5) Viscous fluid dynamics

Deviations from local equilibrium:

(i) $f(x,p) = f_{eq}(x,p) + \delta f(x,p) = f_{eq}\left[1 + (1 \pm f_{eq})\,\phi(x,p)\right]$

Expand δf in powers of gradients of the equilibrium parameters
$T(x)$, $\mu_B(x)$, $u^\mu(x)$

To make the expansion unique, need to find optimal local equilibrium parameters for $f_{eq}(x,p)$: $T(x)$, $\mu_B(x)$, $u^\mu(x)$

$\rightarrow$ Landau matching:

$$\delta n_B \equiv u_\mu \delta N_B^\mu = \int dP\,(u\cdot p)\left(\delta f(x,p) - \delta \bar{f}(x,p)\right) = 0 \qquad \Longrightarrow \frac{\mu_B}{T}(x)$$

$$\delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = \int dP\,(u\cdot p)^2\left(\delta f + \delta \bar{f} + \delta f_g\right) = 0 \qquad \Longrightarrow T(x)$$

fixing the local rest frame:

Landau frame: u^μ = time like eigenvector of $T^{\mu\nu}$: $T^{\mu\nu}u_\nu = e u^\nu$
$\Longleftrightarrow \boxed{u_\nu \delta T^{\mu\nu} = 0}$

Eckart frame: $u^\mu = \dfrac{N_B^\mu}{\sqrt{N_B \cdot N_B}} \Longleftrightarrow \boxed{\delta N_B^\mu = 0}$

We will use the Landau frame.

With $\delta f \neq 0$, we get from kinetic theory

$$N_B^\mu = \int dP\, p^\mu \left(f_{eq} - \bar{f}_{eq} + \delta f - \delta \bar{f}\right) = n_B u^\mu + V^\mu$$

$V^\mu = \Delta^{\mu\nu} N_{B\nu}$ = net baryon flow in l.r.f.

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu \left(\left(f_{eq} + \bar{f}_{eq} + f_{g,eq}\right) + \left(\delta f + \delta \bar{f} + \delta f_g\right)\right)$$
$$= e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + \left(W^\mu u^\nu + W^\nu u^\mu\right)$$

Π = bulk viscous pressure; $\pi^{\mu\nu}$ = shear stress tensor; W^μ = momentum flow in l.r.f.

Using projection we get $\quad (p^\mu = p\cdot u\, u^\mu + \Delta^{\mu\nu} p_\nu \equiv \bar{E} u^\mu + \bar{p}^\mu$

$\bar{p}^\mu \bar{p}_\mu = -\vec{\bar{p}}^{\,2}$

$$V^\mu = \Delta^{\mu\nu} N_{B\nu} = \int dP\, \bar{p}^\mu (\delta f - \delta \bar{f}) \xrightarrow[\text{lrf}]{} \left(0, \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\vec{\bar{p}}}{\bar{E}} (\delta f - \delta \bar{f})\right)$$

$$p + \Pi = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} \Rightarrow \Pi = -\frac{1}{3} \int dP\, p\cdot\Delta\cdot p\, (\delta f + \delta \bar{f} + \delta f_g) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\vec{\bar{p}}^{\,2}}{3\bar{E}} (\delta f + \delta \bar{f} + \delta f_g)$$

$$W^\mu = \Delta^{\mu\nu} T_{\nu\alpha} u^\alpha = \int dP\, (p\cdot u)\, \Delta^{\mu\nu} p_\nu (\delta f + \delta \bar{f} + \delta f_g) \xrightarrow[\text{lrf}]{} \left(0, \int \frac{d^3\bar{p}}{(2\pi)^3} \vec{\bar{p}} (\delta f + \delta \bar{f} + \delta f_g)\right)$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \equiv \Big[\underbrace{\left(\tfrac{1}{2} \Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha\right)}_{\text{transverse to } u^\mu, u^\nu} \underbrace{- \tfrac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta}}_{\text{traceless}}\Big] T^{\alpha\beta}$$

$$= \int dP \left(\bar{p}^\mu \bar{p}^\nu - \tfrac{1}{3} \Delta^{\mu\nu} \bar{p}^2\right)(\delta f + \delta \bar{f} + \delta f_g) \xrightarrow[\text{lrf}]{} \int \frac{d^3\bar{p}}{(2\pi)^3} \left(\bar{p}^i \bar{p}^j - \tfrac{1}{3} \delta^{ij} \vec{\bar{p}}^{\,2}\right)(\delta f + \delta \bar{f} + \delta f_g)$$

for $i,j = 1,2,3$ (0 else)

(no time components in lrf)
traceless

e, n_B and p still are the same integrals of f_{eq} as before — they receive no contributions from δf.

