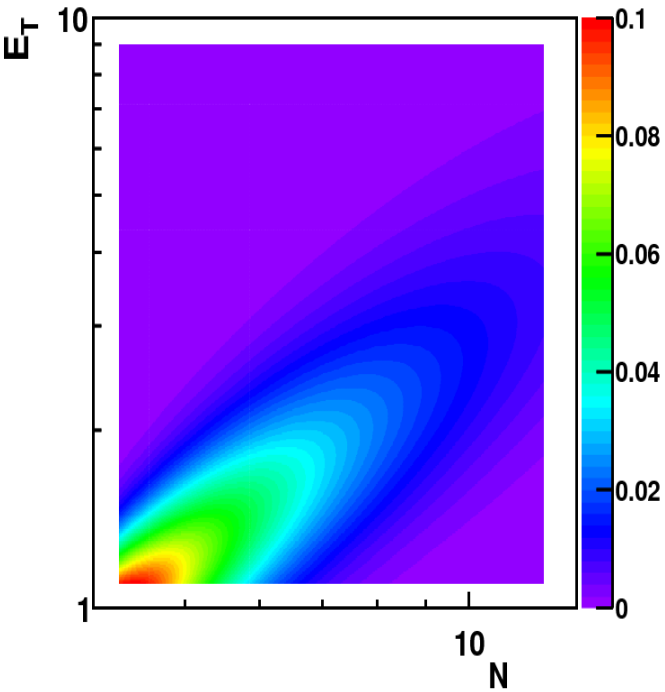
$$p(N, E_{jet}) \sim E_{jet}^{-(\alpha+2)} e^{-\alpha E_0/E_{jet}} \times N^{a-1} e^{-aN/N_0}$$

with

$$\alpha = [\mu \ln\ln(N/N_q)]^{-1} - 4, \quad E_0 = \frac{3N T_0 (1 + N/N_T)}{1 - 4\mu \ln\ln(N/N_q)}$$







Urmossy, arXiv: 1212.0260

3) *Similar Results from T Fluctuations*

C. Beck, G. Wilk, *Eur. Phys. J. A*, **40**, 267 and 299-312, (2009)

If the hadron distribution is *Boltzmann-Gibbs*,

$$f(\epsilon) = A \exp(-\beta \epsilon)$$

but the *temperature fluctuates event-by-event* or *position-to-position* as

$$p(\beta) \propto \beta^{\alpha-1} \exp(-\alpha \beta / \langle \beta \rangle)$$

the *average distribution* becomes the *Tsallis* distribution:

$$\frac{dN}{d^3 p} = \int d\beta \, p(\beta) f_\beta(\epsilon) \propto \left(1 + \frac{\langle \beta \rangle \epsilon}{\alpha}\right)^{-(\alpha + D + 1)}$$

Content

1, From the Smallest to the Largest

- Hadron spectra in e^+e^-, *pp* and *AA* collisions
- *NEW: Multiplicity dependence* of hadron spectra in *pp* collisions

2, Statistical Jet Fragmentation @ LEP & LHC

- Model: *microcanonical jets + multiplicity fluctuations*
- Good agreement with *LEP & ATLAS* data
- *Scale evolution* of the newly obtained *fragmentation functions*

  Evolution of q, T with E_{jet} and Q^2

3, Hadron Spectra in pp Collisions

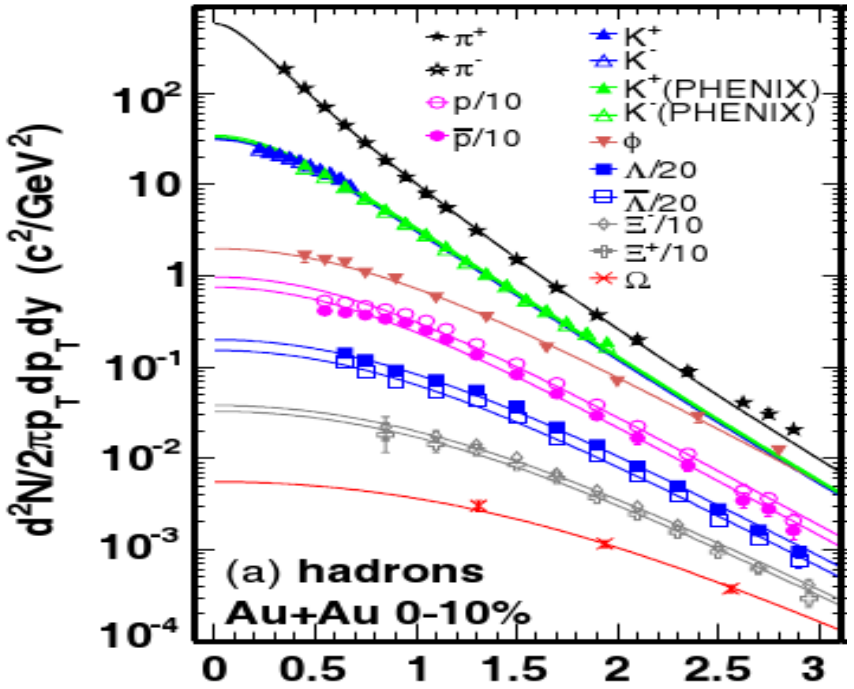
- *Multiplicity* and *collision energy* dependence of *hadron spectra*
- Model: *microcanonical jets + multiplicity* and *jet energy fluctuations*

4, Hadron Spectra and Transverse Flow in AA Collisions

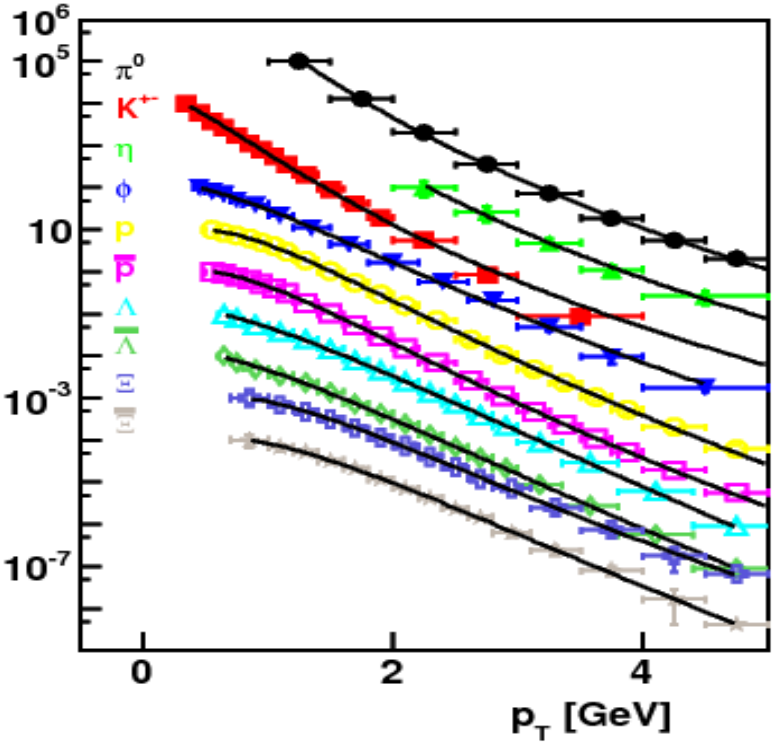
- The adequate *scaling variable* for hadrons
- Spectra from *Quark coalescence + Collective flow + Freeze-out*
- *Non-extensive thermodynamics, resonance* production

4) Hadron Spectra in Heavy-ion Collisions

J. Phys. G: Nucl. Part. Phys. **37** 085104 (2010)



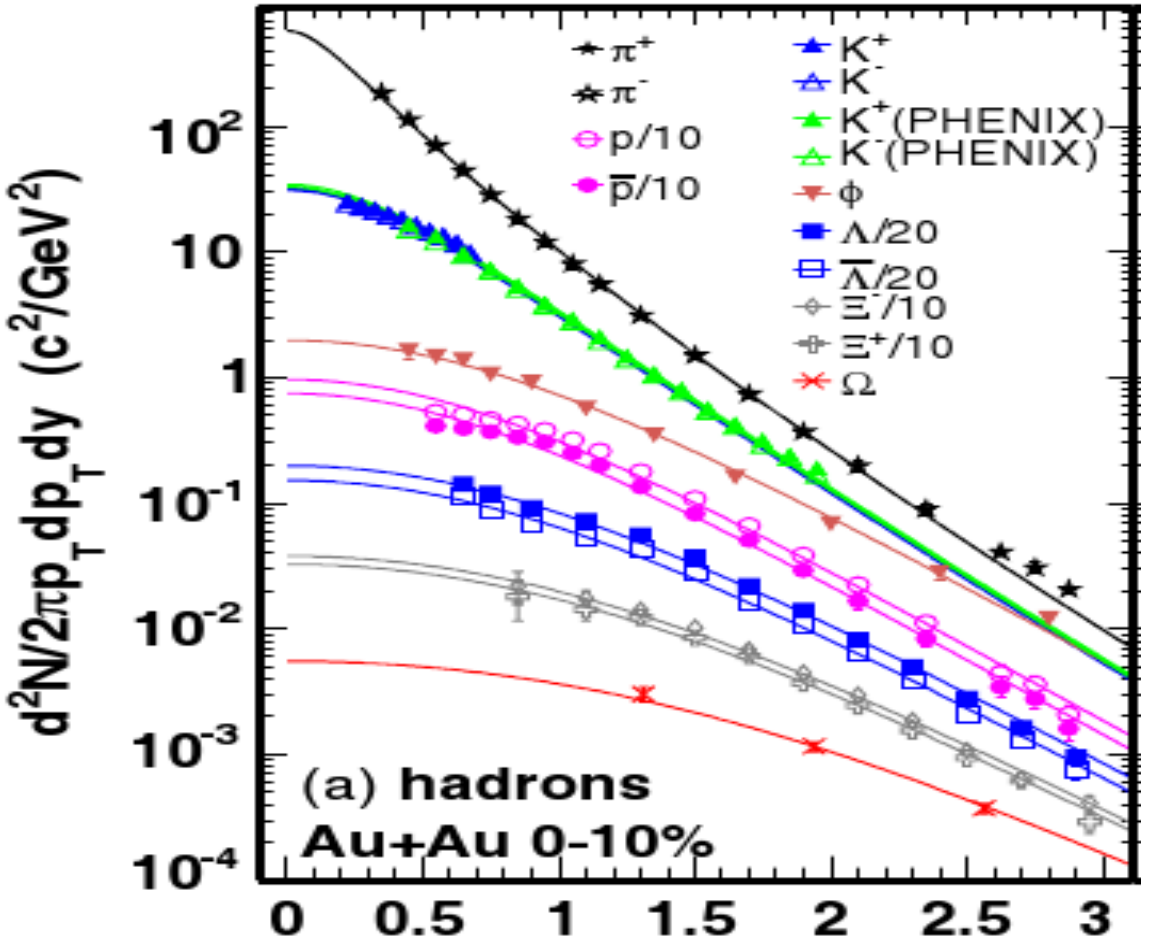
Phys. Lett. B, **689,** 14-17 (2010)



