Supersymmetry in Curved Spacetime

Guido Festuccia

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Why Susy in curved space

Allows to compute exactly many interesting observables such as:

- Partition function Z on a compact manifold $\mathcal{M}$.
- Expectation value of supersymmetric operators.

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For instance:

- The partition function on $S^3 \times S^1$ of $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry. [Romelsberger; Dolan Osborn ...]
- The partition function on S^4 of $\mathcal{N} = 2$ theories. [Pestun;...]
- The partition function on S^3 of $\mathcal{N} = 2$ theories with a $U(1)_R$ symmetry.[Kapustin, Willett, Yaakov;...]

Why Susy in curved space

These results can be extended to less symmetric manifolds:

- $\mathcal{N} = 1, 2, 4$ on $S^3_b \times R$
- $\mathcal{N} = 2$ on S^3_b [Hama, Hosomichi, Lee; Imamura]
- $\mathcal{N} = 2$ on S^4_b [Hama, Hosomichi]

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- $\mathcal{N} = 2$ on S^4_b [Hama, Hosomichi]

Different squashings of S^3 have been considered

- Some preserve an $SU(2) \times U(1)$ isometry others just $U(1) \times U(1)$
- The metric can depend on arbitrary functions or on a finite number of parameters

The partition function on all these backgrounds is a certain function of a single complex parameter b.
[Alday Martelli Richmond Sparks]

Questions

- Which manifolds $\mathcal{M}$ allow for Susy?
- What is the structure of supersymmetric theories on $\mathcal{M}$?
- Dependence of susy observables on the geometry of $\mathcal{M}$.

Outline

A general framework to understand Susy on curved manifolds.

Classification of Susy backgrounds. Survey of results in different number of dimensions.

Dependence of partition functions on geometry

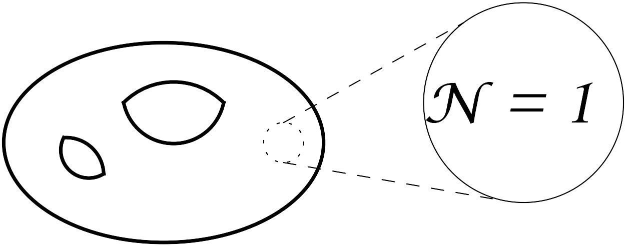
Susy on Curved Manifolds

Consider a supersymmetric theory in flat space.

We want to place it on a manifold $(\mathcal{M}, g)$ so that:

- The short distance limit of the theory is unaffected.
- The theory is invariant under some Supersymmetry.

At short distances the Susy transformations are part of the flat space superalgebra.



The Rigid Limit of SUGRA

Consider an off-shell formulation of Supergravity and give arbitrary background values to the fields in the gravity multiplet:

- The metric $g_{\mu\nu}$
- Various auxiliary fields.
- Set the gravitino $\psi_{\mu\alpha} = 0$

Send $M_p \to \infty$ keeping the background values for the metric and auxiliary fields fixed.

Some supersymmetry is preserved if it is possible to find ζ_α such that the SUSY variation of the gravitino is zero:

$$\delta_\zeta \psi_{\mu\alpha} = 0 \quad \Rightarrow \quad \nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

where $\mathcal{M}_\mu$ depends on the the metric and auxiliary fields.

The Rigid Limit of SUGRA: Comments

- Different backgrounds treated in a unified way.

- Different than Linearized Sugra.

$$\nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

- The "Killing" equation for ζ depends only on the fields in the gravity multiplet

$$\Downarrow$$

Weak dependence on the matter content.

Generalized treatment of different theories.

- We do not impose e.o.m. for the auxiliary fields. Different off shell formulations of SUGRA can lead to distinct deformations.

Example: New Minimal Sugra

In a $\mathcal{N} = 1$ theory with a $U(1)_R$ symmetry consider

- The energy momentum tensor $T_{\mu\nu}$
- The conserved R-current j_μ^R
- The supercurrent $S_{\mu\alpha}$

Together with a string current $C_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{A}^\sigma$ they form the $\mathcal{R}$-multiplet. It couples to the fields in New Minimal Sugra:

- The metric $g_{\mu\nu}$
- The gravitino ψ_α^μ
- An auxiliary $U(1)_R$ connection $A_\mu \sim A_\mu + \partial_\mu a$
- An auxiliary vector V^μ. It is conserved $\nabla_\mu V^\mu = 0$

In the Rigid Limit we set $\psi_\alpha^\mu = 0$ and freeze the metric and auxiliary fields to arbitrary background values.

