Summary:

Density matrix $\rho$ describes wedge w. smooth horizon

Pure state: $|Ex\rangle$: wedge w. singular boundary

Entangled state (purification of $\rho$):
- connected spacetime
- details of spacetime past wedge depend on details at purification.
\[ \sum e^{-E_i} |E_i\rangle \otimes |E_i\rangle_R \]

Product state: 2 disconnected wedges

\[ \sum e^{-E_i} \]

Quantum superposition of disconnected spacetimes = connected

2 wedges connected by entanglement
Another example: 2 CFTs

- States $|E'_1\rangle \otimes |E^2_2\rangle$ describe 2 separate asymptotically AdS spacetimes.

- Witten: thermal state $\Sigma e^{-\beta E_i_1} |E_i_1 \times E_i_1\rangle$ describes black hole.

Maldacena: maximally extended BH w. 2 asymptotic regions described by:

$\Sigma e^{-\beta E_i/2} |E_i\rangle \otimes |E_i\rangle$
Can connect 2 asymptotically AdS spacetimes by entangling 2 CFTs:

\[ = \sum e^{-\beta E_i/2} \]

analogy w. single CFT case:

- Suggests \( \rho_L, \rho_R \) describe only region outside horizon.

- Black hole microstates = typical states in
ensemble \( p \). Also no physics behind horizon??
So FAR: related entanglement structure of vacuum to geometrical structure of AdS

NEXT: use constraints on how entanglement can vary to derive constraints on metric perturbations.
- derive “1st law” for entanglement entropy
- interpret for holographic CFTs

⇒ metric perturbations around pure AdS must satisfy Einstein’s Equations.

Lashkari, McDermott, MVR
Faulkner, Guica, Hartman, Myers, MVR
Start w. general QM. system:

Suppose: \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_A \)

Recall: \( S_A = -\text{tr}(\rho_A \log \rho_A) \)

Then for \( |\psi\rangle \to |\psi\rangle + \delta |\psi\rangle \)

Have: \( \delta S_A = -\text{tr}(\delta \rho_A \log \rho_A) = \delta \langle H_A \rangle \)

where \( H_A \) defined by \( \rho_A = e^{-H_A} \)

Blanco, Casini, Hung, Myers
\[ \delta S_A = S \langle H_A \rangle \]

If \( \rho_A \) is THERMAL \( \rho_A = e^{-\beta H} \)

get \[ \delta S_A = \frac{1}{T} S \langle E \rangle \]

First Law of THERMODYNAMICS
(exact quantum version)
Specialize to $\text{CFT}_{d+1}$ on $\mathbb{R}^{d+1}$, $|\psi\rangle = |0\rangle$, $A =$ ball

\[ \rho = \frac{1}{Z} e^{-2\pi H_\xi} \]  
modular Hamiltonian

$= \text{generator of evolution along } \xi$
RESULT: For perturbations about the vacuum state of a CFT, the entanglement entropy for any ball $B$ on any spatial slice satisfies:

$$\delta S_s_B = \delta\left\langle H_{B}^{\text{vac}} \right\rangle = \delta\left\langle 2\pi \int_{B} d^4 x \frac{R^2 - r^2}{2R} T_{\mu\nu} \right\rangle$$
Specialize to HOLOGRAPHIC CFT

vacuum: $|0\rangle \rightarrow$ pure AdS

small perturbation: $|\psi\rangle \rightarrow M_\psi$

$$ds^2 = \frac{l^2}{z^2} (dz^2 + dx^m dx^m) + z^{d+1} 16 \pi d x^m dx^n$$

(for some $\psi$s)

WHAT DOES $\delta S_B = SE_B$ TELL US ABOUT $M_\psi$?
Holographic Interpretation of $S_B$ and $E_B$

$S_B \rightarrow$ extremal value of some functional on codimension 2 surfaces with boundary $\partial B$

(assume)

e.g. $S_B = \frac{\text{Area}(B)}{4\pi \text{GN}}$ ~ Ryu-Takayanagi

$E_B \rightarrow 2\pi \int_B d^d x \frac{R^2 - r^2}{2R} + \left. \text{grav} \right|_{\infty} \leq \text{Holographic stress tensor}

*can derive given entropy functional

e.g. $t_{\mu\nu} = \frac{d+1}{16\pi \text{GN}} H_{\mu\nu}(z=0)$
Using holographic dictionary,

\[ \delta S_B = \delta S_{E_B} \]

\[ \int_B (\text{energy integrand}) = \int_{\tilde{B}} (\text{entropy integrand}) \]