Theory (Right)*: *Coalescence* of *Tsallis* distributed thermal *quarks
+ *„Blast Wave"* flow profile for the expanding QGP

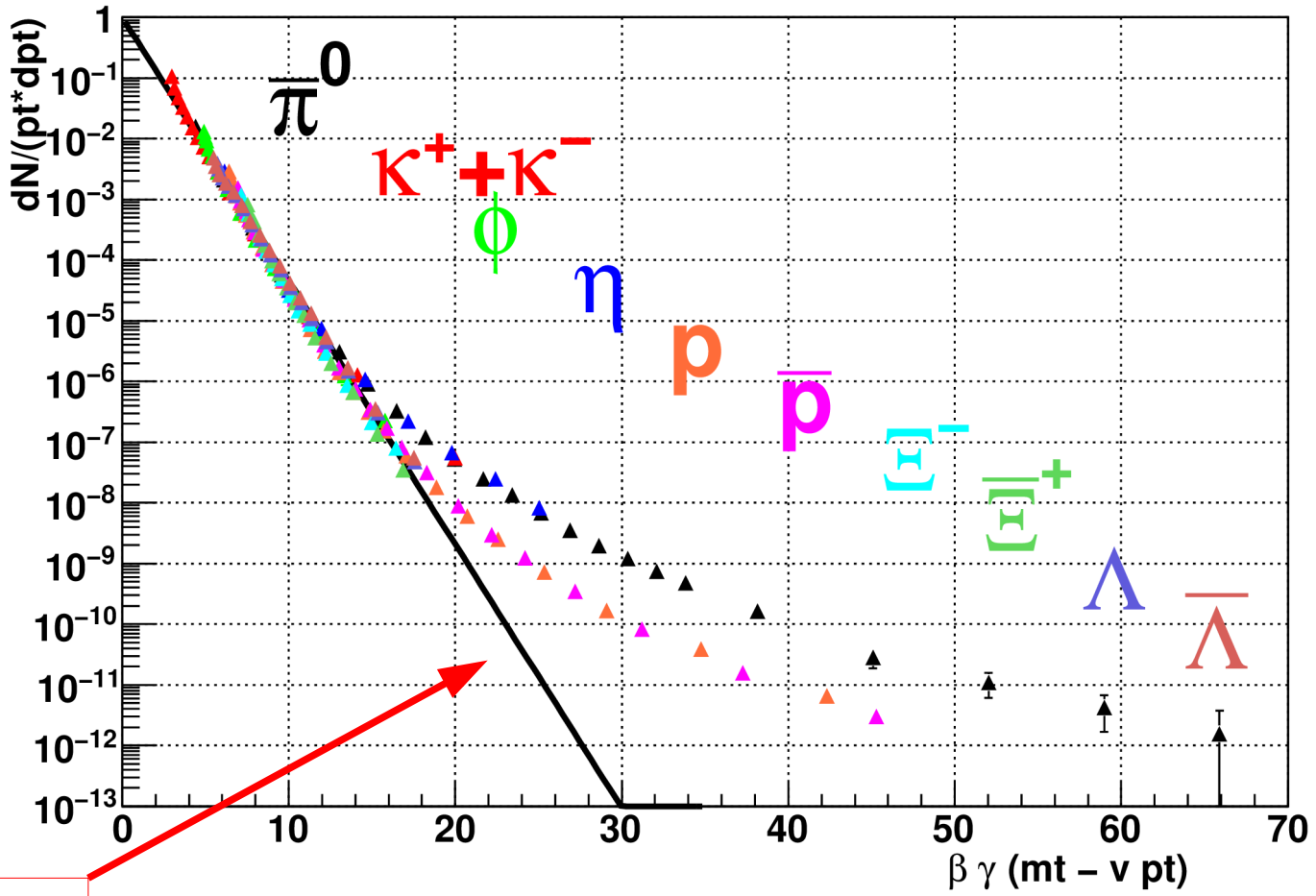
4) What is the Good Scaling Variable?

J. Phys. G: Nucl. Part. Phys. **37** 085104 (2010)



p_T ?

4) What is the Good Scaling Variable?



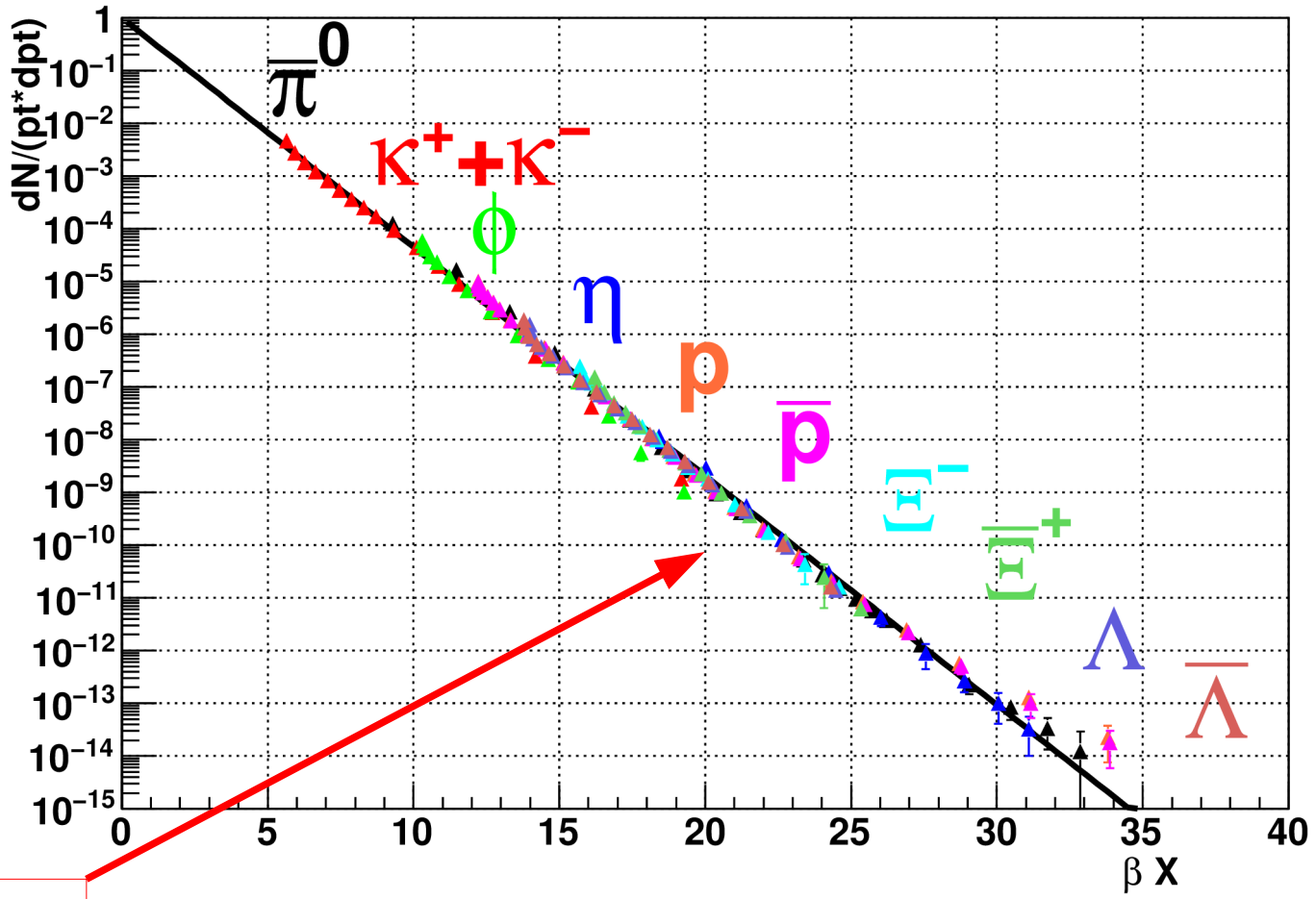
$e^{-E_{co-moving}/T}$

$$E_{co-moving} = \gamma(m_T - v\,p_T)\,?$$

T.S. Biró et.al., *J. Phys. G: Nucl. Part. Phys.*, **37,** 094027, (2010)

4) What is the Good Scaling Variable?

T = 51 ± 10 MeV, q = 1.062 ± 7.65 × 10−3, v = 0.5 ± 0.1



$e^{-X/T}$

$$X = \frac{1}{q-1} \ln\left[1 + \frac{q-1}{T} E_{co-moving}\right]$$:)

T.S. Biró et.al., *J. Phys. G: Nucl. Part. Phys.*, **37,** 094027, (2010)

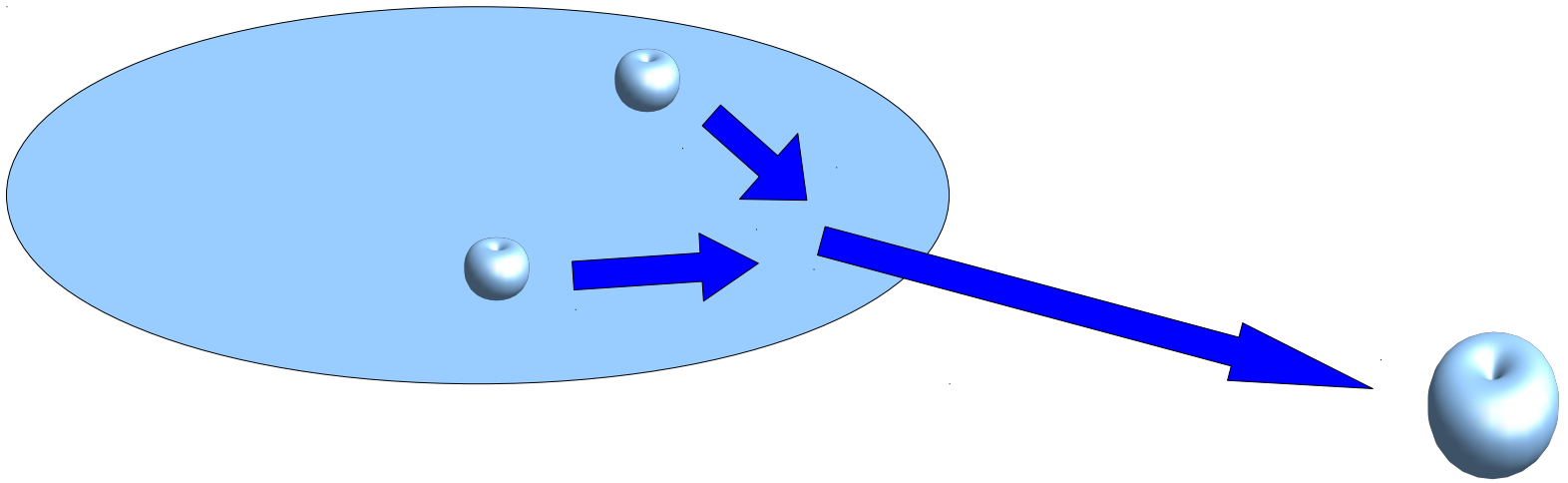
4) *Thermal Re-hadronisation By Quark Coalescence*

