The gradient flow running coupling scheme

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Outline

- Yang-Mills gradient flow
- Finite volume calculation
- Running coupling in finite volume, step scaling
- Continuum results for $SU(3)$ fundamental $N_f = 4$
- Preliminary results for $SU(3)$ fundamental $N_f = 8$

Yang-Mills gradient flow

Luscher considered the following flow in the space of gauge fields

$$\dot{A}_\mu = -\frac{\delta S_{YM}}{\delta A_\mu} = D_\nu F_{\nu\mu}$$

$A_\mu(t, x_1, x_2, x_3, x_4)$ where t is auxiliary "time", dimension length^2

$A_\mu(t)$ is uniquely calculable from $A_\mu(0)$, smoothing operation:

Zeroth order in perturbation theory $A_\mu(t, p) = e^{-p^2 t} A_\mu(0, p)$

Yang-Mills gradient flow

Consider $SU(N)$ gauge theory + N_f fermions in repr R

Path integral usually:

$$\langle \mathcal{O}(A) \rangle = \frac{\int DAd\psi d\bar{\psi}\mathcal{O}(A)e^{-S(A,\psi)}}{\int DAd\psi d\bar{\psi}e^{-S(A,\psi)}}$$

Suggestion of Luscher: path integral over $A_\mu(0,x)$ but observables on $A_\mu(t,x)$ for $t > 0$, flow becomes part of the observable:

$$\langle \mathcal{O}_t(A) \rangle = \frac{\int DA(0)d\psi d\bar{\psi}\mathcal{O}(A(t))e^{-S(A(0),\psi)}}{\int DA(0)d\psi d\bar{\psi}e^{-S(A(0),\psi)}}$$

Yang-Mills gradient flow

Why?

Gradient flow is smoothing/averaging/blocking: $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$ correlation function, $x \to y$ singularities might be tamed?

Let's try with the simplest composite operator first: $E = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}$

Observable $E(t) = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}(t)F_{\mu\nu}(t)$

On the lattice: plaquette from $U_\mu(t)$ smoothed fields

Calculate $\langle E(t)\rangle$ in dimensional regularization in $\overline{\text{MS}}$ scheme $D = 4 - 2\varepsilon$.

Remember: path integral over $A_\mu(t=0)$, observable at $A_\mu(t>0)$.

$$\langle E(t)\rangle = \frac{g_0^2}{2}\langle \partial_\mu A_\nu^a(t)\partial_\mu A_\nu^a(t) - \partial_\mu A_\nu^a(t)\partial_\nu A_\mu^a(t)\rangle + \cdots$$

In momentum space, lowest order: $A_\mu(t,p) = e^{-tp^2} A_\mu(0,p)$

E is quadratic in A_μ: free propagator, two factors of e^{-tp^2}

Gauge sum: factor of $N^2 - 1$.

$$\langle E(t) \rangle = \frac{g_0^2(N^2-1)}{2} \int \frac{d^D p}{(2\pi)^D} e^{-2tp^2} \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu \right) G_{\mu\nu}(p)$$

Free propagator in Feynman gauge: $G_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{p^2}$

$$\langle E(t) \rangle = \frac{g_0^2(N^2-1)(D-1)}{2(8\pi t)^{D/2}} + O(g_0^4)$$

Factor $D-1$ from Euclidean trace, integral over p finite

The Yang-Mills gradient flow

$$\langle E(t)\rangle = \frac{g_0^2(N^2-1)(D-1)}{2(8\pi t)^{D/2}} + O(g_0^4)$$

All of this was tree-level. 1-loop: $1/\varepsilon$ divergence, cancelled by definition of renormalized coupling

$$g_0^2 = g_{\overline{\mathrm{MS}}}^2(\mu)\mu^{2\varepsilon}\left(4\pi e^{-\gamma}\right)^{-\varepsilon}\left(1 - \frac{b_0 g_{\overline{\mathrm{MS}}}^2(\mu)}{\varepsilon} + O(g_{\overline{\mathrm{MS}}}^4)\right)$$

g_0: bare, $g_{\overline{\mathrm{MS}}}$: renormalized, b_0: first β-function coefficient
μ: dimreg scale

In terms of the renormalized coupling

$$\langle E(t)\rangle = \frac{g^2_{\overline{\mathrm{MS}}}(\mu)(N^2-1)(D-1)}{2(8\pi t)^{D/2}} + O(g^4_{\overline{\mathrm{MS}}})$$

In $D = 4$ we have

$$\langle E(t)\rangle = \frac{3g^2_{\overline{\mathrm{MS}}}(\mu)(N^2-1)}{128\pi^2 t^2} + O(g^4_{\overline{\mathrm{MS}}})$$

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We have a finite expression for $t > 0$ (at least to leading order)!

