1) Kinetic theory vs. hydrodynamics

Complementary: hydrodynamics $\longleftrightarrow$ macroscopic

$T(x), \mu_b(x), u^\mu(x)$

(derived quantities:

$e(x), n_b(x), p(x)$)

EoS: $p = p(e, n_b) = p(T, \mu_b)$

kinetic theory $\longleftrightarrow$ microscopic: $f(x,p)$

$f(x^\mu, p^\mu)$: classical kinetic theory: $\rho(E, \vec{p}) \sim \delta(E^2 - E_p^2)$
(g=0)

$$= \frac{1}{2E_p}\left(\delta(E - E_p) + \delta(E + E_p)\right)$$

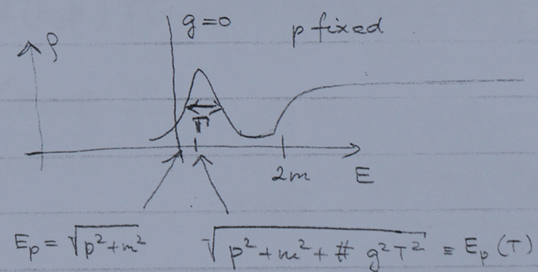
$$E_p = \sqrt{\vec{p}^{\,2} + m^2}$$

$\rightarrow$ on-shell particles, $f(\vec{x}, \vec{p}, t)$, $E = E_p$

quantum kinetic theory: $(g \neq 0)$

(i) weak coupling: $\rho(E, \vec{p}) \sim \dfrac{E^2 \Gamma^2_{(T)}}{(E^2 - \vec{p}^{\,2} - m^2(T))^2 + E^2\Gamma^2}$

$$m^2(T) = m^2 + \#(gT)^2$$
$$\Gamma(T) \sim g^2 T$$
for $g \ll 1$



$f(x,p) \longrightarrow W(x^\mu, p^\nu)$ Wigner function

(ii) strong coupling $\longrightarrow$ no quasiparticles

structureless spectral function

no Boltzmann-like description

Boltzmann equation:

$$m \frac{df}{d\tau} = C \qquad \text{(collision term)}$$

$$\rightarrow \quad \underbrace{m\dot{x}^\mu}_{p^\mu} \frac{\partial f}{\partial x^\mu} + \underbrace{m\dot{p}^\mu}_{mF^\mu} \frac{\partial f}{\partial p^\mu} = C \qquad (\text{E\&M}: mF^\mu = qF^{\mu\nu}p_\nu \text{ Lorentz force})$$

If no longrange forces ($F^{\mu\nu} = 0$)

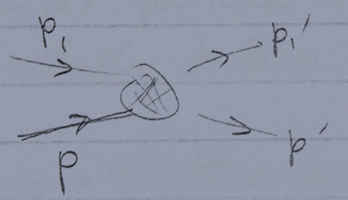
$$\Rightarrow \quad \boxed{p^\mu \partial_\mu f(x,p) = C(x,p)}$$

If system is <u>dilute</u> $\left(f(x_1 x_2, p_1 p_2) \approx f(x_1,p_1)\, f(x_2,p_2)\right)$
and <u>weakly interacting</u> $\left(g \ll 1,\ \rho(E,\vec{p}) \approx \delta(E^2 - E_p^2(T))\right)$

$\Rightarrow$ Boltzmann equation $\boxed{p\cdot\partial f = C[f]}$

$$C(x,p) = \frac{1}{2}\int \frac{d^3p_1}{2E_1}\frac{d^3p'}{2E'}\frac{d^3p_1'}{2E_1'}\, \delta^{(4)}(p+p_1-p'-p_1')\,\sigma(s,\vartheta)$$
$$\times \Big[f(x,p')\,f(x,p_1')\,(1\pm f(x,p))(1\pm f(x,p_1))$$
$$- f(x,p)\,f(x,p_1)\,(1\pm f(x,p'))(1\pm f(x,p_1'))\Big]$$

where $s = (p+p_1)^2 = (p_1+p_1')^2$ and



$$\cos\vartheta = \frac{(p-p_1)\cdot(p'-p_1')}{(p-p_1)^2}$$

(scattering angle in cm system)

The collision term vanishes in two limits:

(1) free streaming ($g = 0 \Rightarrow \sigma = 0$, ideal gas)