***n* on-shell quarks produce a hadron very rapidly:**

$$F_h(P_h) = \int \prod d^3 p_i \;\; f_q(E_1) \ldots f_q(E_n) C(p_i, P_h, M_h)$$

The hadron formation requires *energy-momentum conservation* and that the incoming quarks have *small relative momenta*

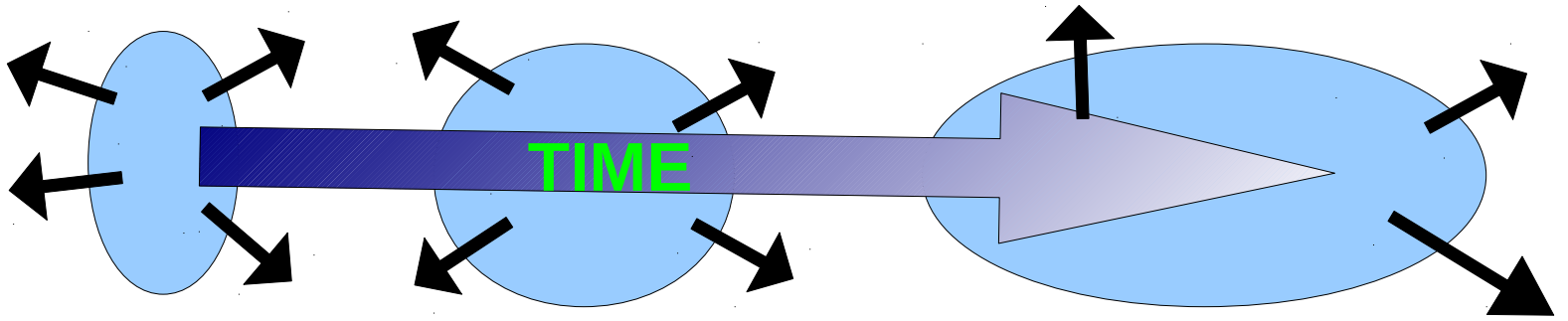
$$C(p_i, P_h, M_h) = \delta^4\left(\sum p_i^\mu - P_h^\mu\right) \prod \delta^3\left(\vec{p}_i - \vec{P}_h / n\right)$$



Hadron Evaporation from the Expanding Plasma

The 3-volume of the QGP evolves in time and spits out hadrons:

$V_3^{QGP}(t) \rightarrow \Sigma^{\mu}$

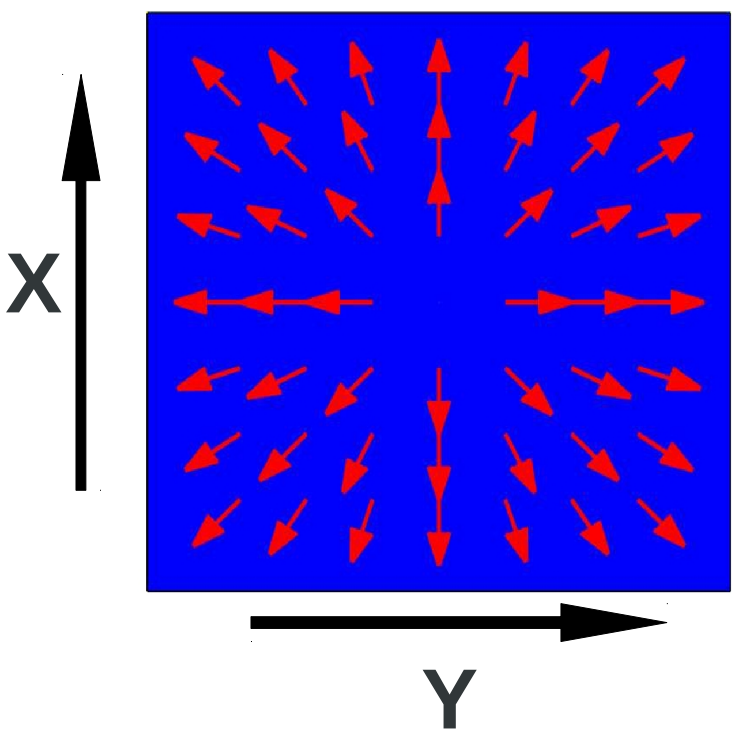


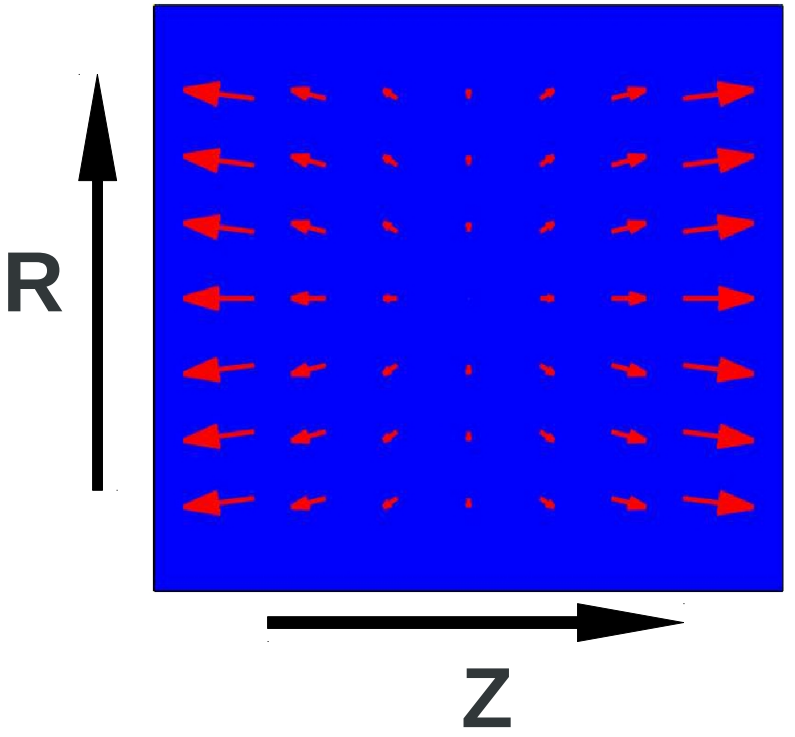
The number of hadrons leaving the plasma with momentum, p^{μ} is the flux of the hadron current through the hadronisation hyper-surface.

$$E\frac{dN}{d^3p} = \int d\Sigma^{\mu} J_{\mu}$$ **with hadron current** $$J^{\mu} = \frac{p^{\mu}}{p^0} F_h(E_h)$$

Cooper-Frye Formula

4) Flow Profile of the Expanding Plasma





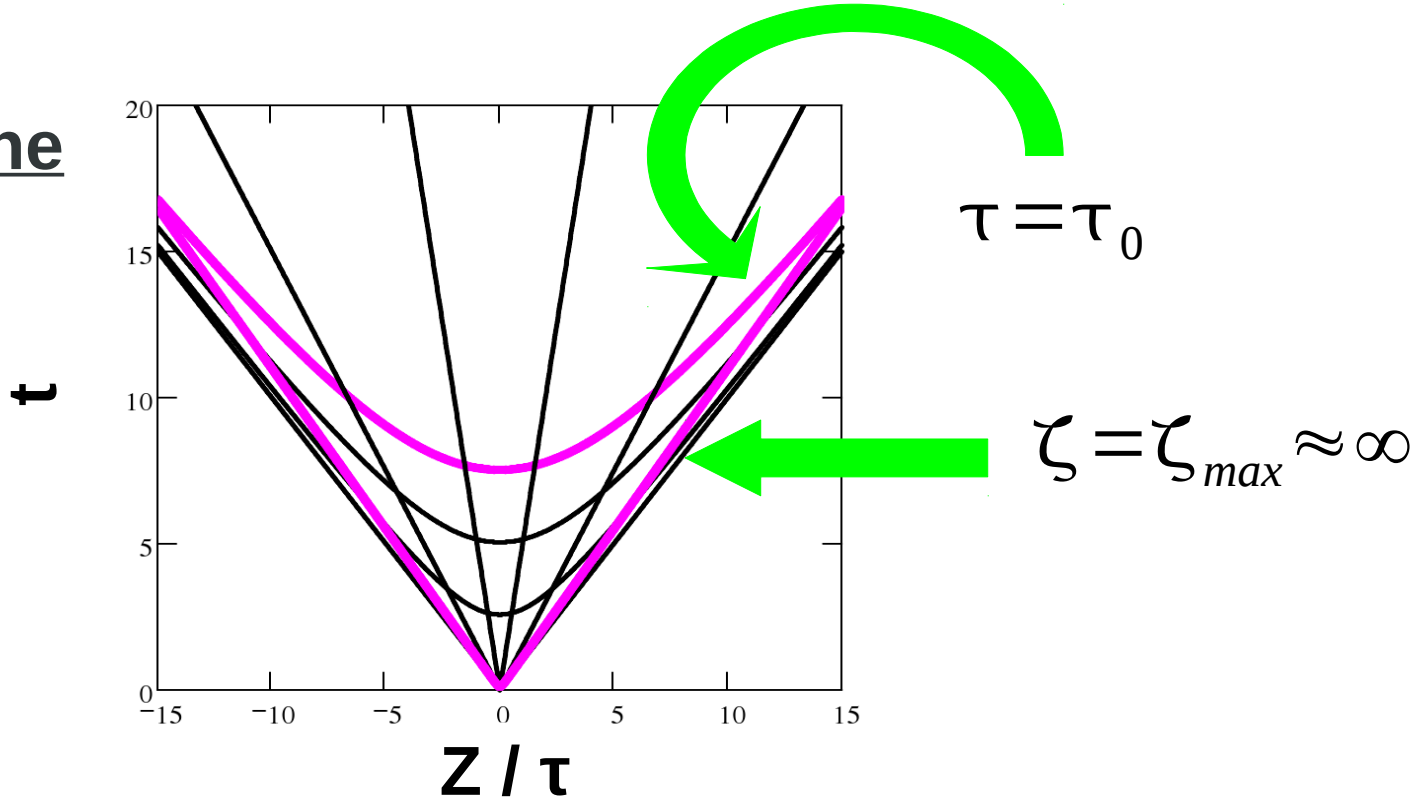
Spacetime coordinates: $r,\ \alpha,\ \zeta=\frac{1}{2}\ln\left(\frac{t-z}{t+z}\right),\ \tau=\sqrt{t^2-z^2}$