Commments:

- Finite to all orders
- Finite non-perturbatively
- Fermions enter at 1-loop

$$\langle E(t) \rangle = \frac{3g_{\overline{\text{MS}}}^2(\mu)(N^2-1)}{128\pi^2 t^2} + O(g_{\overline{\text{MS}}}^4)$$

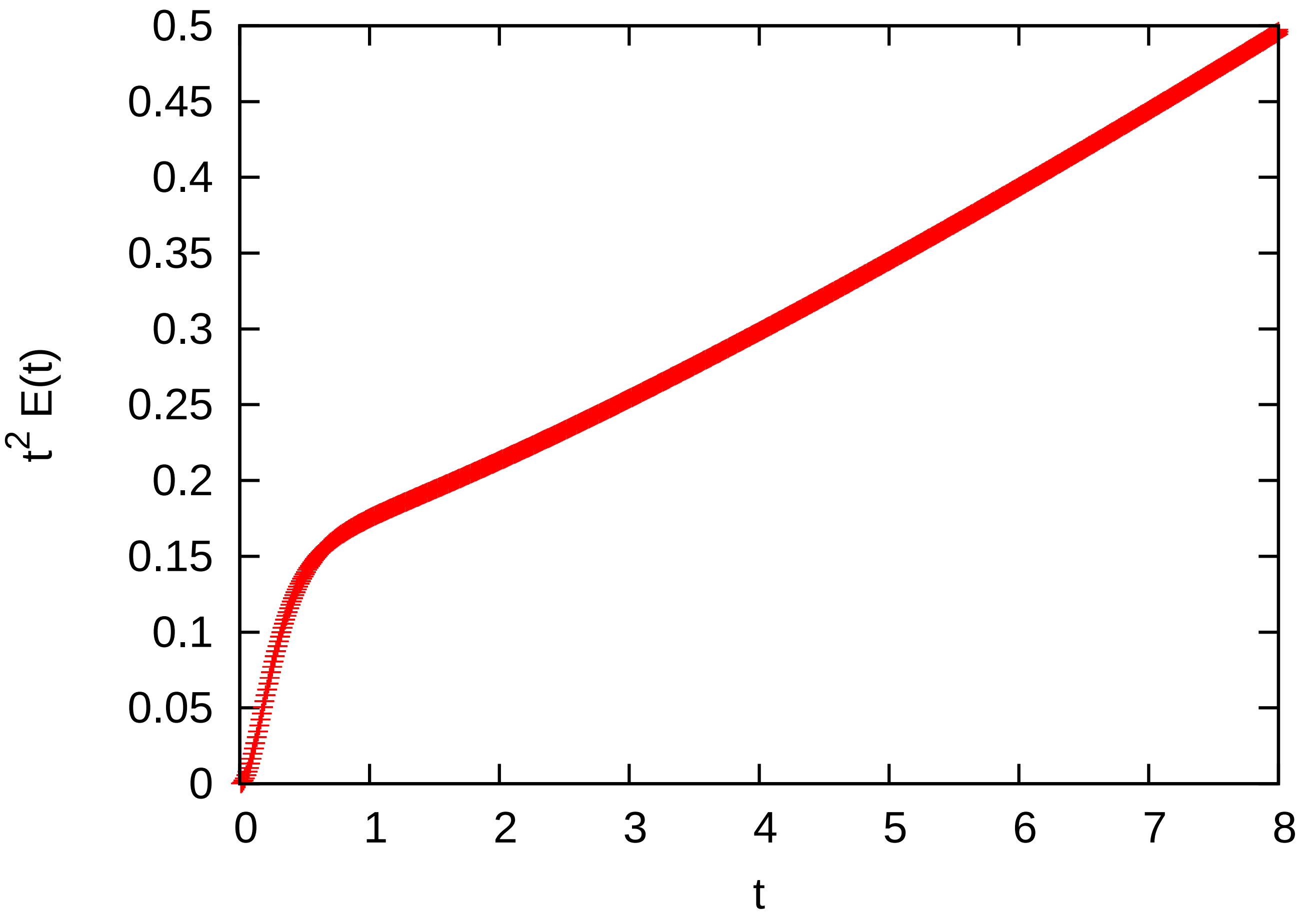
Quick sanity check: at $t = 0$ there should be a divergence when written in terms of g_R, it is there: $1/t^2$.