In the Landau frame $W^\mu = 0$ $\quad$ In this frame one writes $V^\mu = -\frac{n_B}{e+p} q^\mu$ with the "heat flow vector" q^μ

For a baryon-free ($n_B = 0$) system we expect $\delta f = \delta \bar{f}$ and thus $V^\mu = 0$.

Henceforth consider this case ($n_B = V^\mu = 0$).

NB: This does not imply $\delta N^\mu = \int dP\, \bar{p}^\mu \delta f = \delta \bar{N}^\mu$, though.

So we expect in general $\delta N^\mu, \delta \bar{N}^\mu, \delta N_g^\mu \neq 0$.

Viscous fluid equations in comoving coordinates:

Using similar techniques as in ideal fluid case, we find the viscous fluid equations in the Landau frame

$$\boxed{\begin{aligned} \dot{e} &= -(e + p + \Pi)\theta + \pi_{\mu\nu} \sigma^{\mu\nu} \\ (e + p + \Pi)\, \dot{u}^\mu &= \nabla^\mu (p + \Pi) - \Delta^{\mu\nu} \nabla^\sigma \pi_{\nu\sigma} + \pi^{\mu\nu} \dot{u}_\nu \end{aligned}}$$

$\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta \equiv \nabla^{\langle\mu} u^{\nu\rangle}$
velocity shear tensor

Still the same number of equation, but now we have 6 additional unknowns $\underbrace{\Pi}_{1}, \underbrace{\pi^{\mu\nu}}_{10-4-1=5}$!

We need additional equations of motion for $\Pi, \pi^{\mu\nu}$.

Several approaches: (a) Chapman - Enskog: linearize C in terms of δf and solve Boltzmann eq. for δf $\rightarrow$ integro-differential equation

(b) Grad 14-moment method: expand δf as $\sim \varepsilon(x) + \xi_\mu(x) p^\mu + \xi_{\mu\nu}(x) p^\mu p^\nu$ and use this expansion to solve the integro-differential equation

(c) Denicol - Molnar - Niemi - Rischke higher-order moment method $\rightarrow$ generalization of Grad method (PRD 85 (2012) 114047)

(perhaps the most systematic approach)

(d) Israel - Stewart: purely macroscopic approach, using the second law of thermodynamics

(e) Navier - Stokes: macroscopic approach that ignores microscopic relaxation times

(this causes serious problems in a relativistic thy.)

Here I will use the Israel - Stewart approach which also yields the Navier - Stokes approach in a specific limit.

Reference: W. Israel, Ann. Phys. 100 (1976) 310

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 349

Start from $\partial_\mu S^\mu \geq 0$ (2nd law of thermodynamics)

Remember the equilibrium identity

$$S^\mu_{eq} = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_{B,eq} + \beta_\nu T^{\nu\mu}_{eq} \qquad \alpha = \frac{\mu_B}{T}, \quad \beta^\mu = \beta u^\mu = \frac{u^\mu}{T}$$

In near-equilibrium we write

$$S^\mu = S^\mu_{eq} + \Phi^\mu = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_B + \beta_\nu T^{\mu\nu} + Q^\mu(\delta N^\mu_B, \delta T^{\mu\nu})$$

The terms $-\alpha\,\delta N^\mu_B + \beta_\nu \delta T^{\mu\nu}$ contribute linear terms in $q^\mu, \pi^{\mu\nu}, \Pi$; Q^μ contributes higher order terms

We can rewrite the Gibbs-Duhem relation from p. 9,

$$dp = s\,dT + n_B\,d\mu$$

(using thermodyn. identities in equilibrium) as

$$\partial_\mu\left(p(\alpha,\beta)\beta^\mu\right) = N^\mu_{B,eq}\,\partial_\mu\alpha - T^{\mu\nu}_{eq}\,\partial_\mu\beta_\nu$$

to find (using $\partial_\mu N^\mu_B = 0 = \partial_\mu T^{\mu\nu}$)

$$\boxed{\partial_\mu S^\mu = -\delta N^\mu_B\,\partial_\mu\alpha + \delta T^{\mu\nu}\,\partial_\mu\beta_\nu + \partial_\mu Q^\mu}$$