New Minimal Sugra, the Rigid Limit

Consider a flat space $\mathcal{N} = 1$ theory with an $U(1)_R$ symmetry. Coupling to Sugra and taking the rigid limit we obtain:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

- $\mathcal{L}_0$ is the flat space theory minimally coupled to the metric.
- $\mathcal{L}_1$ are terms of order $\frac{1}{r}$ which couple the auxiliary fields to definite components of the R-multiplet

$$\mathcal{L}_1 = -j_\mu^{(R)} (A^\mu) - \mathcal{A}_\mu V^\mu.$$

  At this order the deformation of the flat space theory can be described also when a Lagrangian is not available.
- $\mathcal{L}_2$ are $\frac{1}{r^2}$ terms with curvature or two auxiliary fields.

$$q \left(\frac{1}{4} R + \frac{3}{2} V_\mu V^\mu + 2 V_\mu A^\mu \right) (\phi \bar{\phi})$$

Rigid Variations

The Susy transformation are deformed from their flat space counterparts. E.g. for a chiral multiplet of R-charge q:

$$\delta\phi^i = -\sqrt{2}\zeta\psi^i$$

$$\delta\psi^i_\alpha = -\sqrt{2}\zeta_\alpha F^i - i\sqrt{2}(\sigma^\mu\bar{\zeta})_\alpha(\partial_\mu - iq(A_\mu + \frac{3}{2}V_\mu))\phi^i$$

$$\delta F^i = -i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\left(\nabla_\mu - i(q-1)(A_\mu + \frac{3}{2}V_\mu) - \frac{i}{2}V_\mu\right)\psi^i$$

Setting to zero the gravitino variation gives the Killing spinor equations:

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2}V^\nu\sigma_\mu\bar{\sigma}_\nu\zeta\ , \qquad (\nabla_\mu + iA_\mu)\bar{\zeta} = \frac{i}{2}V^\nu\bar{\sigma}_\mu\sigma_\nu\bar{\zeta}$$

On an Euclidean manifold $\mathcal{M}$ the spinors ζ and $\bar{\zeta}$ are independent and V_μ, A_μ are complex.

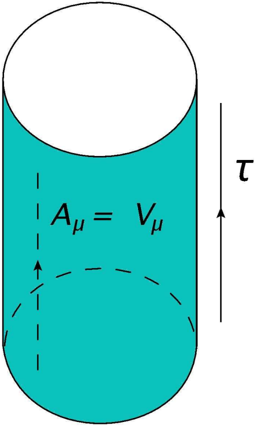
Example: $S^3 \times S^1$ [D. Sen; Romelsberger]

Consider the (Euclidean) cylinder $S^3 \times R$

The isometry group is $SU(2)_\ell \times SU(2)_r \times R$

$$V = -2A = -\frac{i}{r} d\tau$$

- The spinors ζ and $\bar{\zeta}$ are τ independent.
- The spinors ζ and $\bar{\zeta}$ are in $(\frac{1}{2}, 0)$.
- No need for superconformal symmetry.



The resulting superalgebra is $SU(2|1)_\ell \times SU(2)_r \times R$
Because the spinors are τ independent we can compactify R to S^1.

If additional $U(1)_f$ flavor symmetries are present we can add complex background gauge fields $A^f = -\frac{i}{r}\mu_f d\tau$ along S^1.

The partition Function on $S^3 \times S^1$

If additional $U(1)_f$ flavor symmetries are present we can add complex background gauge fields $A^f = -\frac{i}{r}\mu_f d\tau$ along S^1.