Lessons:

- Composite operator became finite if $t > 0$
- $A_\mu(t)$ kind of renormalized field
- No other renormalization necessary beyond usual $g_0 \to g_R$

Smoothing/averaging property of gradient flow $\sim$ renormalization!

What about non-perturbative $\langle t^2 E(t) \rangle$?



This is from a QCD lattice calculation at fixed lattice spacing.

The Yang-Mills gradient flow

Running coupling $\mu = 1/\sqrt{8t}$

$$g^2(1/\sqrt{8t}) = \frac{128\pi^2 \langle t^2 E(t) \rangle}{3(N^2 - 1)}$$

Right hand side evaluated non-perturbatively, definition for left hand side.

All of this in infinite volume.

Finite volume gradient flow scheme

All of this was in infinite volume. Need: $1/L \ll \mu \ll 1/a$

Better: $1/L = \mu \ll 1/a$

Wolff, Luscher, ...

Same idea as in Schroedinger functional $\rightarrow$ step scaling $\rightarrow$ no "finite volume effects"

Need: Yang-Mills gradient flow on 4-torus T^4 i.e. finite volume

Main result

$$\langle t^2 E(t)\rangle = \frac{3(N^2-1)}{128\pi^2} g^2_{\overline{\mathsf{MS}}}(\mu)\left(1+\delta_a(L)+\delta_e(L)\right)$$

$$\delta_a(L) = -\frac{64t^2\pi^2}{3L^4}$$

$$\delta_e(L) = \vartheta^4\left(\exp\left(-\frac{L^2}{8t}\right)\right) - 1 = 8\exp\left(-\frac{L^2}{8t}\right) + 24\exp\left(-\frac{L^2}{4t}\right) + \dots$$

Correction $\delta(L) = \delta_a(L) + \delta_e(L)$ only depends on $c = \sqrt{8t}/L$.

Asymptotic freedom $\rightarrow$ perturbation theory for small L

Periodic gauge field, anti-periodic fermions

Separate zero gauge modes $A_\mu(x) = B_\mu + Q_\mu(x)$

Gauge fixing, ghosts

For small L: integrate out $Q_\mu(x)$, ghosts, fermions in 1-loop, treat B_μ exactly

Integrating out $Q_\mu(x)$: effective action for B_μ

Solve flow for $B_\mu(t)$ and $Q_\mu(t, x)$

Evaluate $\langle E(t) \rangle_B$ by integrating out Q_μ perturbatively and then integrate over B_μ exactly (4-matrix integrals)

Gradient flow running coupling scheme

$$g_R^2(L) = \frac{128\pi^2 \langle t^2 E(t) \rangle}{3(N^2-1)(1+\delta(c))}$$

In principle two scales $g_R^2(t,L)$ let's keep $c = \sqrt{8t}/L$ fixed

1-parameter family of running coupling schemes

By construction all of them run with the universal 1-loop β-function for small g_R

Very easy to measure on the lattice! No expensive fermionic measurements.

$g_R^2(L)$ in terms of $g_{\overline{\mathrm{MS}}}$ contains both even and odd powers, as in finite-T perturbative calculations

Gradient flow running coupling scheme

Numerical implementation for $SU(3)$, $N_f = 4$ fundamental (stout improved staggered) fermions, $c = 0.3$