Using the decomposition of δN^μ_B and $\delta T^{\mu\nu}$ in terms of $q^\mu, \Pi, \pi^{\mu\nu}$, we can rewrite this as

$$\boxed{T\,\partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\mu\nu} + T\,\partial_\mu Q^\mu}$$

where we introduced the thermodynamic forces

$$X \equiv -\theta = -\nabla\cdot u \qquad \text{scalar force}$$

$$X_\mu \equiv \frac{\nabla_\mu T}{T} - \dot{u}_\mu = -\frac{n_B T}{e+p}\nabla_\mu\left(\frac{\mu_B}{T}\right) \qquad \text{vector force}$$

$$X_{\mu\nu} \equiv \sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle} \qquad \text{tensor force}$$

The second law of thermodynamics now requires

$$\boxed{\Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\mu\nu} + \partial_\mu Q^\mu \geq 0}$$

(A) Relativistic Navier-Stokes theory

(See Landau-Lifshitz, Fluid Dynamics)

Ignore second-order terms, set $Q^\mu = 0$

$\Rightarrow$ 2nd law is automatically satisfied if we demand

$$\boxed{\begin{aligned} \Pi &= -\zeta\,\theta && \zeta = \text{bulk viscosity} \\ q^\nu &= -\kappa\,\frac{nT^2}{e+P}\,\nabla^\nu\!\left(\frac{\mu_B}{T}\right) && \kappa = \text{heat conductivity} \\ \pi^{\mu\nu} &= 2\eta\,\sigma^{\mu\nu} && \eta = \text{shear viscosity} \end{aligned}} \qquad (*)$$

such that $\partial_\mu S^\mu = \dfrac{\Pi^2}{\zeta} - \dfrac{q^\alpha q_\alpha}{2\kappa T} + \dfrac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0$

(remember: q^μ is spacelike!)

problem: the instantaneous response $(*)$ of δN_B^μ, $\delta T^{\mu\nu}$ to the thermodynamic forces is <u>acausal</u>!

$\Rightarrow$ superluminal signal propagation, unstable solutions of viscous hydro EoM ↯

(B) Causal viscous relativistic fluid dynamics (Israel-Stewart theory)

keep Q^μ, parametrize it to second order in $\delta T^{\mu\nu}$

ignoring q^μ again, this is done as follows

$$\boxed{Q^\mu = -\left(\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T}}$$

Then

$$T\,\partial\cdot S = \Pi\left[-\theta - \beta\dot{\Pi} - \Pi T\,\partial_\mu\!\left(\frac{\beta_0 u^\mu}{2T}\right)\right] + \pi^{\alpha\beta}\left[\sigma_{\alpha\beta} - \beta_2\dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta}\,T\,\partial_\mu\!\left(\frac{\beta_2 u^\mu}{2T}\right)\right.$$

This is positive definite if we set the expressions in [...] proportional to Π, $\pi_{\alpha\beta}$ such that $\partial \cdot S$ looks like in NS theory.

This gives

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right]$$

$$\Delta^{\alpha\beta}_{\mu\nu}\dot{\pi}^{\mu\nu} = -\frac{1}{\tau_\pi}\left[\pi^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + \pi^{\alpha\beta}\eta T\partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)\right]$$

$\Rightarrow$ dynamical relaxation equations for Π, $\pi^{\mu\nu}$ with relaxation times $\tau_\Pi = \zeta\beta_0$ and $\tau_\pi = 2\eta\beta_2$

$\Rightarrow$ Causality restored!

Using $\gamma_\Pi = T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)$, $\gamma_\pi = T\partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)$

this can be rewritten as

$$\boxed{\begin{aligned} \dot{\Pi} &= -\frac{1}{\tau_\Pi'}\left[\Pi + \zeta'\theta\right] \\ \Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} &= -\frac{1}{\tau_\pi'}\left[\pi^{\mu\nu} - 2\eta'\sigma^{\mu\nu}\right] \\ \text{where}\quad \tau_\Pi' &= \frac{\tau_\Pi}{1+\zeta\gamma_\Pi} \qquad \tau_\pi' = \frac{\tau_\pi}{1+\eta\gamma_\pi} \\ \zeta' &= \frac{\zeta}{1+\zeta\gamma_\Pi} \qquad \eta' = \frac{\eta}{1+\eta\gamma_\pi} \end{aligned}} \qquad (✡)$$

$\rightarrow$ fast expansion (γ_Π, γ_π large) effective reduces the viscosities and relaxation times

✡ must be solved together with the viscous hydro equations.