$\Rightarrow$ solution $f(\vec{x}, \vec{p}; t) = f\left(\vec{x} - \frac{\vec{p}}{E}(t - t_0), \vec{p}; t_0\right)$

(2) extremely strong coupling ($g \to \infty$, $\sigma \to \infty$, ideal fluid)

$$f(x,p) \xrightarrow{\text{collisions}} f_{eq}(x,p) = \frac{1}{e^{(p \cdot u(x) - \mu(x))/T(x)} + a}$$

$a = \begin{cases} 0 & \text{Boltzmann} \\ 1 & \text{Fermi} \\ -1 & \text{Bose} \end{cases}$

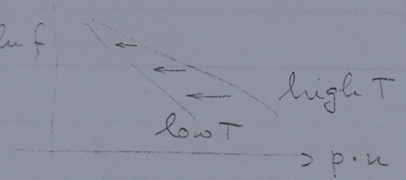
For the local equilibrium distribution $f_{eq}\left(\frac{p \cdot u(x)}{T(x)}, \frac{\mu(x)}{T(x)}\right)$

the gain and loss terms cancel in the collision term

If the system expands and accelerates, $T(x)$ decreases

and $f_{eq}\left(\frac{p \cdot u(x)}{T(x)}, \frac{\mu(x)}{T(x)}\right)$ changes shape (steeper slope in

a log-plot:



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This requires particles to change their momenta (particles drift from higher to lower energies, on average)

Even for $g \to \infty$, this cannot happen instantaneously (quantum mechanics!)

$\longrightarrow$ $f(x,p) = f_{eq}(x,p) + \delta f(x,p)$

For $g \to \infty$ or small expansion rates $\longrightarrow$ δf small

For $g \to 0$ or large expansion rates $\longrightarrow$ δf large

Will show: $\delta f = 0$ $\to$ ideal fluid dynamics
δf small $\to$ viscous fluid dynamics | δf large: hydrodynamic theory [illegible]

2) Ideal fluid dynamics

$$f(x,p) = f_{eq}(x,p) = \frac{1}{e^{p\cdot u(x)/T(x)}+a}$$ (neglect chem. potentials mostly)

$u^\mu(x)$ = local fluid velocity $\longrightarrow u^\mu_{lrf} = (1,\vec{0}) \equiv \bar{u}^\mu$

$u^\mu u_\mu = 1.$

In l.r.f. $f_{eq}(\bar{x},\bar{p}) = \frac{1}{e^{\bar{p}\cdot\bar{u}/T}+a} = \frac{1}{e^{\bar{E}/T}+a} = f_{eq}(\bar{E})$

Moment equations:

Integrate Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C(x,p)$$

with integration measure $dP \equiv \frac{1}{(2\pi\hbar)^3}\frac{d^3p}{E_p} = 2\theta(p^0)\,\delta(p^2-m^2(T))\,\frac{d^4p}{(2\pi\hbar)^3}$

Write $g^{\mu\nu} = (g^{\mu\nu}-u^\mu u^\nu) + u^\mu u^\nu = u^\mu u^\nu + \Delta^{\mu\nu}$

timelike projector in lrf — spacelike projector in lrf

$$p^2 = p^\mu p_\mu = p_\mu u^\mu u^\nu p_\nu + p_\mu \Delta^{\mu\nu} p_\nu = (p\cdot u)^2 + p\cdot\Delta\cdot p$$
$$= \bar{E}^2 - \vec{\bar{p}}^{\,2}$$

$$d^4p = d^4\bar{p}\ ,\quad \theta(p^0) = \theta(\bar{p}^0) = \theta(\bar{E})$$

$$\Rightarrow \int dP\ A(p)\, f_{eq}\left(\frac{p\cdot u}{T}\right) = \int \frac{d^4p}{(2\pi)^3}\, A(p)\, 2\theta(p^0)\,\delta(p^2-m^2)\, f_{eq}$$

$$= \int \frac{d^4\bar{p}}{(2\pi)^3}\, A(\bar{p})\, 2\theta(\bar{E})\, \underbrace{\delta\left(\bar{E}^2-(\bar{p}^2+m^2)\right)}_{\frac{1}{2E_p}\left(\delta(\bar{E}-E_p)+\delta(\bar{E}+E_p)\right)}\, f(\bar{E})$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3\,\bar{E}}\, f_{eq}(\bar{E})\, A(\bar{p})$$ $\Longrightarrow$ in local equilibrium these moments can be easily worked out in lrf coords.