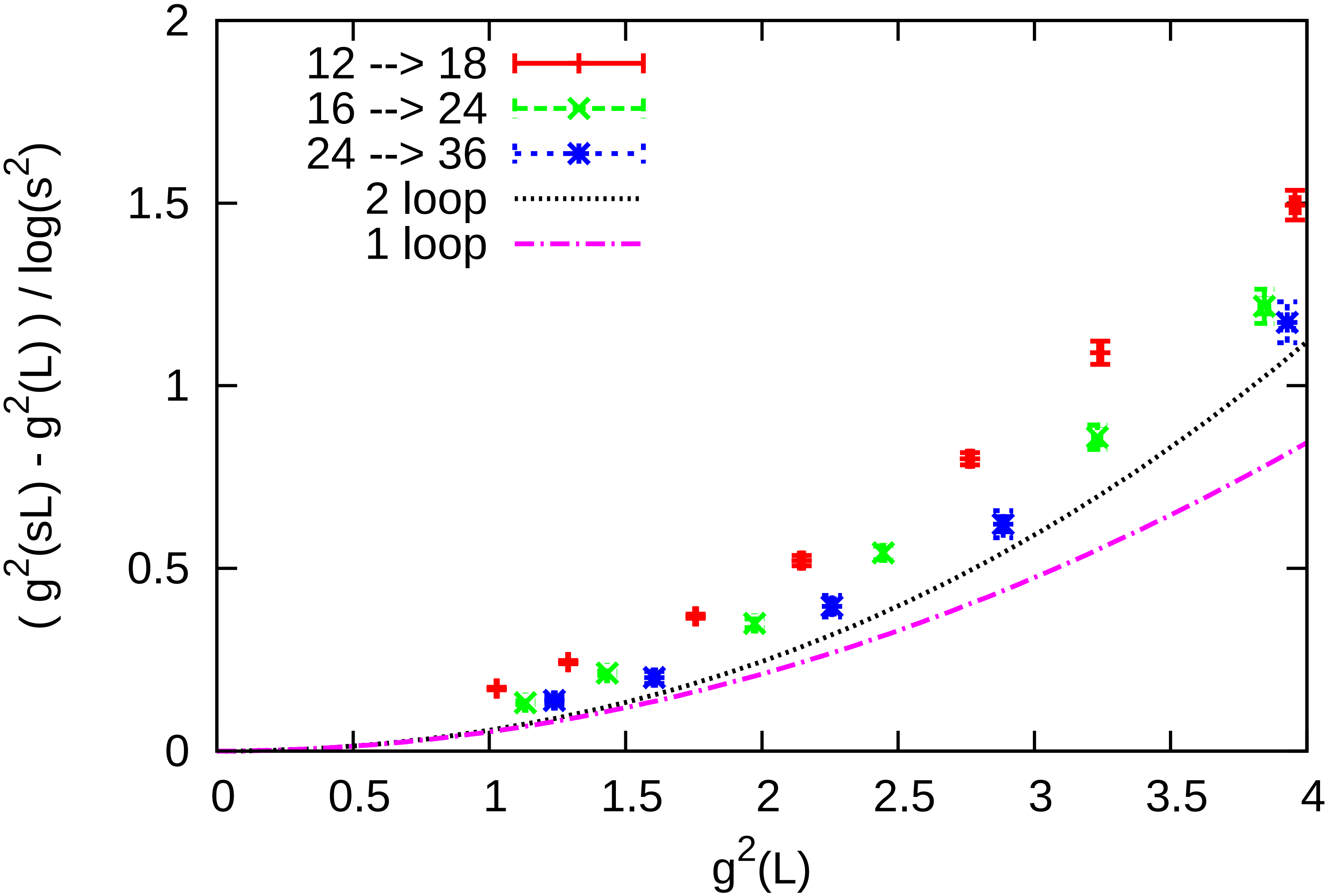
Calculate discrete β-function, $L \to sL$, $c = \sqrt{8t}/L = 0.3$