Flow profile:

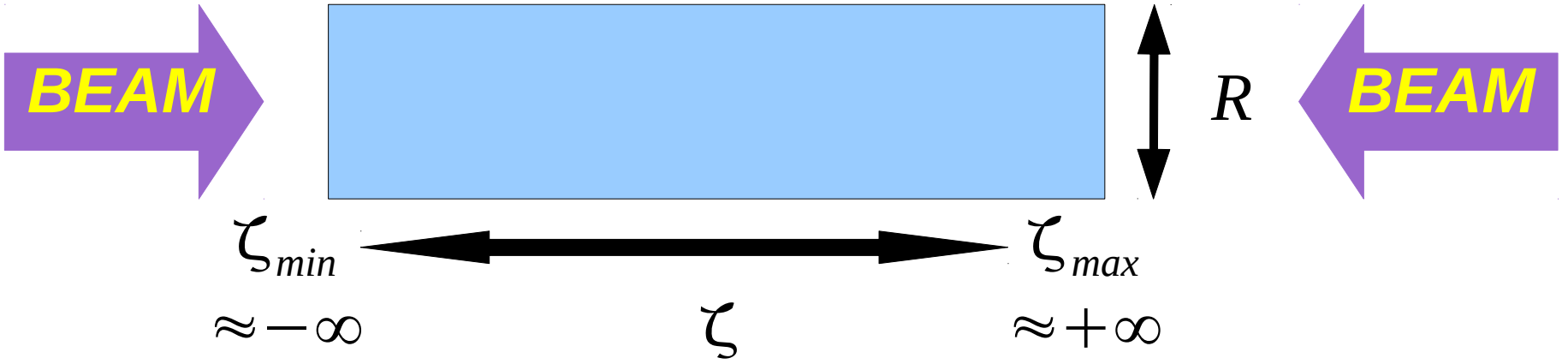
$$u^{\mu}(\zeta,\alpha)=(\gamma\cosh\zeta,\gamma\sinh\zeta,\gamma v\cos\alpha,\gamma v\sin\alpha),\quad \gamma=\frac{1}{\sqrt{1-v^2}}$$

4) The Hadronisation hyper-surface

In the t-z plane



In the r-ζ plane



4) Used thermal hadron distributions:

1. $F_h(E) = A\, e^{-\beta E}$ ***Boltzmann-Gibbs***

2. $F_h(E) = A\,(1+(q-1)E/T)^{-1/(q-1)}$ ***Cut power-law***

with *E*, the hadron energy in the frame co-moving with the plasma flow,

$$E = p_\mu u^\mu$$

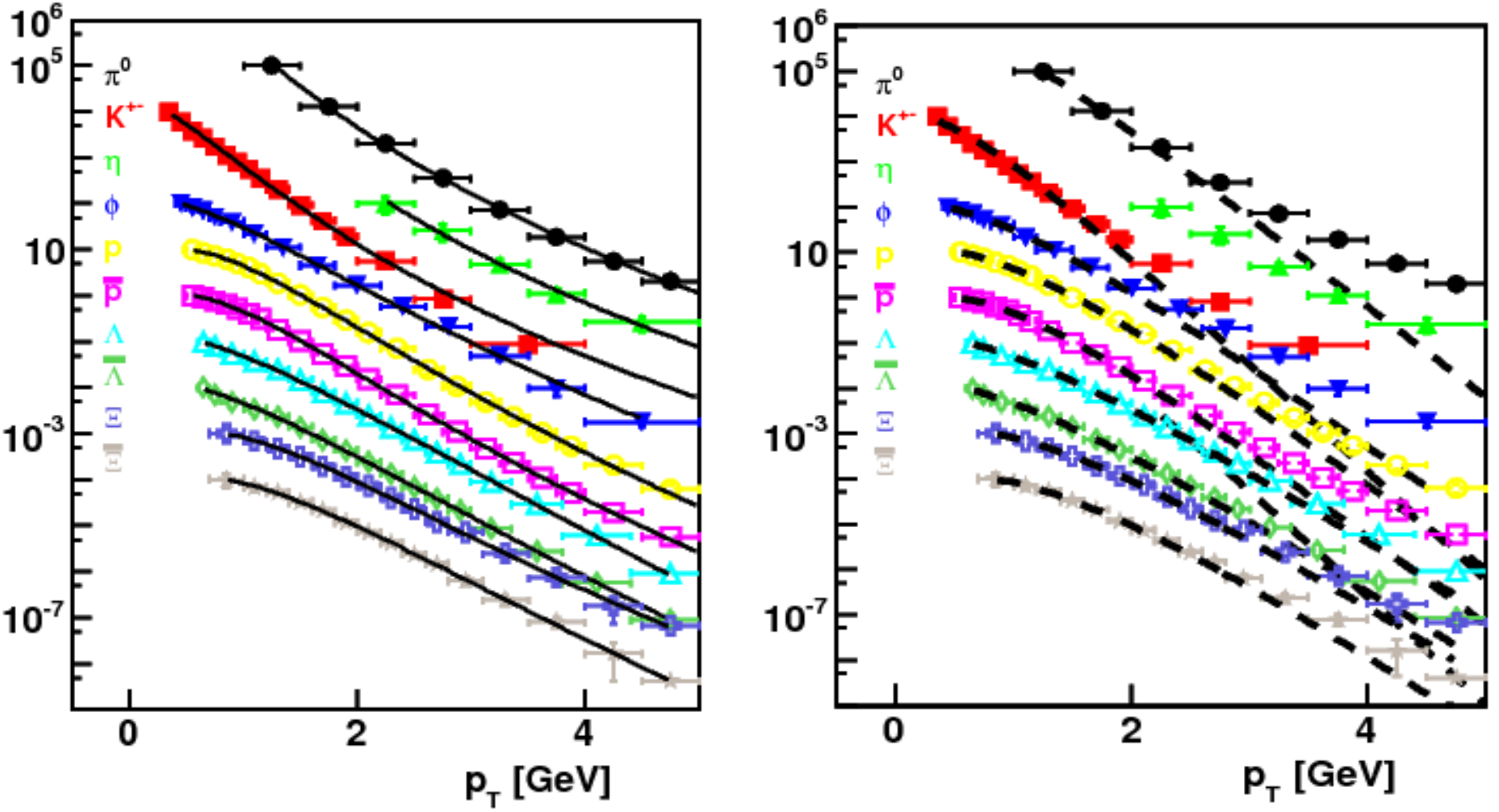
The resulting transverse hadron yielsd are:

1. $$p^0 \frac{dN}{d^3 p} \sim \xi m_T K_1(\beta\gamma m_T) I_0(\beta\gamma v p_T) + (1-\xi) p_T K_0(\beta\gamma m_T) I_1(\beta\gamma v p_T) \sim e^{-\beta\gamma(m_T - v p_T)}$$

2. $$p^0 \frac{dN}{d^3 p} \sim \frac{\xi m_T G_0(p_T) + (1-\xi) p_T G_2(p_T)}{(1+(q-1)\beta\gamma(m_T - v p_T))^{1/(q-1)}} \sim p_T^{-q/(q-1)}$$

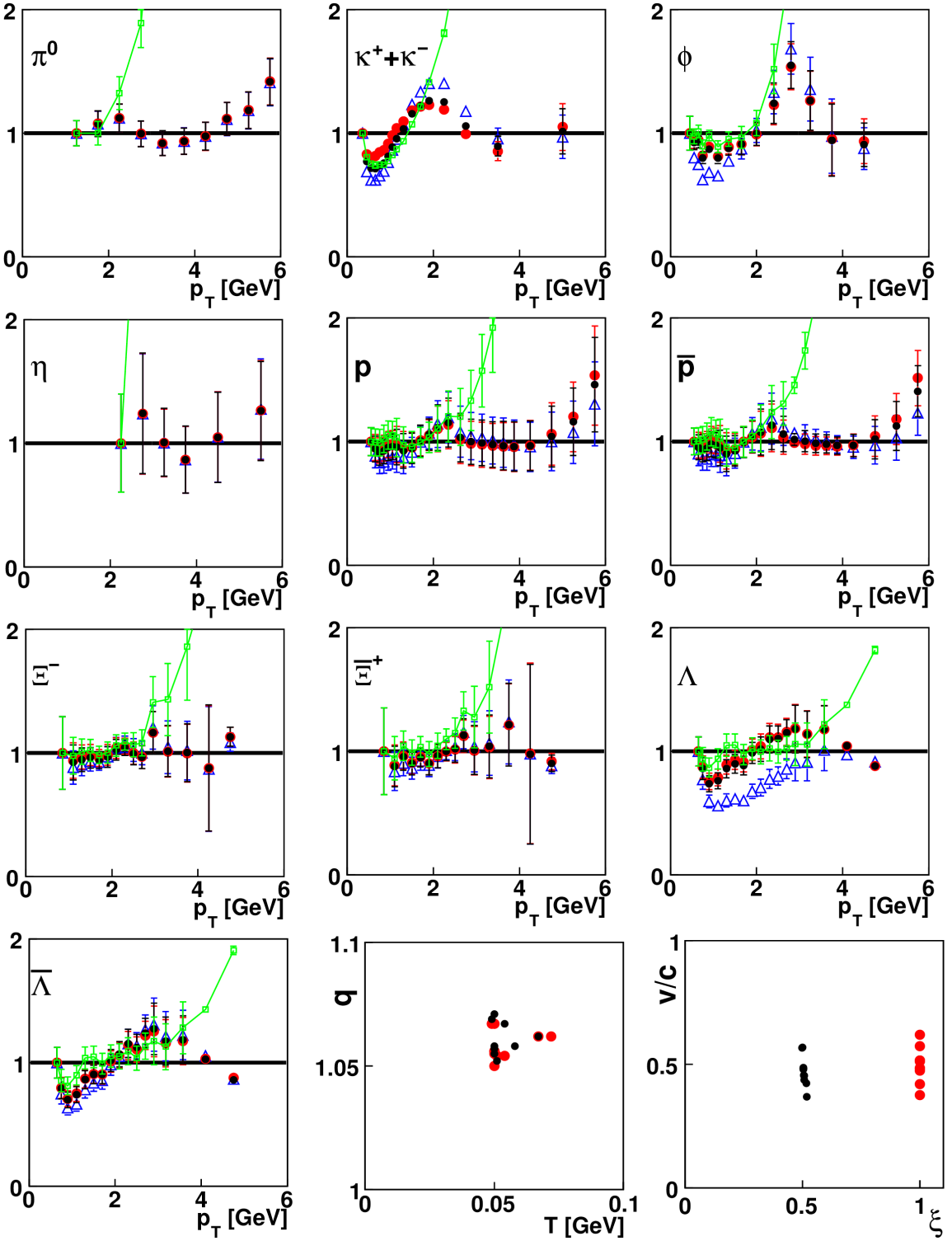
4)