Define $\hat{I}^{\mu\nu\ldots\sigma}[f] \equiv \int dP\, p^\mu p^\nu \ldots p^\sigma f(x,p)$

(i) 0^{th} moment:

$$\int dP\, p^\mu \partial_\mu f = \int dP\, C(x,p)$$

$$\Rightarrow \underbrace{\partial_\mu \int dP\, p^\mu f}_{N^\mu \text{ particle number current}} = \partial_\mu \int_{\vec{p}} v^\mu f = \int dP\, C \qquad \int_{\vec{p}} \equiv \int \frac{d^3p}{(2\pi)^3} \qquad v^\mu = \frac{p^\mu}{E_p}$$

Now net baryon number is conserved:

$$\partial_\mu (N^\mu - \bar{N}^\mu) = 0 \quad \Rightarrow \quad \int dP (C - \bar{C}) = 0$$

zeroth moment of collision kernel for quarks minus antiquarks vanishes. (must hold for any approx. of C!)

(ii) 1^{st} moment

$$\int dP\, p^\lambda\, p^\mu \partial_\mu f = \int dP\, p^\lambda C$$

$$\partial_\mu \underbrace{\int dP\, p^\mu p^\lambda f(x,p)}_{T^{\mu\lambda}(x)} = \int dP\, p^\lambda C(x,p)$$

Since energy and momentum are conserved, we have

$$\partial_\mu (T^{\mu\nu}_q + T^{\mu\nu}_{\bar{q}} + T^{\mu\nu}_g) = 0 \quad \Rightarrow \quad \int dP\, p^\lambda \left(C_q + C_{\bar{q}} + C_g \right) = 0$$

first moment of sum of collision kernels vanishes (in any approximation!)

(iii) Higher moments (needed in some derivations of viscous hydrodynamics)

$$\text{e.g. } \partial_\mu \underbrace{\int dP\, p^\mu p^\nu p^\lambda f}_{F^{\mu\nu\lambda}} = \int dP\, p^\nu p^\lambda C \equiv P^{\nu\lambda}$$

Since there are no other collision invariants, $P^{\nu\lambda} \neq 0$ in general.
But $F^{\mu\nu}{}_{\nu} = m^2 j^\nu$ and $(P_q - P_{\bar{q}})^\mu{}_\mu = 0$ since $p^2 = m^2$ and $\int dP (C - \bar{C}) = 0$

For ideal fluids $(f(x,p) \equiv f_{eq}(x,p))$ the moments have simple form:

$$N_B^\mu(x) = n_B(x)\, u^\mu(x) \quad \Big| \quad u \cdot N_B = n_B(x) = 3 \int_{\vec{p}} d^3\bar{p} \left(f_q(\bar{E}) - \bar{f}_q(\bar{E}) \right)$$

$$f_q = \frac{3_c \cdot 2_s \cdot N_f}{e^{(\bar{E}-\mu_q)/T} + 1} \qquad \bar{f}_q = \frac{3_c \cdot 2_s \cdot N_f}{e^{(\bar{E}+\mu_q)} + 1}$$

$$T^{\mu\nu}(x) = e(x)\, u^\mu(x)\, u^\nu(x) - p(x)\, \Delta^{\mu\nu}(x)$$

projection technique

$$e = u_\mu T^{\mu\nu} u_\nu = \int dP (u \cdot p)^2 f(u \cdot p) = \int d^3\bar{p}\, \bar{E} (f + \bar{f} + f_g)$$

$$f_g = \frac{8_c \cdot 2_s}{e^{\bar{E}/T} - 1}$$

$$p = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} = \int \frac{d^3p}{E} \frac{p^2}{3} (f + \bar{f} + f_g)$$

NB: $\boxed{T^{\mu\nu} u_\nu = e u^\mu}$ $\Rightarrow$ u^ν = timelike eigenvector of $T^{\mu\nu}$, with eigenvalue e

Trace of $T^{\mu\nu}$: $\quad T^\mu_\mu(x) = e(x) - 3p(x)$

For a classical system of massless particles: $\boxed{T^\mu_\mu = 0}$
conformal symmetry!