$$\frac{g^2(sL) - g^2(L)}{\log(s^2)}$$

For $s = 3/2$ $12 \to 18$, $16 \to 24$, $24 \to 36$

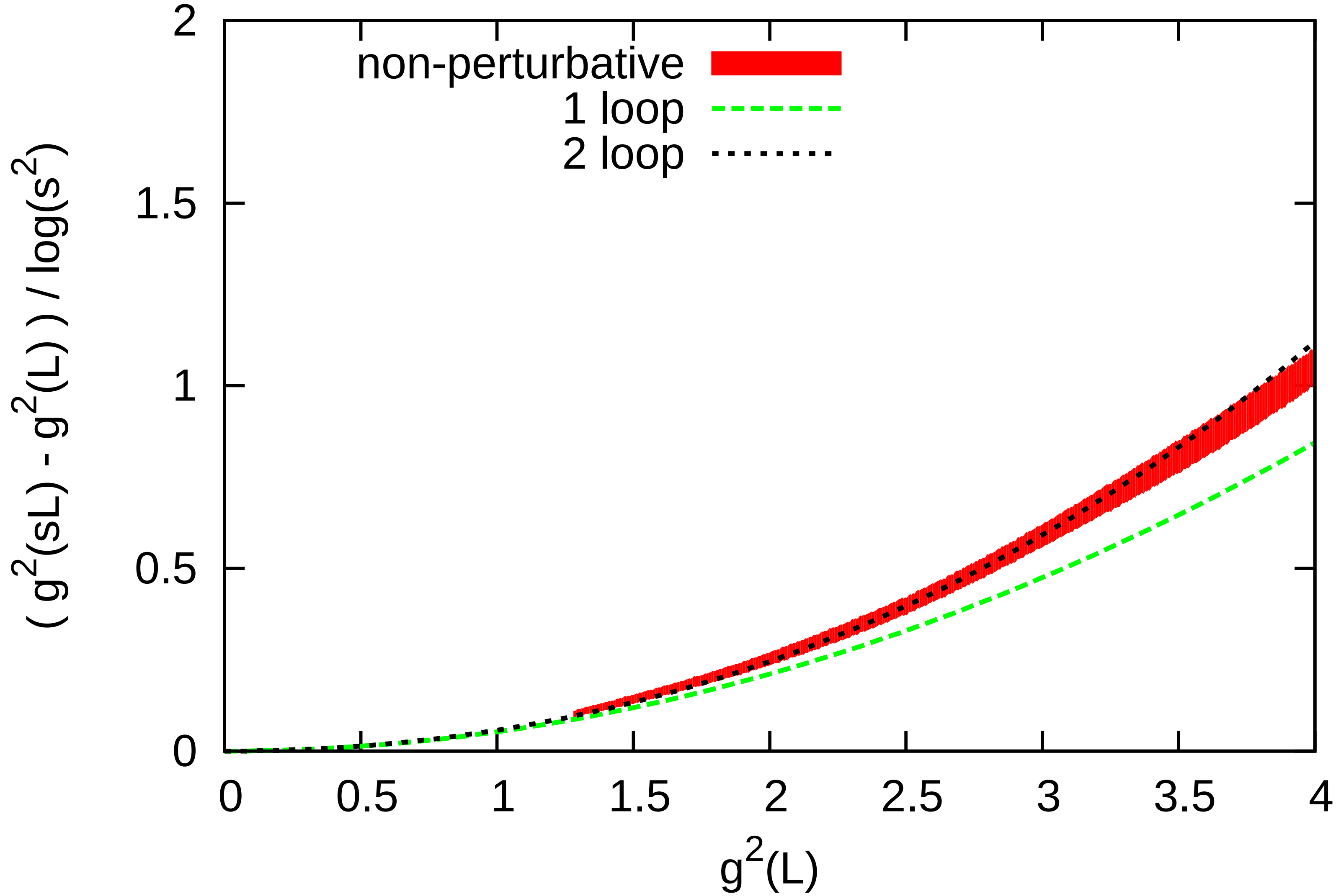
Continuum limit: $L/a \to \infty$

Results, $s = 3/2$



Bare g_0 or $\beta = 6/g_0^2$ moves us along the x-axis

Results, $s = 3/2$, continuum extrapolation



Backup slides with continuum extrapolation.

