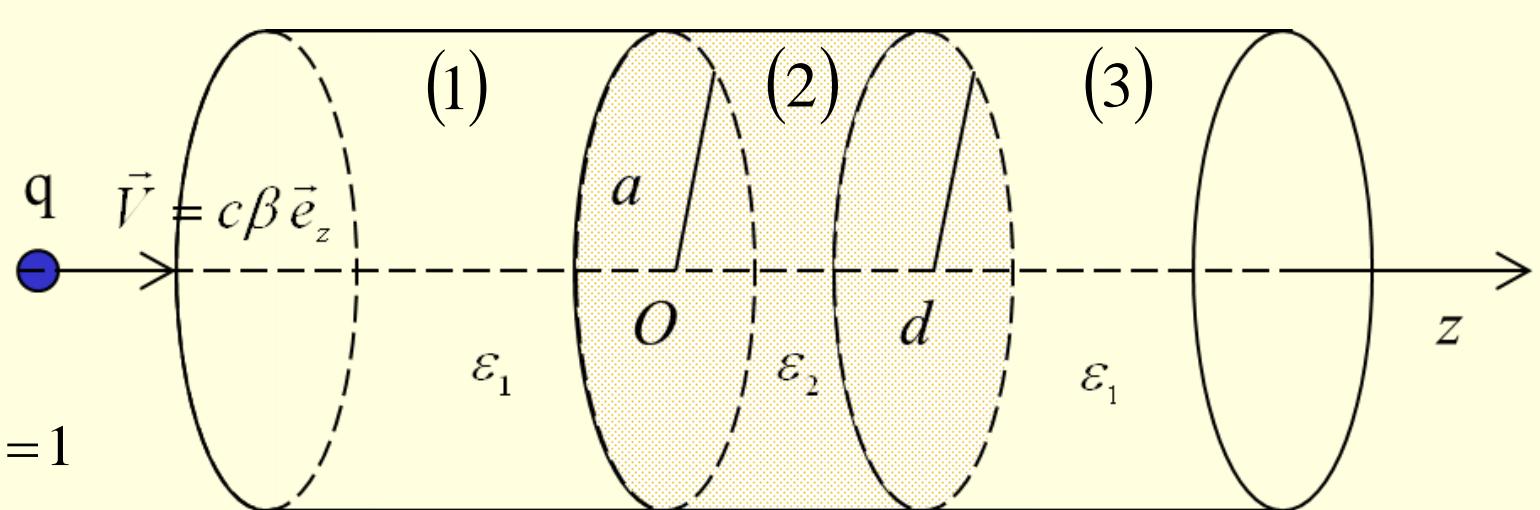


Electromagnetic Field of a Charge Intersecting Boundary Surfaces in a Waveguide

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FORMULATION OF THE PROBLEM



$$\vec{E}_j = \vec{E}_j^q + \vec{E}_j^b, \quad j = 1, 2, 3 \quad (1)$$

The “forced” field (called by V.L. Ginzburg [*]) is the field of the charge in a regular waveguide. It contains **Cherenkov radiation (CR)** if the charge velocity exceeds the Cherenkov threshold $\beta > \beta_{C1,2} = 1/\sqrt{\epsilon_{1,2}}$.

* V.L. Ginzburg and V.N. Tsytovich, “Transition Radiation and Transition Scattering”, Hilger, London, p. 445 (1990).

The “free” field is connected with the influence of the boundaries. It includes **transition radiation (TR)**.

$$E_{zj}^b = \frac{2qc^2}{\pi a^4 \varepsilon_j} \sum_{n=1}^{\infty} \frac{\chi_n^2 J_0\left(\frac{\chi_n r}{a}\right)}{J_1^2(\chi_n)} I_{zj}^b, \quad j=1,2,3, \quad \varepsilon_3 \equiv \varepsilon_1 \quad (2)$$

$$I_{zj}^b = \frac{i\beta(\varepsilon_2 - \varepsilon_1)}{(1 - \beta^2 \varepsilon_1)(1 - \beta^2 \varepsilon_2)} \int_{-\infty}^{+\infty} \exp(-i\omega t) [A_{nj} \exp(i k_{zj} z) + B_{nj} \exp(-i k_{zj} z)] d\omega, \quad (3)$$