Trace over the Hilbert space. The μ_f are complex chemical potentials for the $U(1)_f$.

$$Z = Tr(-1)^F \exp\left(-\beta H - \frac{\beta}{r}\sum_f \mu_f Q_f\right)$$

Gets contributions only from short representations of $SU(2|1)_l$.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\delta_{\alpha\beta}\Big(H + R/r\Big) + \ldots$$

The $U(1)$ in $SU(2|1)$ is $H + R/r$. The states that contribute to the trace have $H = \frac{1}{r}(2J_l^3 - R)$.

The partition Function on $S^3 \times S^1$

- The values of H are quantized. Z is independent of small deformations of the lagrangian. It is the same in the UV and IR.
- Free field computations in the UV are possible [Romelsberger].
- Used to test dual descriptions in the IR. [Romelsberger; Dolan, Osborn; Spiridonov,Vartanov;...]
- For superconformal theories it reduces to the superconformal index [Kinney, Maldacena, Minwalla, Raju]
- The dependence on μ_f is holomorphic

Classification of SUSY geometries

Classifying Supersymmetric Manifolds

Consider the 4d New Minimal Sugra Killing spinor equation.

On which Riemannian Manifolds $(\mathcal{M}, g)$ are there solutions of

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2}V^\nu \sigma_\mu \bar{\sigma}_\nu \zeta \ , \qquad (\nabla_\mu + iA_\mu)\bar{\zeta} = \frac{i}{2}V^\nu \bar{\sigma}_\mu \sigma_\nu \bar{\zeta}$$

for some choice of background fields A_μ and V_μ?

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- One Killing spinor ζ exists if and only if $\mathcal{M}$ is complex.
- Two Killing spinors of opposite R-charge ζ and $\bar{\zeta}$ are present only on torus fibrations over a Riemann surface Σ.
- Two Killing spinors of the same R-charge require $SU(2)$ holonomy (compact case) or one of the cases below.
- Four supercharges are only present on R^4 or $S^3 \times R$ or $H^3 \times R$ (and some of their compactifications)

One supercharge in 4d with $U(1)_R$

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2}V^\nu \sigma_\mu \bar{\sigma}_\nu \zeta$$

A solution ζ is everywhere nonzero hence we can form the tensor

$$J^\mu{}_\nu = \frac{2i}{|\zeta|^2}\zeta^\dagger \sigma^\mu{}_\nu \zeta$$

- $J^\mu{}_\nu$ is an almost complex structure $J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu_\rho$
- $J^\mu{}_\nu$ is metric compatible. $g_{\rho\lambda}J^\rho{}_\mu J^\lambda{}_\nu = g_{\mu\nu}$
- $J^\mu{}_\nu$ is integrable because ζ is Killing.

The triple $(\mathcal{M},\ g_{\mu\nu},\ J^\mu{}_\nu)$ defines an Hermitian manifold.

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Conversely on any Hermitian manifold $\exists$ a solution ζ such that

$$J^\mu{}_\nu = \frac{2i}{|\zeta|^2}\zeta^\dagger \sigma^\mu{}_\nu \zeta$$

[Klare, Tomasiello, Zaffaroni; Dumitrescu, Seiberg, GF]

One supercharge in 4d with $U(1)_R$

In the Kähler case we could set $V_\mu = 0$ and get $(\nabla_\mu - iA_\mu)\zeta = 0$

The holonomy of the Levi-Civita connection is in $U(1)_l \times SU(2)_r$ we can twist away its $U(1)_l$ part using A_μ.

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In the general case:

- The auxiliary vector field V_μ encodes the failure of $\mathcal{M}$ to be Kähler.

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + (*W)_\mu, \quad W_{ij\bar{j}}, \quad dW = 0$$

- The Chern connection has holonomy in $U(1)_l \times SU(2)_r$. A_μ twists away its $U(1)_l$ part.

The Superalgebra generated by ζ is $\{Q_\zeta, Q_\zeta\} = 0$.

Q_ζ is a scalar under holomorphic complex coordinate changes followed by appropriate R-transformations.