Cut power-law **and** ***exponential fits*** **to identified particle spectra, produced in** ***AuAu --> h+X*** **at** ***√s = 200 GeV*** **.**



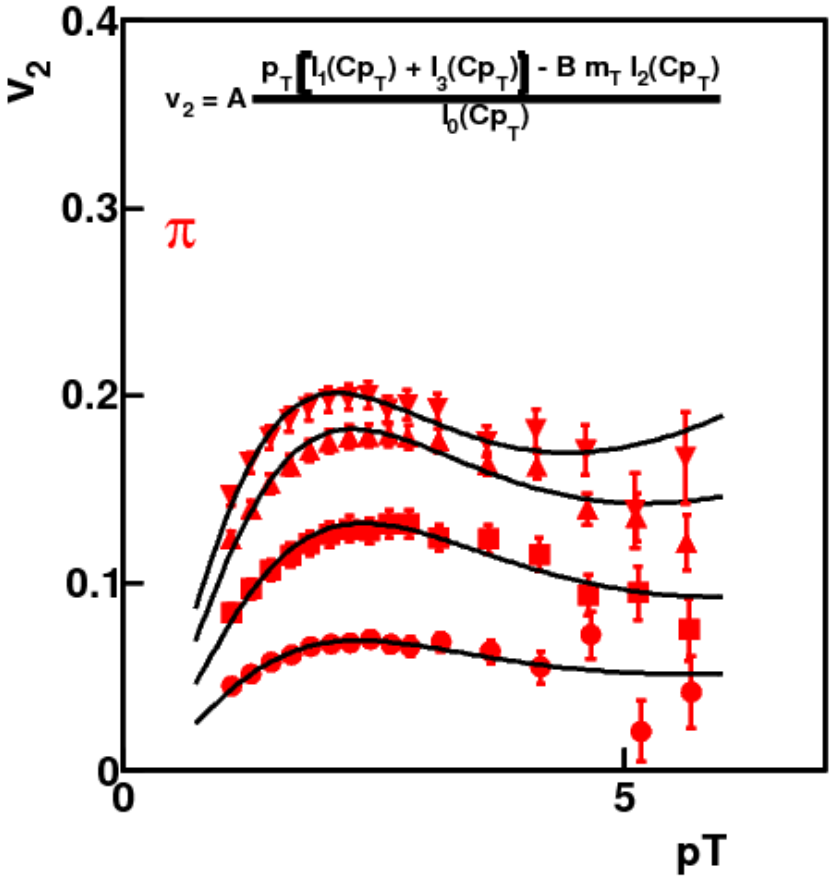
Data / Theory plots for
Identified particle spectra
In *Au + Au - > h + X*
Reactions at
$s = (200\ GeV)^2$
Collision energy

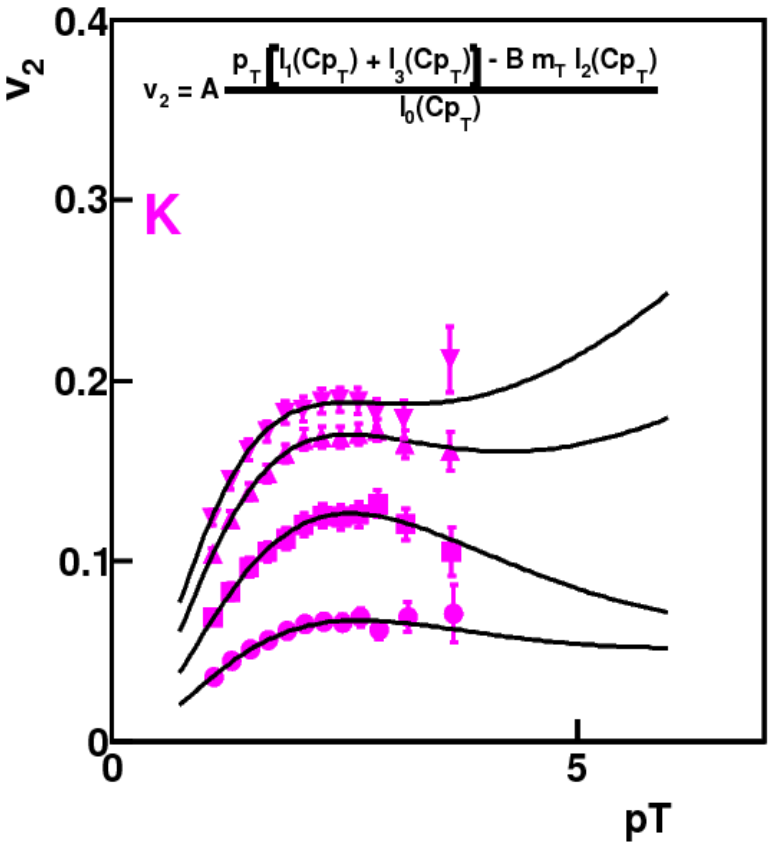
- ■ Boltzmann–Gibbs
- ▲ power-law,
  surface terms only,
  $\xi = 0$.
- ● power-law,
  volume terms only,
  $\xi = 1$.
- ● power-law,
  ξ = surface/volume
  ratio is fitted.
  ($\xi \approx 0.5$)



K. Urmossy, T. S. Biro, *Phys. Lett. B*, **689**: 14-17, 2010.

4) V2 in Au Au - > h X at s = (200 GeV)^2





4)

Non-Extensive Thermodynamics

4) How to Obtain the Tsallis Distribution (a)

Generalise the free-energy functional in the Maximum Entropy variational ansatz

$$S[f(\epsilon)] - \beta \int \epsilon f(\epsilon) \;=\; max$$

Deform the entropy*, *S[f] **(F. Caruso and C. Tsallis., Phys Rev E 78, 021101 (2008); C. Tsallis, Eur. Phys. J. A, 40, 257-266 (2009))**

In a *linear spin-chain*, because of *entenglement*, the entropy of N adjacent spins becomes proportional to N (in the limit of N-->∞) if a new entropy functional is introduced:

$$S_{BG} = -\int f \ln f \quad \rightarrow \quad S_{TS} = -\int f \ln_q f$$

$$\ln_q f = \frac{f^{1-q} - 1}{q - 1}$$

The sollution of the variational problem is the Tsallis distribution:

$$f(\epsilon) = A\left(1 + (q-1)\epsilon / T\right)^{-1/(q-1)}$$

4) How to Obtain the Tsallis Distribution (b)

Introduce *special N-body interactions* of the type

$$E = E_1 + E_2 + \ldots\ a\,(E_1\,E_2 + E_1\,E_3 + \ldots) + \ldots + a^{N-1}\,E_1 * \ldots * E_N$$

which is equivalent to

$$L(E) = L(E_1) + L(E_2) + \ldots + L(E_N) \quad \text{with} \quad L(E) = (1/a)\,\ln(1 + a\,E)$$

Now the variational problem becomes

$$-\int f \ln f - \beta \int L[\epsilon] f \;=\; max$$

with the Tsallis distribution as the sollution:

$$f(\epsilon) = A\left(1 + (q-1)\epsilon / T\right)^{-1/(q-1)},$$

- K. Urmossy et. al., *EPJ Web of Conferences,* **13**, 05003 (2011)
- T. S. Biro et. al., *J. Phys. G,* **36** 064044 (2009)
- T. S. Biro et. al., *Eur. Phys. J. A* **40** 325-340 (2009)

4) *What is T?* *Equilibration of 2 Systems*

The sum of the energies *E_1* and *E_2* of 2 non-interacting sub-systems is the total energy *E* of the whole system

$$E = E_1 + E_2$$

In *equilibrium* the total energy is fixed, while the total entropy is maximal:

$$S(E) = S_1(E_1) + S_2(E_2) = max$$

$$E = E_1 + E_2 = fix$$

Separating the variables, both sides are equal to the inverse temperature:

$$\frac{\partial S_1(E_1)}{\partial E_1} = \frac{1}{T} = \frac{\partial S_2(E_2)}{\partial E_2}$$