$\Rightarrow$ $\boxed{p = \frac{e}{3}}$ for a conformally invariant system.

(CMB, QGP, early universe where $T \gg m_i$, etc.)

NB Strictly speaking, the ideas of an ideal fluid (requires $g \to \infty$, $\sigma \to \infty$) and of classical on-shell particles are mutually inconsistent. Also, an ideal fluid will always show deviations from conformal symmetry, since interactions break the scale invariance ($T^\mu_\mu \neq 0$, trace "anomaly")
But this only complicates the kinetic microscopic description — macroscopic hydrodynamics, with nonpert. EOS $p(e, n_B)$, remains valid.

3) Ideal fluid equations of motion

$\partial_\mu N^\mu_B = 0$	1 equ.	unknowns:	$n_B(x)$, $e(x)$, $p(x)$ } 3
$\partial_\mu T^{\mu\nu} = 0$	4 equs.		$u^\mu(x)$ } 3
$p = p(e, n_B)$	EoS (1 equ.)		
	6 equs.		6 unknowns ✓

(i) EoM in local rest frame (best for understanding the physics)

Write $\partial_\mu = u_\mu u^\nu \partial_\nu + \Delta_\mu{}^\nu \partial_\nu \equiv u_\mu D + \nabla_\mu$

time derivative in l.r.f.

spatial gradient in l.r.f.

Denote $Df \equiv \dot{f}$

(a) $\partial_\mu N^\mu_B = 0 = \partial_\mu (n_B u^\mu) = u^\mu \partial_\mu n_B + n_B \underbrace{\partial_\mu u^\mu}_{\theta \,=\, \text{local expansion rate}} = \dot{n}_B + n_B \theta$

note: $\partial_\mu u^\mu = \partial \cdot u = \nabla \cdot u = \nabla_\mu u^\mu$

since $\dot{u}_\mu$ is $\perp u^\mu$: $\dot{u}_\mu u^\mu = \frac{1}{2} D(\underbrace{u^\mu u_\mu}_{1}) = 0$

$$\Rightarrow \boxed{\dot{n}_B = - n_B \theta}$$ (similar for any other conserved density!)

n_B only changes because of expansion/contraction of the fluid

(b) $\partial_\mu T^{\mu\nu} = 0 = \partial_\mu \left[(e+p) u^\mu u^\nu - p g^{\mu\nu} \right] = u^\nu \underbrace{u^\mu \partial_\mu}_{D} (e+p) + (e+p)\theta u^\nu$

$+ (e+p) \underbrace{u^\mu \partial_\mu u^\nu}_{\dot{u}^\nu} - \partial^\nu p$

time-like component: project with u_ν:

(A) $D(e+p) + (e+p)\theta + (e+p) \underbrace{u_\nu D u^\nu}_{\frac{1}{2} D(u\cdot u) = 0} - Dp = 0 \Rightarrow \boxed{\dot{e} = -(e+p)\theta}$

energy density changes by expansion only, but faster than n_B, due to work done by pressure

(B) Plug this back in

$$\cancel{u^\nu}(\cancel{\dot e}+\cancel{\dot p}\ +\cancel{(e+p)\theta}) + (e+p)\dot u^\nu \overbrace{-\cancel{u^\nu Dp} - \nabla^\nu p}^{-\partial^\nu p} = 0$$

$$\Rightarrow \boxed{\dot u^\nu = \frac{\nabla^\nu p}{e+p}} \quad \left(\text{Newton's second law: } a = \frac{F}{m}\right)$$

pressure gradients are the driving force for hydrodynamic expansion

For an EOS of type $p = c_s^2 e$ $\quad \left(c_s^2 = \frac{\partial p}{\partial e}\right)$
this reduces to

$$\boxed{\dot u^\nu = \frac{c_s^2}{1+c_s^2}\,\frac{\nabla^\nu e}{e}}$$

$\rightarrow$ scale invariance: magnitude of e doesn't matter.

acceleration driven by c_s^2 ("stiffness" of EOS)

- large c_s^2: fast acceleration, "stiff" EOS
- low c_s^2: slow acceleration, "soft" EOS

Special situation: 1st order phase transition $p(e) = $ const. in mixed phase

$$\Rightarrow \frac{\partial p}{\partial e} = c_s^2 = 0 \quad \text{(sound cannot propagate)}$$

$\rightarrow$ unaccelerated ("self-similar") expansion of mixed phase.