For a conformal system $\boxed{\zeta\gamma_\Pi = \eta\gamma_\pi = \frac{4}{3}\theta\tau_\pi}$

6) Anisotropic viscous fluid dynamics

Different expansion scheme:

$$f(x,p) = f_{eq}(x,p) + \delta f(x,p)$$

$$= f_{RS}(x,p;\xi,\Lambda) + \delta\tilde{f}(x,p)$$

where $f_{RS}(x,p;\xi(x),\Lambda(x)) = f_{iso}\left(\frac{\sqrt{\vec{p}^{\,2}+m^2+\xi(x)p_z^2}}{\Lambda(x)}\right)$

f_{iso} is an isotropic thermal "energy" distribution with effective temperature Λ, but now evaluated for an "energy" evaluated anisotropically from the momenta

$\xi > 0$: $P_L < P_T$

$\xi = 0$: $P_L = P_T$ isotropic case

$\xi < 0$: $P_L > P_\perp$

$$\xi = \frac{\langle p_T^2\rangle}{2\langle p_L^2\rangle} - 1$$

This is an anisotropic distribution that captures deviations from isotropic local equilibrium that correspond to a very specific subclass of shear viscous effects.

By comparing with Navier-Stokes theory one finds

$$\xi = \frac{10}{T\tau}\frac{\eta}{s} \quad \text{for small } \eta/s$$

$$\simeq \frac{10}{T_0(\tau/\tau_0)^{2/3}}\frac{\eta}{s} \quad \text{for Bjorken expansion}$$

(deformation decreases with time)

By Landau matching one obtains for massless partons

$$T(\Lambda,\xi) = R^{1/4}(\xi)\,\Lambda$$

isotropic distr. with temperature Λ

$$e(\Lambda,\xi) = R(\xi)\,e(\Lambda)$$

$$n(\Lambda,\xi) = \frac{n(\Lambda)}{\sqrt{1+\xi}}$$

$$p_\perp(\Lambda,\xi) = \frac{1}{2}\left(T^{xx}+T^{yy}\right) = R_\perp(\xi)\,p(\Lambda)$$

$$p_L(\Lambda,\xi) = T^{zz} = R_L(\xi)\,p(\Lambda)$$

where $R(\xi) = \frac{1}{2}\left(\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$

$$R_\perp(\xi) = \frac{3}{2\xi}\left(\frac{1+(\xi^2-1)R(\xi)}{\xi+1}\right)$$

$$R_L(\xi) = \frac{3}{\xi}\left(\frac{(\xi+1)R(\xi)-1}{\xi+1}\right)$$

are known analytically

The anisotropic distribution function $f_{RS}\left(\frac{\sqrt{\vec{p}^{\,2}+\xi p_z^2}}{\Lambda}\right)$ gives rise to an energy-momentum tensor

$$T^{\mu\nu}_{RS} = e(\Lambda,\xi)\,u^\mu u^\nu - p_\perp(\Lambda,\xi)\,\Delta^{\mu\nu} + \left(p_L(\Lambda,\xi)-p_\perp(\Lambda,\xi)\right)z^\mu z^\nu$$

where $z^\mu \xrightarrow[\text{e.r.f}]{} (0,0,0,1)$

We can write this as

$$T^{\mu\nu}_{RS} = T^{\mu\nu}_{ideal} + \pi^{\mu\nu}_{RS} = e\,u^\mu u^\nu - p\,\Delta^{\mu\nu} + \underbrace{+(p-p_\perp)\Delta^{\mu\nu} + (p_L-p_\perp)z^\mu z^\nu}_{\pi^{\mu\nu}_{RS}}$$

In addition to $\pi^{\mu\nu}_{RS}$, $\delta\tilde{f}$ will generate an additional $\tilde{\pi}^{\mu\nu}$ such that

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{RS} + \tilde{\pi}^{\mu\nu}$$

gives the complete viscous shear stress tensor

(If the particles are not massless, there will also be viscous bulk pressure contributions Π_{RS} and $\tilde{\Pi}$.)

The difference between Israel-Stewart and anisotropic hydro is how one describes the evolution of $\pi^{\mu\nu}_{RS}$.

In anisotropic hydrodynamics, one uses the Boltzmann equation in relaxation-time form,

$$p^\mu \partial_\mu f(x,p) = -\frac{p\cdot u}{\tau_{relax}}\left(f(x,p) - f_{eq}(x,p)\right)$$

and then uses the 0th and 1st moment of this equation to derive equations of motion for the deformation and effective temperature parameters

$$\xi(\vec{x}_\perp, \eta, \tau) \quad , \quad \Lambda(\vec{x}_\perp, \eta, \tau)$$

together with (viscous) hydrodynamic equations for the flow velocity.

The solutions for ξ, Λ obtained this way appear to account for the viscous dynamics associated with the large initial $p_L/p_\perp$ anisotropy more efficiently than the IS equations.