Gives nonlocal constraint on \( M_\Psi \) (one for each bulk point)
Turn this into a local constraint:

We can find a \( d \)-form \( \chi_B(H) \) such that:

\[
\int_{B} \chi_B = \delta E_B^{\text{grav}} \quad \int_{\tilde{B}} \chi_B = \delta S_B^{\text{grav}}
\]

\[
d\chi_B = \zeta^t \cdot S E_{tt} \cdot V^\Sigma
\]

Then:

\[
\delta S_B^{\text{grav}} - \delta E_B^{\text{grav}} = \int_{\Sigma_B} \chi_B = \int_{\Sigma_B} d\chi_B = \int_{\Sigma_B} \zeta^t \cdot S E_{tt} \cdot V^\Sigma
\]
\[ \delta S_{B}^{\text{grav}} - \delta E_{B}^{\text{grav}} = \int_{\Sigma_{B}} \dot{\chi}_{B} \cdot \delta E_{tt} \cdot \nu_{z} \]

L.S. vanishes for all B

\[ \delta E_{tt} = 0 \] everywhere

Repeats for balls in Lorentz frame specified by velocity vector \( u^{\mu} \):

\[ u^{\mu} u^{\nu} \delta E_{\mu\nu} = 0 \]

\[ \Rightarrow \delta E_{\mu\nu} = 0 \]

LINEARIZED GRAVITATIONAL EQUATIONS!
Example: \( S_{\text{grav}}^B = \frac{\text{Area}(\bar{B})}{4\pi G} \)

\[
\delta E_{\text{grav}} = \frac{3}{2} \int_B d\mathbf{x} \left( R^2 - x^2 \right) H_{ii}
\]

\[
\delta S_{\text{grav}} = \int_{\tilde{B}} d\mathbf{x} \left( R^2 H_{ii} - x^i x^j H_{ij} \right)
\]

\[
\chi = \mathbf{v}_\Sigma \left( \xi^t \partial_2 h_{ii} + (2z^{-1} \xi^t + z) \cdot h_{ii} \right) + x^i \cdot \mathbf{v}_\Sigma \wedge d\mathbf{z} \left( \xi^t (\partial_i h_{ij} - \partial_j h_{ii}) + x^i h_{ij} - x^j h_{ji} \right)
\]

\[
\xi^t = \frac{1}{2} \left( R^2 - x^2 - z^2 \right) \quad \delta E_{tt}
\]

\[
\mathbf{v}_\Sigma = dx^1 \wedge ... \wedge dx^d
\]

\[
\delta E_{tt} = 2 \partial_2^2 H_{ii} + 4 \partial_2 H_{ii} + 2 \partial_i \partial_j h_{ij} - 2 \partial^2 h_{ii} = 0
\]
Other components:

\[ \delta E_{zz}, \delta E_{zm} \] satisfied everywhere if satisfied at \( z=0 \) and other equations hold.

\[ \delta E_{zz}\big|_{z=0} = \delta E_{zm}\big|_{z=0} \] follow from

\[ \partial_{\mu} t^{\mu} = (t^{\mu})_{\text{grav}} = 0 \]
**Summary:**

\[ SS_B^{\text{CFT}} = SE_B^{\text{CFT}} \]

\[ SS_B^{\text{grav}} = SE_B^{\text{grav}} \]

\[ \delta E_{ab} = 0 \]

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This direction is a special case of Iyer-Wald's first law for black holes:

For a stationary black hole with bifurcate Killing horizon, arbitrary perturbations (satisfying the linearized gravitational e.o.m.) obey

\[ SS_{BH} = SE_{BH} \]

Wald entropy can argue this must agree with \( SS_B \) (though EE not computed by Wald functional in general)

We took \( x = SQ - \xi \cdot \Theta \) from Iyer-Wald

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Interior is a hyperbolic black hole
OPEN QUESTIONS:

- How much do universal entanglement constraints determine dynamics at the nonlinear level?
  
  e.g. $\Delta S \leq \Delta E$

- Is there physics behind the horizon of a B.H. microstate geometry? If so, how is it encoded in CFT?

- Can we use what we've learned to understand generalizations of AdS/CFT to more general (e.g. cosmological) spacetimes?