$$A_{n1} = 0, \quad (4a)$$

$$B_{n1} = \frac{1}{q_n \Delta_n} \left(g_n^+ r_n^+ e^{-ik_{z2}d} + g_n^- r_n^- e^{ik_{z2}d} - 2\varepsilon_1 \sqrt{\omega^2 \varepsilon_2 - \omega_n^2} p_n^+ e^{i\omega d / \beta c} \right), \quad (4b)$$

$$A_{n2} = \frac{1}{q_n \Delta_n} \left(-g_n^- p_n^+ e^{i\omega d / \beta c} + g_n^+ p_n^- e^{-ik_{z2}d} \right), \quad (5a)$$

$$B_{n2} = \frac{1}{q_n \Delta_n} \left(-g_n^+ p_n^+ e^{i\omega d / \beta c} + g_n^- p_n^- e^{ik_{z2}d} \right), \quad (5b)$$

$$k_{z1,2} = \frac{1}{c} \sqrt{\omega^2 \varepsilon_{1,2} - \omega_n^2}, \quad \text{Im } k_{z1,2} > 0 \quad \omega_n = \frac{\chi_n c}{a}, \quad (6)$$

χ_n is the n^{th} zero of the Bessel function ($J_0(\chi_n) = 0$)



$$B_{n3} = 0, \quad A_{n3} = \frac{1}{q_n \Delta_n} \left(\frac{-g_n^+ r_n^- e^{-ik_{z2}d + i\omega d/\beta c} - g_n^- r_n^+ e^{ik_{z2}d + i\omega d/\beta c}}{+ 2\varepsilon_1 \sqrt{\omega^2 \varepsilon_2 - \omega_n^2} p_n^-} + \right) e^{-ik_{z1}d}, \quad (7)$$

$$g_n^\pm = \varepsilon_1 \sqrt{\omega^2 \varepsilon_2 - \omega_n^2} \pm \varepsilon_2 \sqrt{\omega^2 \varepsilon_1 - \omega_n^2} \quad (8)$$

$$\Delta_n = (g_n^+)^2 e^{-ik_{z2}d} - (g_n^-)^2 e^{ik_{z2}d}, \quad (9)$$

$$q_n = \left(\omega^2 + \frac{\beta^2 \omega_n^2}{1 - \beta^2 \varepsilon_1} \right) \left(\omega^2 + \frac{\beta^2 \omega_n^2}{1 - \beta^2 \varepsilon_2} \right), \quad (10)$$

$$\alpha_n = \omega^2 (1 - \beta^2 \varepsilon_1 - \beta^2 \varepsilon_2) + \beta^2 \omega_n^2,$$

$$p_n^\pm = \alpha_n \pm \omega \beta^3 \varepsilon_2 \sqrt{\omega^2 \varepsilon_1 - \omega_n^2}, \quad r_n^\pm = \alpha_n \pm \omega \beta^3 \varepsilon_1 \sqrt{\omega^2 \varepsilon_2 - \omega_n^2}. \quad (11)$$

We analyze the electromagnetic field (EMF) of the charge moving uniformly and traversing a waveguide discontinuity:

a dielectric plate in a vacuum
and a vacuum cavity in a dielectric.

The case of dielectric plate situated in a waveguide was partially considered in [*].

* K.A. Barsukov, Zh. Tekh. Fiz. 30, 9 (1960) 1337.

Methods of analysis

We investigate the exact solution with analytical and computational methods.

Analytical research into the field components of each mode is carried out with methods of the complex variable function theory.

Computations are based on original algorithm using some separation of the integration path and the singularities of the integrands. As distinct from our previous works

T.Yu. Alekhina and A.V. Tyukhtin, Phys. Rev. E 83 (2011) 066401.

T.Yu. Alekhina and A.V. Tyukhtin, Journal of Physics: Conference Series 357, 012010 (2012).