(ii) entropy conservation in ideal fluid dynamics (in the absence of shocks)

- Fundamental law of thermodynamics

$$Ts = p - \mu_B n_B + e$$
$$s = \frac{e+p}{T} - \frac{\mu_B}{T} n_B$$

entropy current: $S^\mu = s u^\mu = \underbrace{(e+p)\beta u^\mu}_{p\beta^\mu + \beta_\nu T^{\mu\nu}} - \alpha n_B$

$\beta = \frac{1}{T}$

$\alpha = \frac{\mu_B}{T}$

$\beta^\mu = \frac{u^\mu}{T}$

$$\Rightarrow \boxed{S^\mu = p(\alpha,\beta)\beta^\mu - \alpha n_B^\mu + \beta_\nu T^{\mu\nu}} \quad \text{(fund. law. th. dyn.)}$$

- Gibbs-Duhem relation

$$dp = s\,dT + n_B\,d\mu$$

- 1^{st} law of thermodynamics

$$ds = \beta\,de - \alpha\,dn_B$$

$\Rightarrow$ entropy production rate

$$\partial_\mu S^\mu = \partial_\mu (s u^\mu) = \dot{s} + s\theta$$

1^{st} law, fund. rel.

$$= \beta\dot{e} - \alpha\dot{n}_B + \left(\beta(e+p) - \alpha n_B\right)\theta = \beta\underbrace{\left(\dot{e} + (e+p)\theta\right)}_{0} - \alpha\underbrace{\left(\dot{n}_B + n_B\theta\right)}_{0} = 0$$

$\Rightarrow$ $\boxed{\partial \cdot S = 0}$ entropy is conserved in a ideal fluid

(iii) Ideal fluid equations in global reference frame:

write $u^\mu = \gamma(1, \vec{v})$ $\quad \vec{v}$ = 3-velocity $\quad \gamma = \frac{1}{\sqrt{1-v^2}}$

$N_B^\mu = n_B u^\mu = \gamma n_B (1, \vec{v}) \equiv R(1, \vec{v})$ $\quad R = N_B^0$ lab frame density

$T^{00} = (e+p)u^0 u^0 - p g^{00} = \gamma^2(e+p) - p \equiv \mathcal{E}$ $\quad$ lab frame energy dens.

$\vec{M} \equiv (T^{01}, T^{02}, T^{03}) = (e+p)u^0\vec{u} = \gamma^2(e+p)\vec{v}$ $\quad$ lab. frame mom. dens.

$$\partial \cdot N_B = 0 = \partial_t R + \vec{\nabla}\cdot(R\vec{v})$$
$$\partial_\mu T^{\mu 0} = 0 = \partial_t \mathcal{E} + \vec{\nabla}\cdot((e+p)\vec{v})$$
$$\partial_\mu T^{\mu i} = 0 = \partial_t M^i + \vec{\nabla}\cdot(M^i\vec{v}) + \partial_i p$$

$\Rightarrow$ Generic form: $\boxed{\partial_t \mathcal{U} + \sum_{j=1}^{3} \partial_j F_j(\mathcal{U}) = 0}$ $\quad (*)$

$$\mathcal{U} = R, \mathcal{E}, \vec{M}$$

Solution of equs. of the form $(*)$ require flux-corrected transport algorithm

We also need EOS $p(e, n_B)$.

To implement it must find e and $\vec{v}$ from $T^{\mu\nu}$:

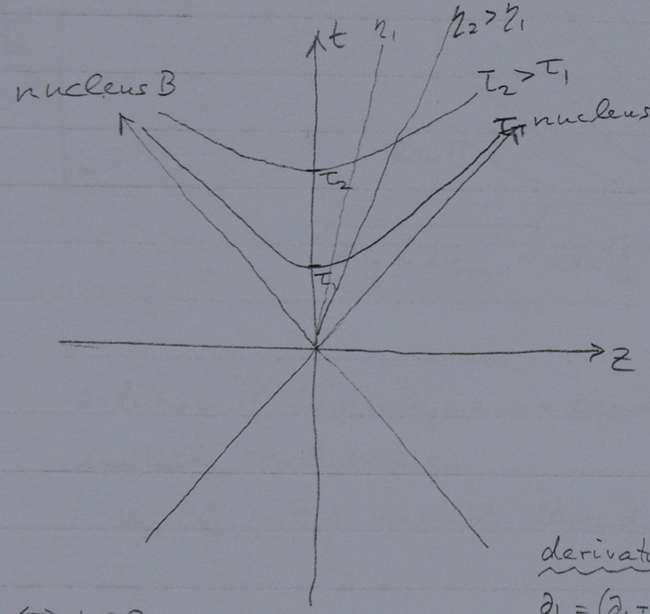
$$\left.\begin{aligned} n_B &= R\sqrt{1-v^2} = \frac{N^0}{\gamma} \\ e &= \mathcal{E} - \vec{M}\cdot\vec{v} \end{aligned}\right\} \text{needs } \vec{v}$$