Gradient flow scheme

Works very well for $SU(3)$ and $N_f = 4$

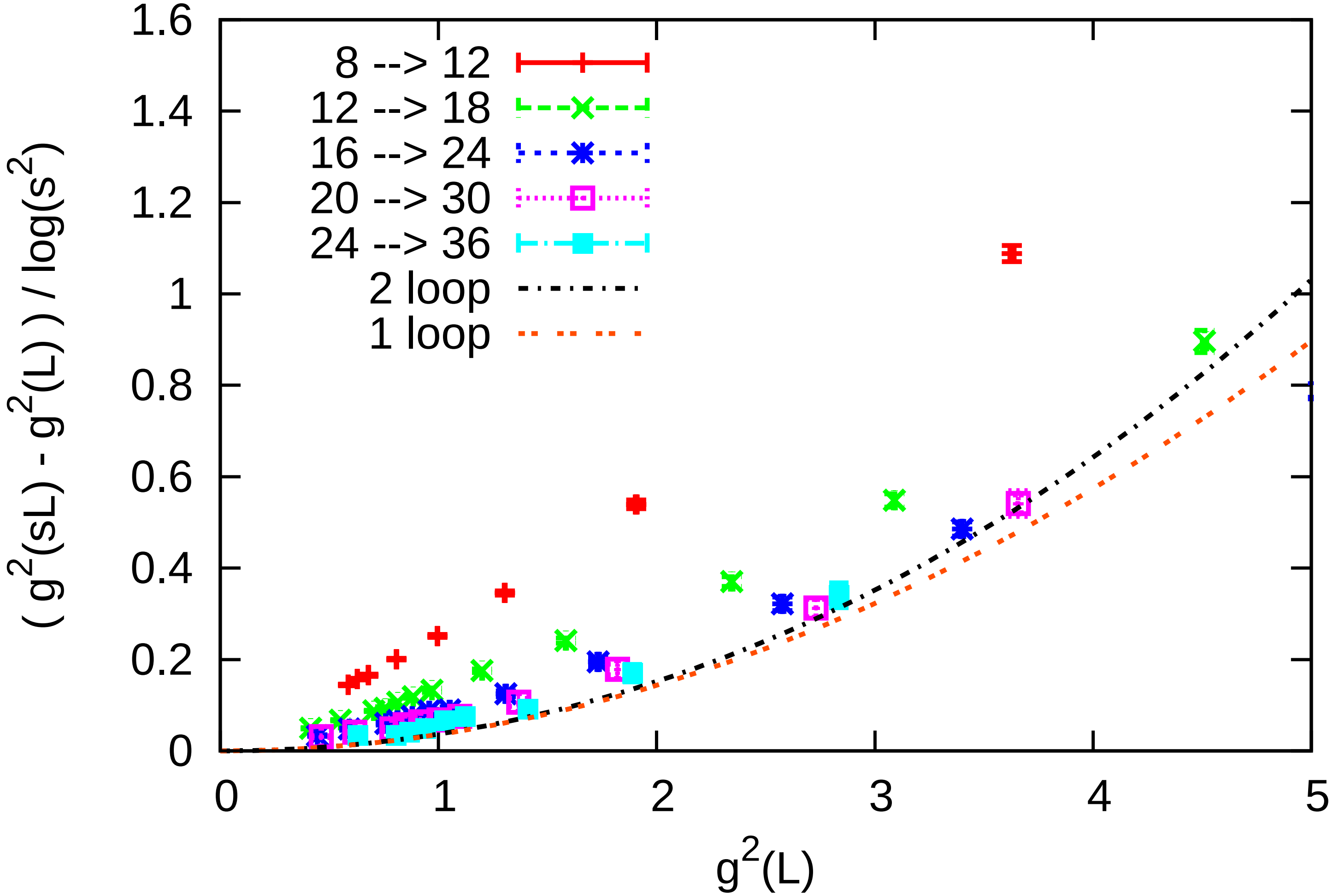
Let's see $SU(3)$ and $N_f = 8$

$s = 3/2$ $\quad$ $8 \rightarrow 12,\ 12 \rightarrow 18,\ 16 \rightarrow 24,\ 20 \rightarrow 30,\ 24 \rightarrow 36$

$c = \sqrt{8t}/L = 0.3$

Exactly the same setup as $N_f = 4$

Results for $SU(3)$ and $N_f = 8$, preliminary



Working on continuum limit … no sign of fixed point!

Outlook

Fermion flow $\psi(t)$ — Luscher

Schroedinger functional + gradient flow — Fritzsch, Ramos

Lots of other applications ...