4) *What is T?* *Equilibration of 2 Systems*

Suppose, that because of special interactions, the energy of 2 sub-systems, E_1 and E_2 and the total energy E are related as

$$L(E)=L(E_1)+L(E_2), \qquad L(E)=\frac{1}{a}\ln(1+aE)$$

In *equilibrium* the total energy is fixed, while the total entropy is maximal:

$$S(E)=S_1(E_1)+S_2(E_2) \;=\; max$$

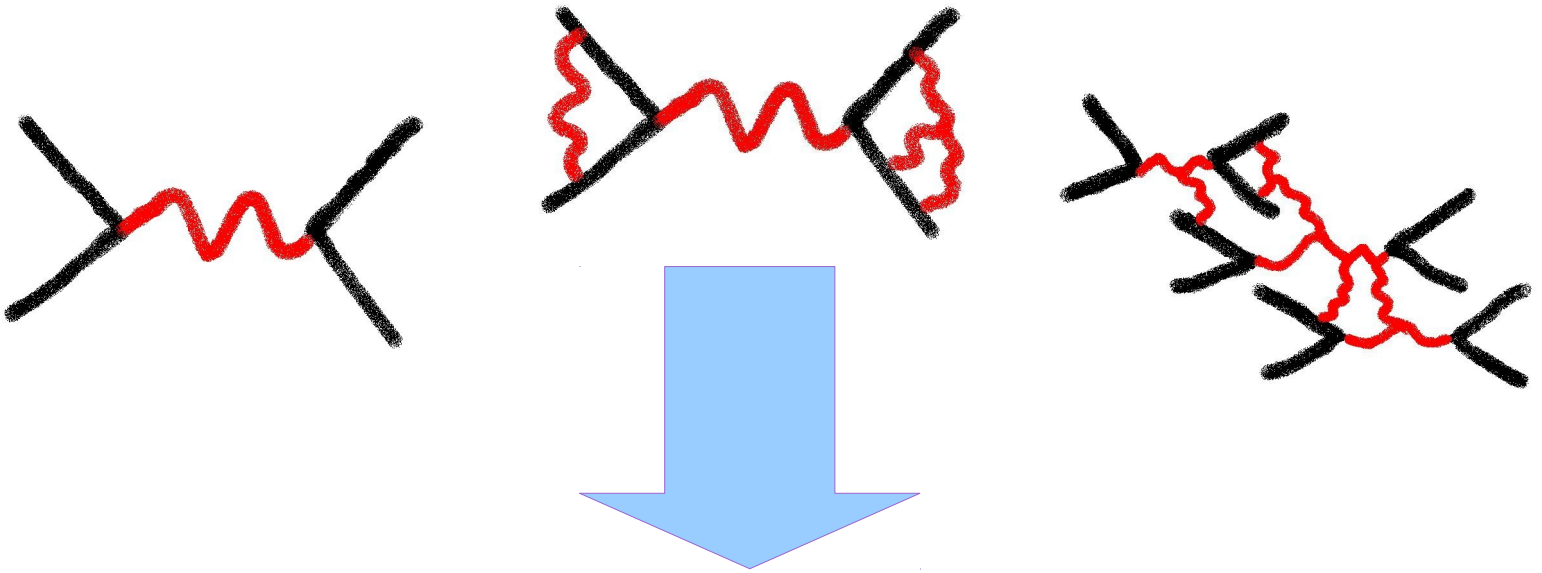
$$L(E)=L(E_1)+L(E_2) \;=\; fix$$

Separating the variables, both sides are equal to the inverse temperature:

$$\frac{\partial S_1(E_1)}{\partial L(E_1)} \;=\; \frac{1}{T} \;=\; \frac{\partial S_2(E_2)}{\partial L(E_2)}$$

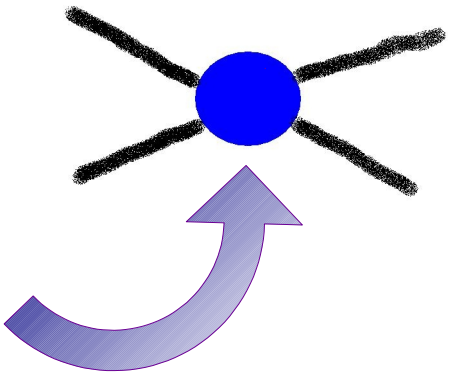
T.S. Biró, P. Ván, *Physical Review E*, **83**, 061187 (2011)

4) Equilibration of Such a System



We try out a simple rule

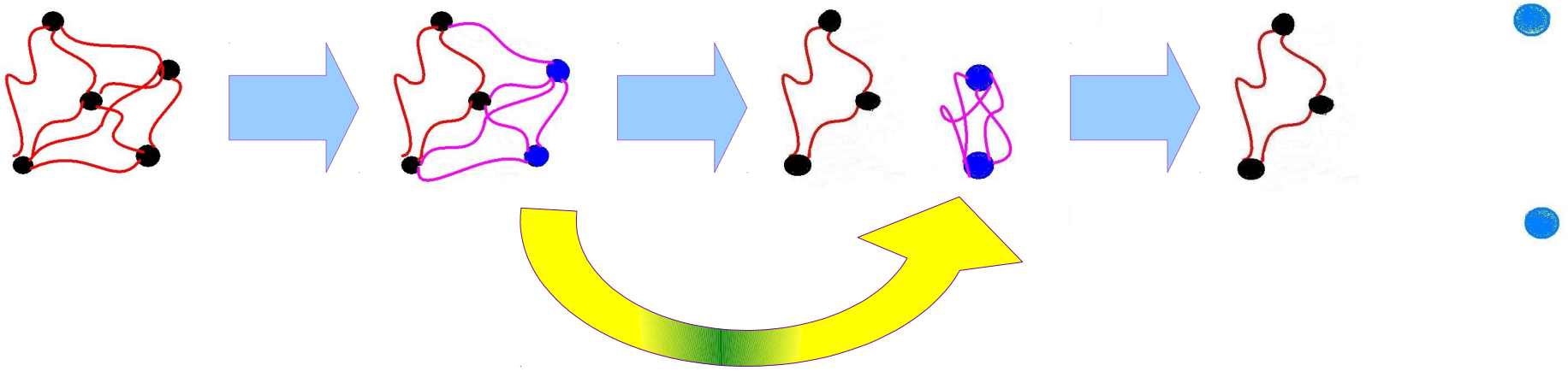
$$\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times\delta(E_1\oplus E_2-E_3\oplus E_4)$$



With

$$E_1\oplus E_2=$$
$$=E_1+E_2+aE_1E_2$$

4) Hadronisation is modelled by a $q\bar{q} \rightarrow Resonance \rightarrow \pi^+\pi^-$



$$\vec{P}_R = \vec{p}_1 + \vec{p}_2 \qquad M_R^2 = E_R^2 - \vec{P}_R^2$$

Probability of color neutrality ~1/9

$$E_R = E_N + E_{N-2}$$

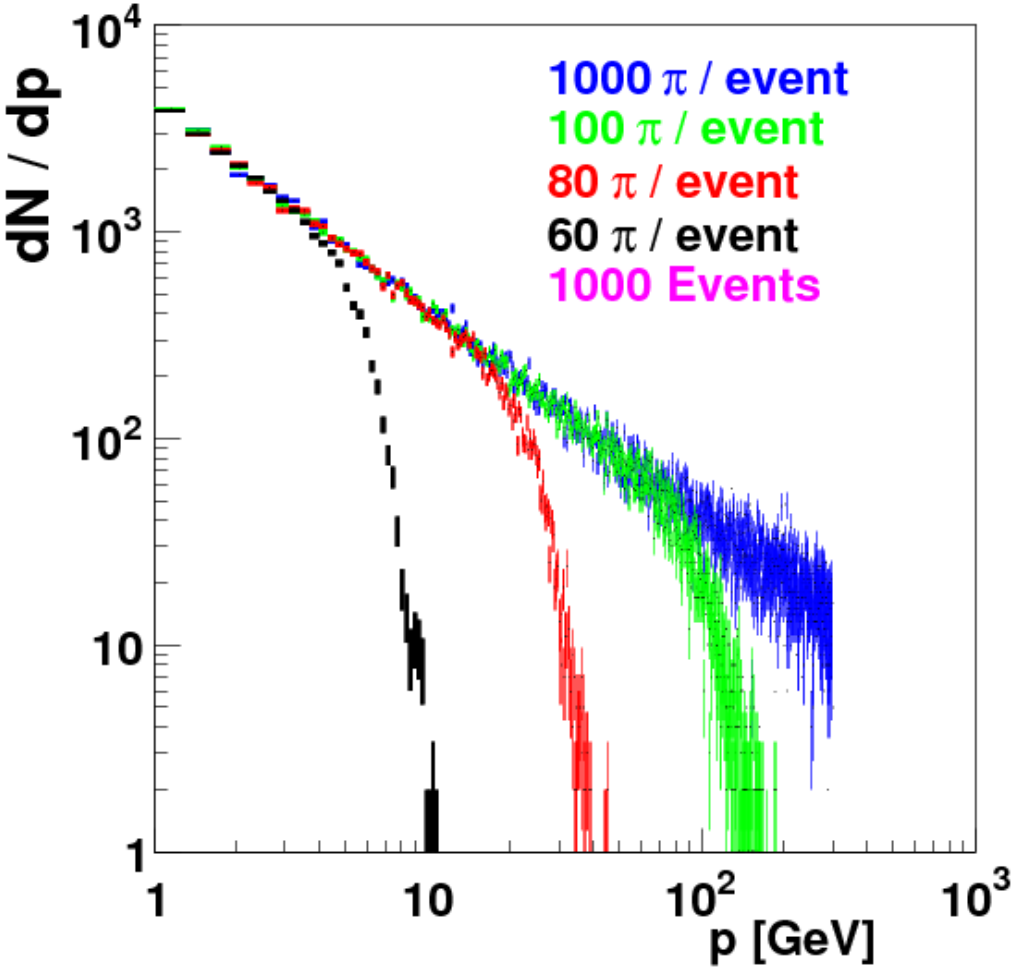
Large interaction energy put into the mass of the Resonance

$$E_R \sim (1 + a\bar{\epsilon})^{N-2}(2\bar{\epsilon} + a\bar{\epsilon}^2)$$

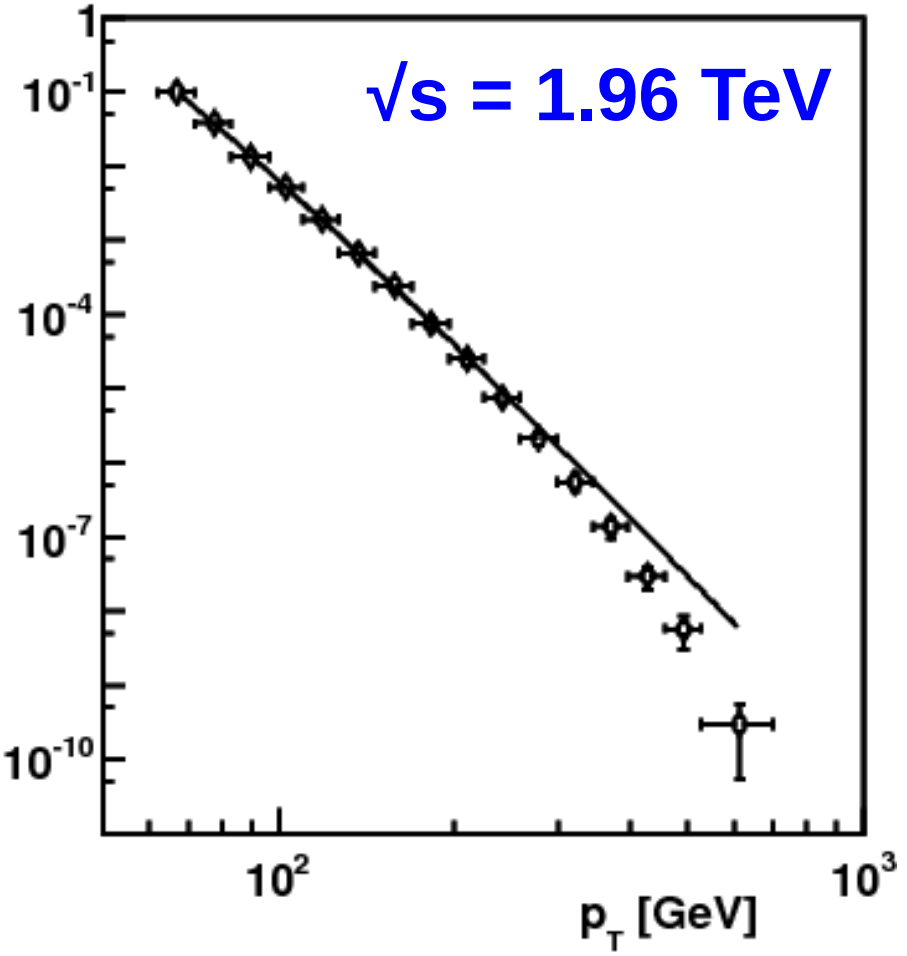
4) Finite *N*, *E* – > cut in the spectrum at high E