Since $\vec{M} \parallel \vec{v}$, we only need to compute v and get $\vec{v}$ from $\vec{v} = v\frac{\vec{M}}{M}$

$$\vec{M} = \gamma^2(e+p)\vec{v} = \mathcal{E}\vec{v} + p\vec{v} \Rightarrow M = (\mathcal{E}+p)v \Rightarrow \boxed{v = \frac{M}{\mathcal{E}+p(\underbrace{e,n_B}_{\text{from above}})}}$$

$$\Rightarrow v = \frac{M}{\mathcal{E}+p\left(\mathcal{E}-Mv, R\sqrt{1-v^2}\right)} \Rightarrow \text{1-d zero search for } v.$$

4) A special solution: boost-invariant 1-dimensional expansion (Bjorken solution)



Milne coordinates

$$\left.\begin{aligned} t &= \tau \operatorname{ch}\eta \\ z &= \tau \operatorname{sh}\eta \end{aligned}\right\} \tau^2 = t^2 - z^2 \quad \text{"longitudinal proper time"}$$

$$\eta = \frac{1}{2}\ln\frac{t+z}{t-z} \quad \text{"space-time rapidity"}$$

$$= \frac{1}{2}\ln\frac{1+z/t}{1-z/t}$$

$z=0 \Leftrightarrow \eta = 0$

$z = \pm t \Leftrightarrow \eta = \pm\infty$ (light rays)

derivatives:

$$\partial_t = (\partial_t\tau)\partial_\tau + (\partial_t\eta)\partial_\eta = \operatorname{ch}\eta\,\partial_\tau - \frac{\operatorname{sh}\eta}{\tau}\partial_\eta$$

$$\partial_z = (\partial_z\tau)\partial_\tau + (\partial_z\eta)\partial_\eta = -\operatorname{sh}\eta\,\partial_\tau + \frac{\operatorname{ch}\eta}{\tau}\partial_\eta$$

Under boosts: $\Lambda^\mu{}_\nu(v_L) = \begin{pmatrix} \operatorname{ch}y_L & 0 & 0 & \operatorname{sh}y_L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \operatorname{sh}y_L & 0 & 0 & \operatorname{ch}y_L \end{pmatrix}$ $\quad y_L = \frac{1}{2}\ln\frac{1+v_L}{1-v_L}$

$v_L = \tanh y_L$

$$\tau' = \tau, \quad \eta' = \eta - y_L \quad \text{rapidities are additive!}$$

Matching momentum coordinates:

$E = m_\perp \operatorname{ch}y \qquad m_\perp^2 = E^2 - p_z^2 = m^2 + \vec{p}_\perp^{\,2}$

$p_z = m_\perp \operatorname{sh}y \qquad y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{1+v_z}{1-v_z}$

Under boosts

$m_\perp' = m_\perp$

$y' = y - y_L$

Boost-invariance:

In phase-space:

$$f(\vec{x},\vec{p},t) = f(\vec{x}_\perp,\eta;\vec{p}_\perp,y;t) = f(\vec{x}_\perp,\vec{p}_\perp,y-\eta;\tau)$$