Background gauge fields

If the flat space theory has some global symmetry (say $U(1)$) it possesses a conserved current j_μ part of a linear multiplet

$$(J,\ j^\mu,\ j_\alpha,\ \bar{j}_{\dot\alpha})$$

We can couple it to a background gauge multiplet

$$(D,\ a_\mu,\ \lambda_\alpha = 0,\ \bar\lambda_{\dot\alpha} = 0)$$

This background preserves a supercharge ζ if

$$f_{\mu\nu}^{(0,2)} = 0, \qquad D = -\frac{1}{2} J^{\mu\nu} f_{\mu\nu}$$

Hence a_μ is connection on a holomorphic line bundle

Two Supercharges in 4d

If a second solution $\bar{\zeta}$ is present there are further restrictions on the metric. Consider the complex vector field

$$K^\mu = \bar{\zeta}\bar{\sigma}^\mu\zeta, \qquad Re(K^\mu) = X^\mu, \qquad Im(K^\mu) = Y^\mu$$

- K^μ is Killing.
- $X^\mu X_\mu = Y^\mu Y_\mu$ and $X^\mu Y_\mu = 0$
- $J^\mu{}_\nu$ is determined by K^μ and the metric.
- If $[X, Y] \neq 0$ the manifold is locally isometric to $S^3 \times R$
- If $[X, Y] = 0$ the two Killing vector fields X and Y generate translations on a T^2 fibered over a Riemann surface Σ

Two Supercharges 4d/3d

The superalgebra generated by ζ and $\bar{\zeta}$ is

$$\{Q_\zeta, Q_\zeta\} = 0, \quad \{Q_{\bar{\zeta}}, Q_{\bar{\zeta}}\} = 0, \quad \{Q_{\bar{\zeta}}, Q_\zeta\} = \delta_K$$

$$[\delta_K, Q_\zeta] = [\delta_K, Q_{\bar{\zeta}}] = 0$$

By reducing along one direction on the T^2 we obtain the following:

Any $\mathcal{N} = 2$ field theory with a $U(1)_R$ symmetry in 3d can be placed on a circle bundle over Σ preserving two supercharges.

All squashed 3-spheres in the literature are in this class.

Dependence of Observables on Geometry

Dependence of Observables on Geometry

- Start with analyzing the partition function of $\mathcal{N} = 1$ theories on compact complex manifolds $\mathcal{M}_4$
- Here we will consider the limit of large $\mathcal{M}_4$. A linearized analysis around flat space is applicable
- Only classical considerations.

Linearized Analysis

Choose a single supercharge Q_ζ in flat space $\Rightarrow$ choice of $J^\mu{}_\nu$.

Small variations of the geometry $\delta g_{\mu\nu}$, $\delta J^\mu{}_\nu$ and of $W_{\mu\nu\rho}$ couple to the R-multiplet while changes in the background gauge fields δa_μ couple to the corresponding linear multiplet.

$$\delta\mathcal{L} = -\frac{1}{2}\delta g^{\mu\nu} T_{\mu\nu} + \delta A^\mu j_\mu^{(R)} + \delta V^\mu \mathcal{A}_\mu - \delta a_\mu j^\mu + \delta D J \ .$$

Some terms in $\delta\mathcal{L}$ are Q-exact and do not contribute to Z.

Q_ζ is a scalar under complex coordinate changes $\Rightarrow$ the results holds also at the nonlinear level.

Deformations of Complex structures

Choose a c.s $J^\mu{}_\nu$ on $\mathcal{M}_4$ and deform it by adding $\delta J^\mu{}_\nu$.

In complex coordinates adapted to $J^\mu{}_\nu$ the requirement that $J + \delta J$ is an almost complex structure implies that at linear level

$$\delta J^i{}_j = \delta J^{\bar{i}}{}_{\bar{j}} = 0$$

The remaining components are constrained by the integrability requirement

$$\partial_{\bar{j}} \delta J^i{}_{\bar{i}} - \partial_{\bar{i}} \delta J^i{}_{\bar{j}} = 0$$

δJ generated by diffeomorphisms are trivial $\delta J^i{}_{\bar{i}} \sim \delta J^i{}_{\bar{i}} + 2i\partial_{\bar{i}}\epsilon^i$
Hence, at first order, deformations of $J^\mu{}_\nu$ are determined by

$$\Theta^i = \delta J^i{}_{\bar{i}} d\bar{z}^{\bar{i}}, \qquad [\Theta^i] \in H^{0,1}(\mathcal{M}_4, T^{1,0}(\mathcal{M}_4))$$