Interaction measure *a = 1*

Pions from parton cascade

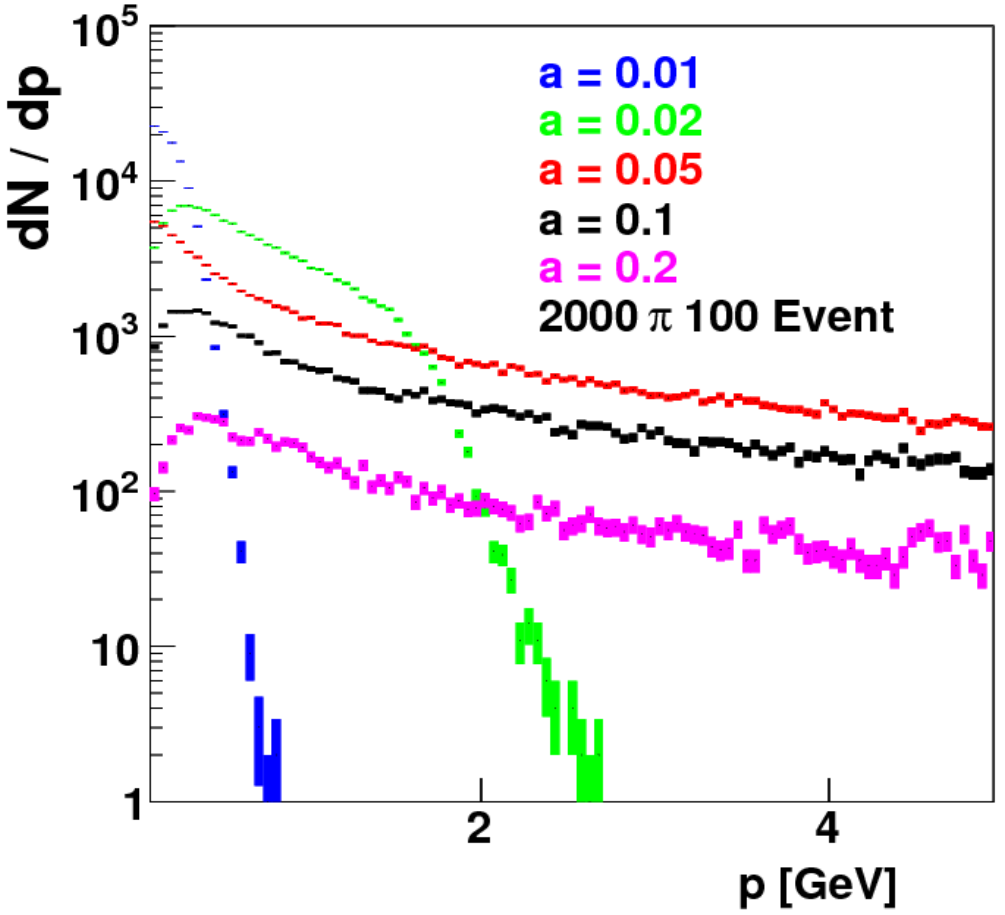


Jets at *CDF@Tevatron*

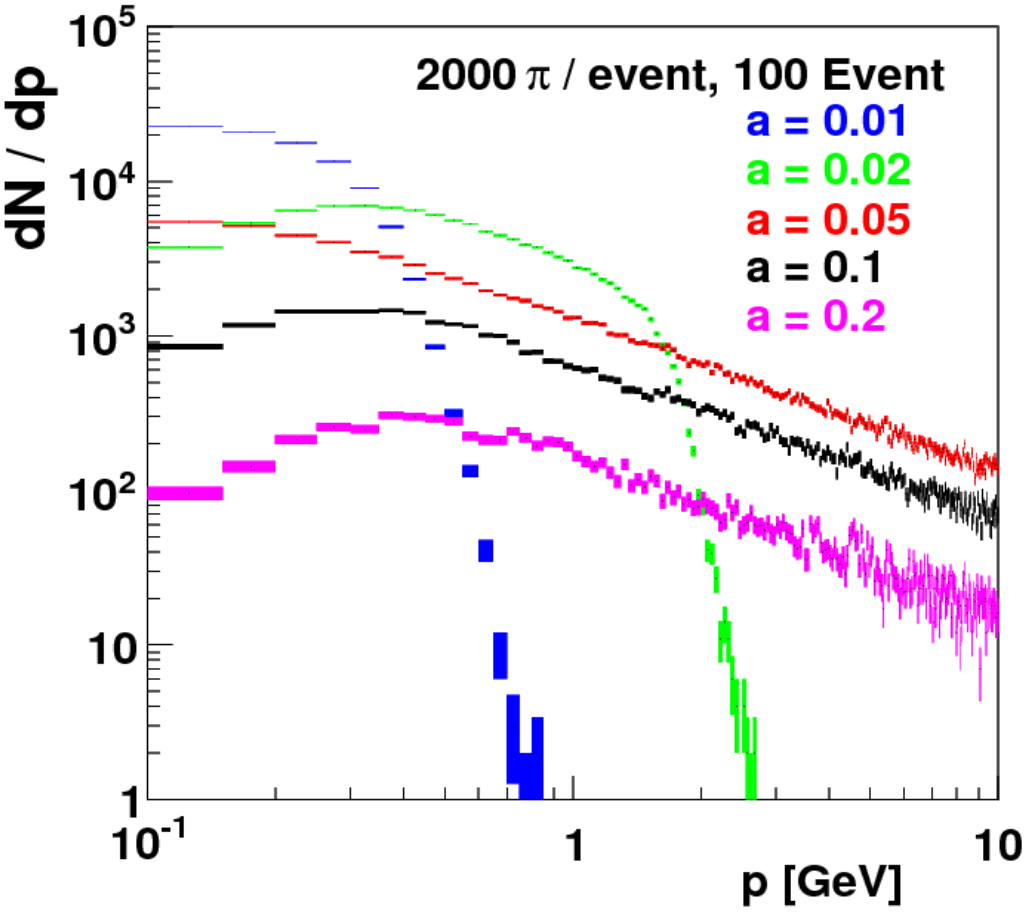


4) The Spectrum changes from Boltzmann to Tsallis as interactions (*a*) grow

Log – Lin plot



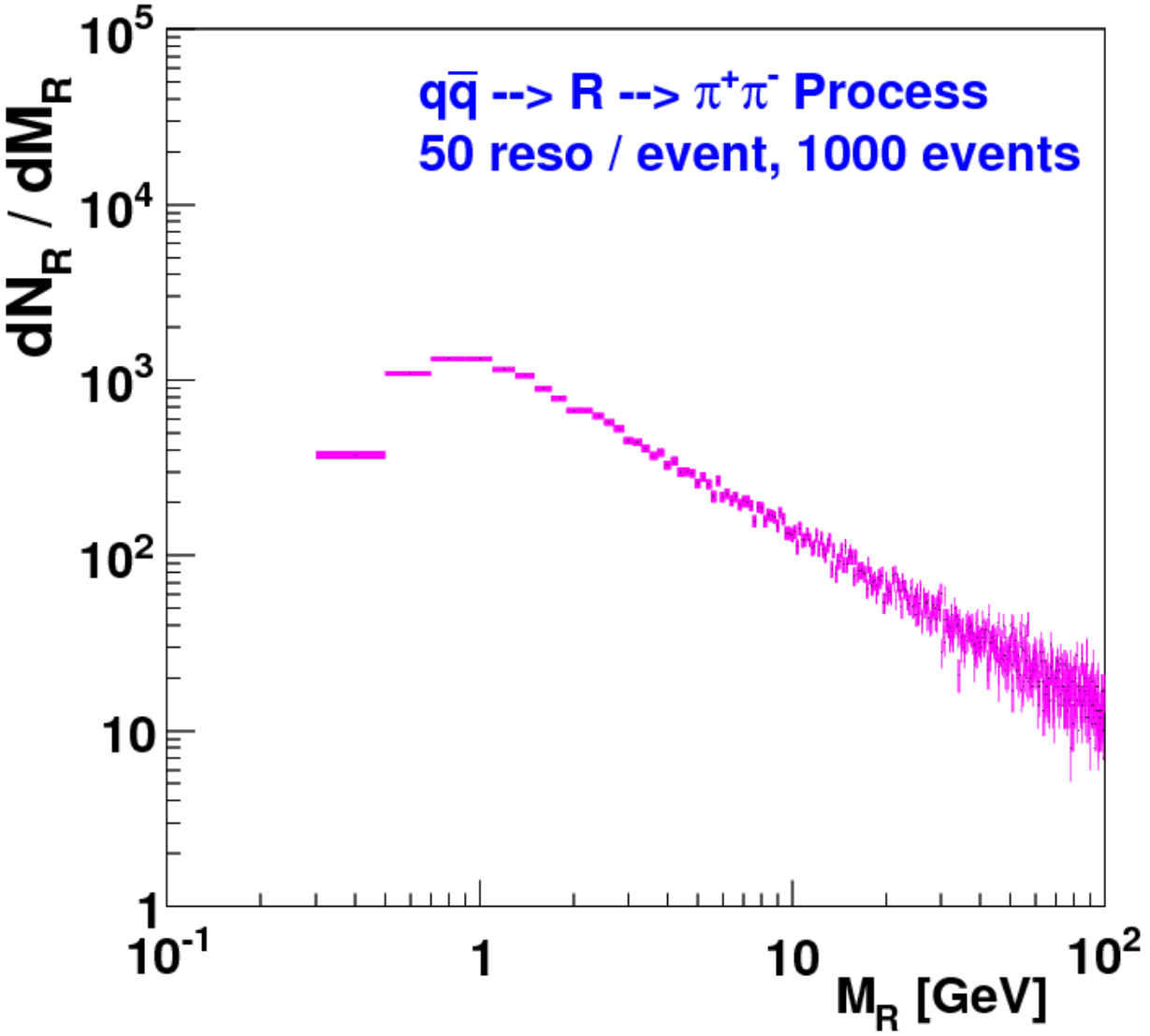
Log – Log plot



4)