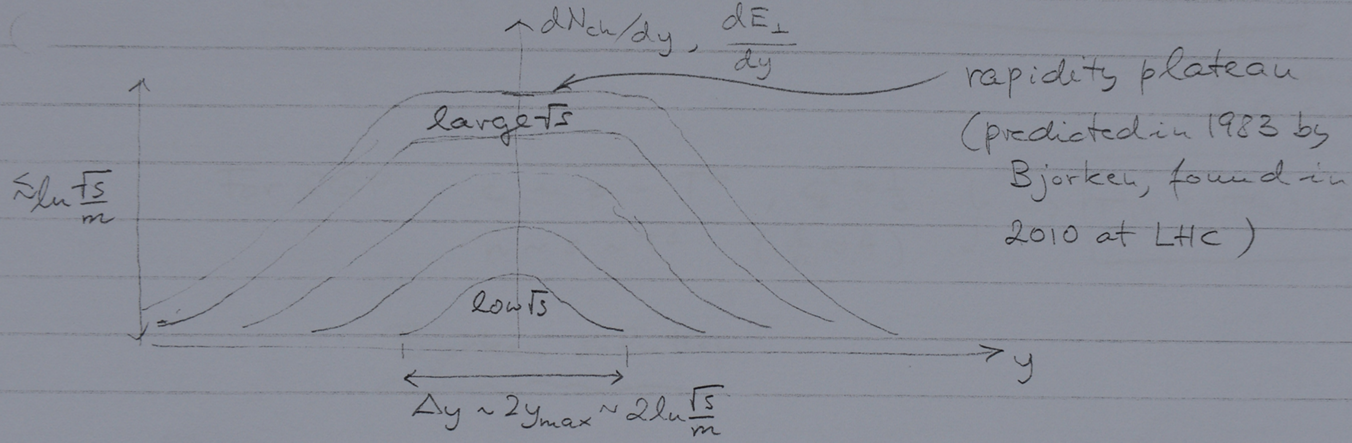
boost-invariant combination

In coordinate space:

$$e(x) = e(\vec{x}_\perp,\eta;\tau) = e(\vec{x}_\perp,\tau) \qquad \text{no } \eta\text{-dependence}$$

Why boost-invariance? Explain Bjorken's idea.

Phenomenological evidence:



Solution of hydrodynamic equations with boost-invariant initial conditions:

$u_\mu(\vec{x}_\perp,\eta,\tau)$ cannot depend on $\eta \Rightarrow \boxed{y_{flow} = \eta}$

$(y_{flow} - \eta = 0 \ \forall \tau)$

$$\boxed{u^\mu = \mathrm{ch}\,y_\perp(\mathrm{ch}\,\eta, v_x, v_y, \mathrm{sh}\,\eta)} \quad \text{where } \mathrm{th}\,y_\perp(\tau,\vec{x}_\perp) = v_\perp(\tau,\vec{x}_\perp,\eta=0)$$

general boost-invariant form

In Milne coordinates: $u^m \equiv (u^\tau, u^x, u^y, u^\eta) = \mathrm{ch}\,y_\perp(1, v_x, v_y, 0)$

$\Rightarrow u^\eta = 0 = \text{const},$

$\boxed{\dot{u}^\eta = 0}$ (since $\nabla_\eta p = 0$)

$\Rightarrow$ this reduces the dimensionality of the problem by 1.

If we ignore also transverse expansion
(nuclei = infinitely large, transversally homogeneous discs)
$v_x = v_y = 0$

$$\Rightarrow \quad \theta = \partial_\mu u^\mu = \frac{1}{\tau} \quad , \quad D = u^\mu \partial_\mu = \frac{d}{d\tau}$$

$$\Rightarrow \quad \frac{dn_B}{d\tau} = -\frac{n}{\tau} \quad \longrightarrow \quad \boxed{n_B(\tau) = n_B(\tau_0)\,\frac{\tau_0}{\tau}}$$

linear growth of volume

$$\frac{ds}{d\tau} = -\frac{s}{\tau} \quad \Rightarrow \quad \boxed{s(\tau) = s(\tau_0)\,\frac{\tau}{\tau_0}}$$

$$\frac{de}{d\tau} = -\frac{e+p}{\tau} = -(1+c_s^2)\frac{e}{\tau} \quad \Rightarrow \quad \boxed{e(\tau) = e(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}}$$

work done
by long. pressure

For QGP
$$\left.\begin{array}{ll} e \sim p \sim T^4 , & c_s^2 \approx \frac{1}{3} \\ n \sim s \sim T^3 & \left(\frac{s}{n} \approx 4\right) \end{array}\right\} \Rightarrow \boxed{T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1/3}}$$

Bjorken solution