Deformations of metric and background gauge fields

Variations of the metric are constrained by the change in complex structure

- $\delta g_{i\bar{j}}$ are unconstrained
- $\delta g_{ij} = \frac{i}{2}\left(g_{i\bar{k}}\Delta J^{\bar{k}}{}_j + g_{j\bar{k}}\Delta J^{\bar{k}}{}_i\right)$

In the same way for Abelian background gauge fields we must have

$$\partial_{\bar{i}}\delta a_{\bar{j}} - \partial_{\bar{j}}\delta a_{\bar{i}} = 0$$

modulo gauge transformations $\delta a_\mu = \partial_\mu \epsilon$. Hence the holomorphic line bundle moduli are in $H^{0,1}(\mathcal{M}_4)$.

The deformed Lagrangian

We can express the deformation of the Lagrangian $\delta\mathcal{L}$ in terms of the variations of $\delta J^{\mu}{}_{\nu}$ $\delta g_{\mu\nu}$ and δa_{μ}. (We set $W = d\tilde{B}$)

$$\delta\mathcal{L} = Q_{\zeta}(\mathcal{I}) + \delta J^{i}{}_{\bar{i}}\mathcal{O}^{\bar{i}}_{i} + \delta a_{\bar{i}}\mathcal{J}^{\bar{i}}$$

- $\delta g_{i\bar{j}}$ appears in Q_{ζ} exact terms.
  $Z(\mathcal{M}_4)$ does not depend on the Hermitian metric.
- Varying $W = d\tilde{B}$ does not change $Z(\mathcal{M}_4)$.
  Dependence on $W_{\mu\nu\rho}$ is at most cohomological.

Invariance under diffeomorphisms and gauge transformations implies that for $\delta J^{i}{}_{\bar{i}} = 2i\partial_{\bar{i}}\epsilon^{i}$ and $\delta a_{\bar{i}} = \partial_{\bar{i}}\epsilon$

$$\delta\mathcal{L}_{\text{trivial}} = Q_{\zeta}(\mathcal{I}') + \text{total der}$$

- The partition function depends holomorphically on the moduli of the complex structure and of the holomorphic line bundle.
  Z can however be singular.

Example $S^3 \times S^1$

Display $S^3 \times S^1$ as a complex manifold by a quotient of $C^2 – \{(0,0)\}$.

$$(z_1, z_2) \sim (pz_1, qz_2), \qquad 0 < |p| \leq |q| < 1$$

p, q are complex structure moduli.

We will denote this branch of the moduli space of complex structures on $S^3 \times S^1$ by $\mathcal{M}_4^{p,q}$.

- There exists an Hermitian metric that allows to preserve 2 supercharges for any (p, q).
- For $p = q^*$ we can preserve four supercharges.

Example $S^3 \times S^1$

The partition function on $S^3 \times S^1$ is the supersymmetric index

$$\mathcal{I}(p, q, u) = Tr_{S^3}\left((-1)^F p^{J_3+J_3'-R/2} q^{J_3-J_3'-R/2} u^{Q_f}\right)$$

- The fugacities p, q can be identified with the moduli of $\mathcal{M}_4^{p,q}$
- u the fugacity for Q_f is an holomorphic line bundle modulus.
- The index is meromorphic in p, q and u.
- It does not depend on the choice of Hermitian metric.

Conclusions

Placing supersymmetric theories on different manifolds preserving Susy provides a new set of tools to study the dynamics of strongly coupled theories.

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Placing supersymmetric theories on different manifolds preserving Susy provides a new set of tools to study the dynamics of strongly coupled theories.

Turning on background values for the fields in the supergravity multiplet and taking the rigid limit allows a general description of rigid SUSY in curved space.

The $(\mathcal{M}, g)$ allowing for SUSY can be identified independently from the matter content.

This allows a classification of supersymmetric geometries.

We can study the dependence on the geometry of supersymmetric observables. We find they are "almost" topological.

Thank You!