Mass Distribution of Resonances

With interaction measure *a = 1*



as usual

$$M_R^2 = E_R^2 - \vec{P}^2$$

$$E_N = E_{N-2} + E_R$$

but

$$E_N = E_1 \oplus E_2 \oplus \ldots \oplus E_N$$

4) *Entropy and Energy Functionals with the Corresponding Equilibrium Distributions*

$$S[f(\epsilon)] - \beta\, C[f(\epsilon)] \;=\; max$$

$S[f]$	$C[f]$	$f_{eq}(\epsilon)$
$-\int f \ln f$	$\int \epsilon f$	$A \exp\{-\beta\,\epsilon\}$
$-\int f \ln f$	$\int L(\epsilon) f$	$A \exp\{-\beta L(\epsilon)\}$
$\int f \ln_q f$	$\int \epsilon f$	$A\left[1+(q-1)\beta\,\epsilon\right]^{-1/(q-1)}$
$\int f \ln_q f$	$\int L(\epsilon) f$	$A\left[1+(q-1)\beta L(\epsilon)\right]^{-1/(q-1)}$

T. S. Biro et. al.,
Eur. Phys. J. A **40** 325-340 (2009)

C. Tsallis

T.S. Biró et. al.,
Phys. Rev. E, 83, 061187, (2011)

Deformed logarithm (C. Tsallis, Eur. Phys. J. A, 40, 257-266 (2009) ***):***

$$\ln_q(z) = \frac{z^{1-q}-1}{q-1}$$

Take Home Message:

I. ***May Be***

Tsallis Statistics	~	***Conventional Statistics + Fluctuations***

II. ***Multiplicity fluctuations hide event-by-event physics from our eyes.***

NEW! ***Hadron Spectra is Measured in***

Fix Multiplicity pp, pA Events!

F. Sikler's talk,

Phys. J. C (2012), arXiv:1207.4724

Back-up Slides.....

How to Obtain the Tsallis Distribution (c)

SuperStatistics **(*C. Beck, G. Wilk,* *Eur. Phys. J. A*, 40, 267 and 299-312, (2009)*):***

If the hadron distribution is *Boltzmann-Gibbs*,

$$f(\epsilon) = A \exp(-\beta \epsilon)$$

but the *temperature fluctuates* *event-by-event* or *position-to-position* as

$$p(\beta) \propto \beta^{\alpha-1} \exp(-\alpha \beta / \langle \beta \rangle)$$

the *average distribution* becomes the *Tsallis* distribution:

$$\frac{dN}{d^3 p} = \int d\beta \, p(\beta) f_\beta(\epsilon) \propto \left(1 + \frac{\langle \beta \rangle \epsilon}{\alpha}\right)^{-(\alpha + D + 1)}$$

How to Obtain the Tsallis Distribution (c)

Or similarly,

If the hadron distribution is *Boltzmann-Gibbs*,

$$f(\epsilon) = A \exp(-\beta \epsilon), \qquad E/n \;=\; DT$$

but the *multiplicity fluctuates* *event-by-event while E = constant*

$$p(n) \;\propto\; n^{\alpha-1} \exp(-\alpha n/\langle n \rangle)$$

the *average distribution* becomes the *Tsallis* distribution:

$$\frac{dN}{d^3 p} = \int dn\, p(n) f_n(\epsilon) \;\propto\; \left(1 + \frac{D\langle n \rangle}{\alpha E}\epsilon\right)^{-(\alpha + D + 1)}$$

How to Obtain the Tsallis Distribution (c)

Moreover,

If the hadron distribution is *Boltzmann-Gibbs*,

$$f(\epsilon) = A \exp(-\beta \epsilon), \qquad E/n \;=\; DT$$

but the *total transverse energy fluctuates* *event-by-event while n = fix*

$$p(E) \;\propto\; E^{-\alpha-2} \exp(-\alpha \langle E \rangle / E)$$

the *average distribution* becomes the *Tsallis* distribution:

$$\frac{dN}{d^3 p} = \int dE\, p(E) f_E(\epsilon) \;\propto\; \left(1 + \frac{D n}{\alpha \langle E \rangle} \epsilon\right)^{-(\alpha + D + 1)}$$

Now What Is the T Parameter?

Alas, from measurements, we do not see the *interactions* and *internal fluctuations* inside the quark-matter, however, we measure the *mean energy per particle* (for $\epsilon(p) = p$ disp. rel.**):**

$$\frac{E}{N} = \frac{\int \epsilon f_{TS}(\epsilon)}{\int f_{TS}(\epsilon)} = \frac{DT}{1-(q-1)(D+1)}$$

Thus, the fitted *Tsallis T is much smaller then the* fitted *Boltzmann T*!

Fluctuations of the total transverse energy can describe pp data

If the distribution of the total transverse energy is

$$p(E) \propto E^{-\alpha-2}\exp(-\alpha\langle E\rangle/E)$$

where the *mean energy* and the *width* of the distribution *varies with n* as

$$\alpha = \frac{1}{\mu \ln\ln(N/N_q)} - (D+1)$$

$$\langle E\rangle = \frac{D\,T_0(1+N/N_T)}{1-(D+1)\mu \ln\ln(N/N_q)}$$

This prediction could be tested experimentally ...

Experimentally multiplicity averaged spectra are measured

Miltiplicity distributions show KNO-scaling (Koba-Nielsen-Olesen)

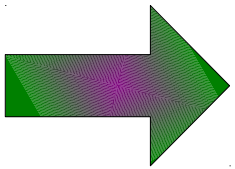
$$p(N)=\frac{1}{\langle N(s)\rangle}\Psi\left(\frac{N-N_0}{\langle N(s)\rangle}\right)$$

- A. Rényi, Foundations of Probability, Holden-Day (1970).
- A. M. Polyakov, Zh. Eksp. Teor. Fiz. 59, 542 (1970).
- Z. Koba, H. B. Nielsen, P. Olesen, Nucl. Phys. B 40, 317 (1972).
- S. Hegyi, Phys. Lett. B: 467, 126-131, 1999.
- S. Hegyi, Proc. ISMD 2000, Tihany, Lake Balaton, Hungary, 2000
- Yu.L. Dokshitzer, Phys. Lett. B, 305, 295 (1993); LU-TP/93-3 (1993).

A kírérletekkel konzisztens konkrét függvényalak:

$$p(N) \propto (N-N_0)^{\alpha-1} e^{-\beta(N-N_0)}$$

Amiből az átlag hadron eloszlás:

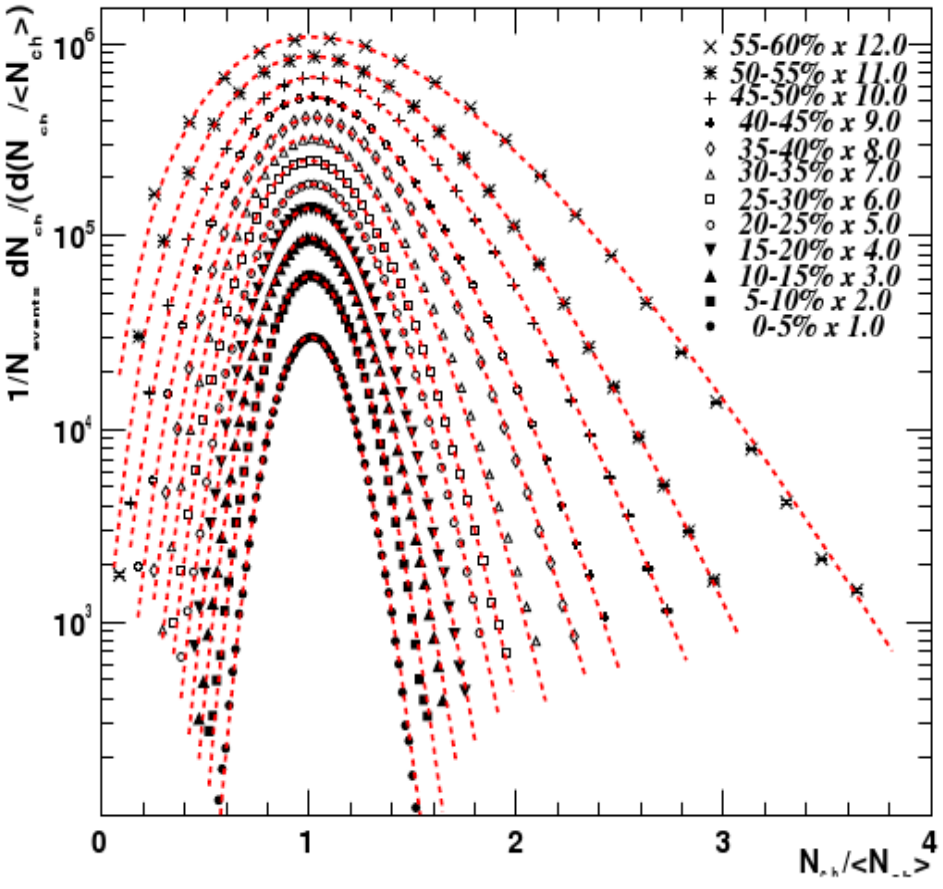


$$\frac{d\sigma}{d^D x}=\sum f_N(x)\, N\, p(N) \propto \frac{(1-x)^{D(N_0-1)-1}}{\left(1-a\ln(1-x)\right)^b}$$

Urmossy et. al., Phys. Lett. B, 701: 111-116 (2011), arXiv:1101.3023

3) Multiplicity Fluctuations

Multiplicity distributions in *AuAu* collisions @200 AGeV



PHENIX Collab., arXiv:0805.1521v1

p(N) = NBD