$SU(3)$ with $N_f = 12, 16$

$N_f = 16$ should be conformal

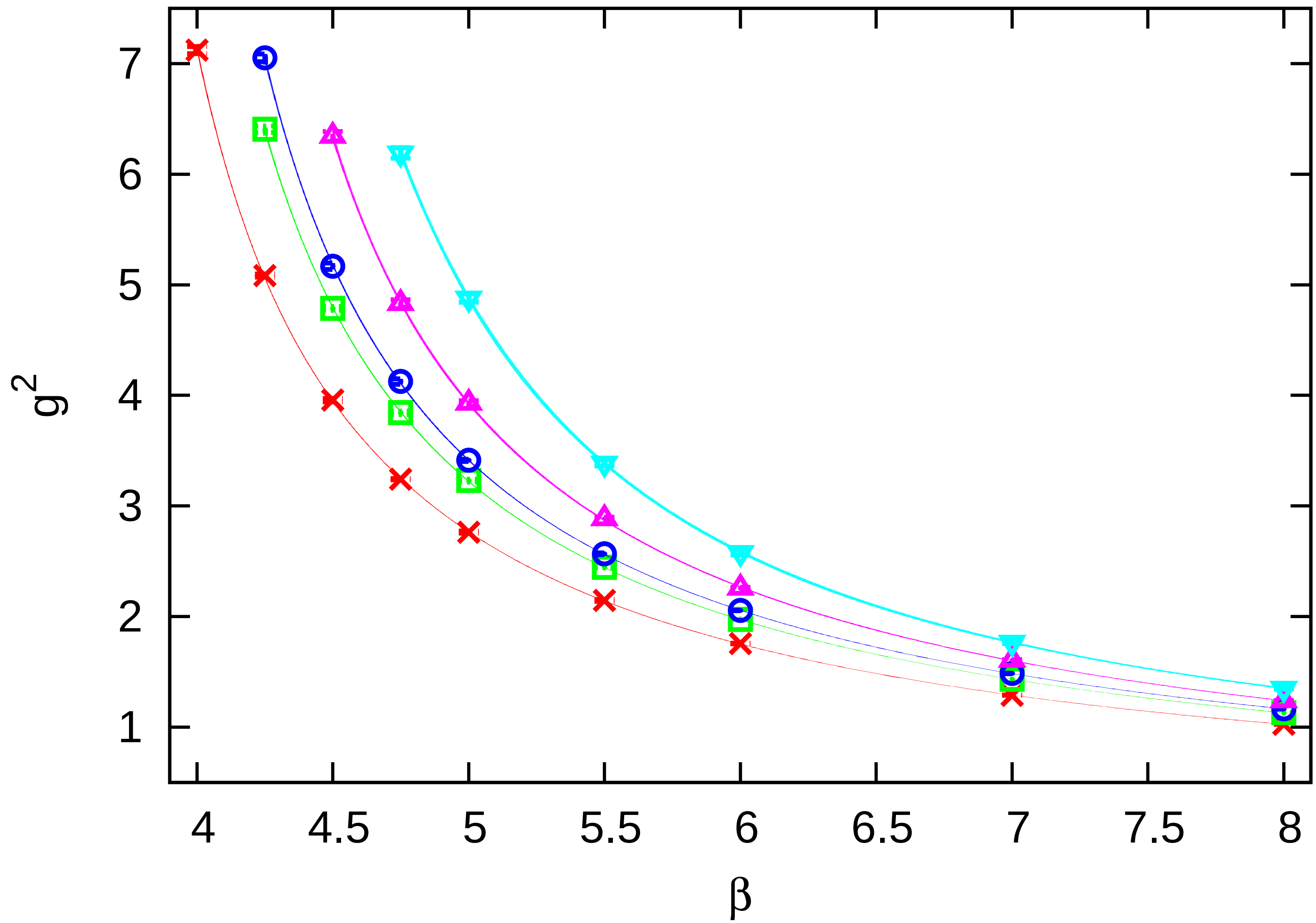
$N_f = 12$ currently various groups and various approaches don't agree, would be good to know

Summary

- Yang-Mills gradient flow is a great new tool
- New look at renormalization
- Cheap gluonic measurement, high precision
- 1-parameter family, c can be optimized
- β-function for $SU(3)$ $N_f = 4, 8$

Thank you for your attention!

Results, $s = 3/2$, continuum extrapolation



Parametrization of $g_R(\beta, L/a)$ as a function of β for fixed L/a

Results, $s = 3/2$, continuum extrapolation