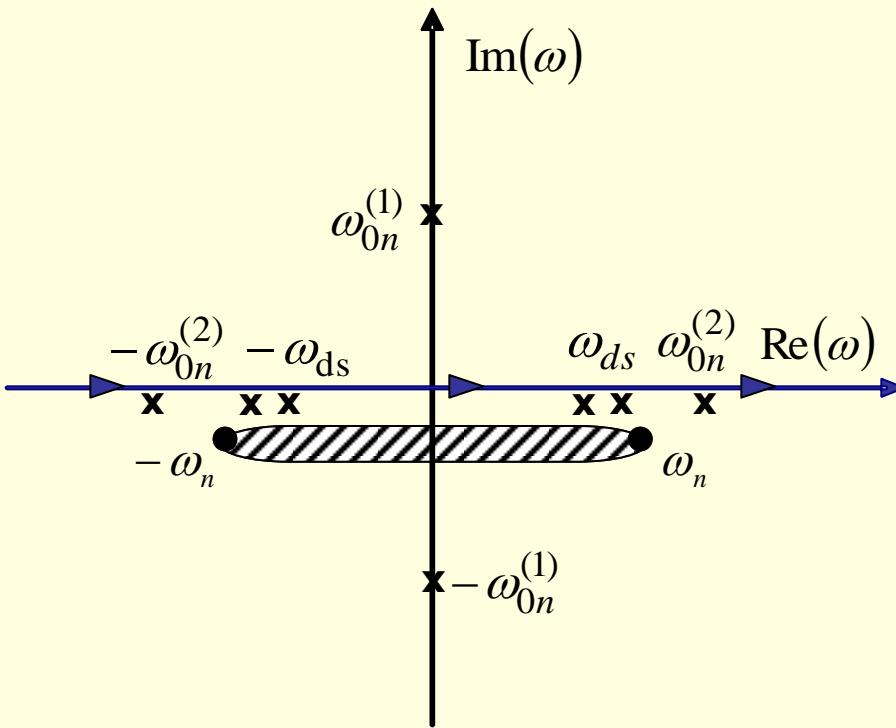
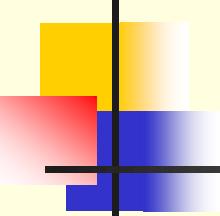
T.Yu. Alekhina and A.V. Tyukhtin, Phys. Rev. ST – AB 15, 091302 (2012).

T.Yu. Alekhina and A.V. Tyukhtin, Phys. Rev. ST – AB 16, 081301 (2013).

we do not transform the initial integration contour (a real axis) but displace the singularities from the integration path by taking into account small absorption in the media.



Medium 3: analysis of A_{n3} in a complex plane of (ω) at $\beta > \beta_{c2}$



Branch points : $\pm \tilde{\omega}_n^{(1)} = \pm \omega_n - i\delta_1$

Branch cut : $\text{Re} \sqrt{\omega^2 - (\tilde{\omega}_n^{(1)})^2} = 0$ F. (10)

Poles: $\pm \omega_{0n}^{(1)} = \pm \frac{i\beta\omega_n}{\sqrt{1-\beta^2}}, \quad q_n|_{\omega_{0n}^{(1),(2)}} = 0$

$$\pm \omega_{0n}^{(2)} = \pm \frac{i\beta\omega_n}{\sqrt{1-\varepsilon_2\beta^2}}, \quad \beta < \frac{1}{\sqrt{\varepsilon_2}}$$

$$\pm \omega_{0n}^{(2)} = \pm \frac{\beta\omega_n}{\sqrt{\varepsilon_2\beta^2 - 1}} - i\delta_2, \quad \beta > \frac{1}{\sqrt{\varepsilon_2}} \quad \text{F. (9)}$$

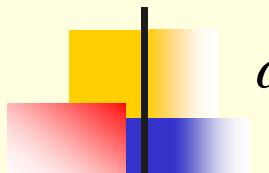
$$\pm \omega_{dj}, \quad j = 1, \dots, s, \quad \Delta_n|_{\omega_{dj}} = 0$$

A dielectric plate in a vacuum

$$\varepsilon_1 = 1, \varepsilon_2 > 1$$

Poles ω_{dj} :

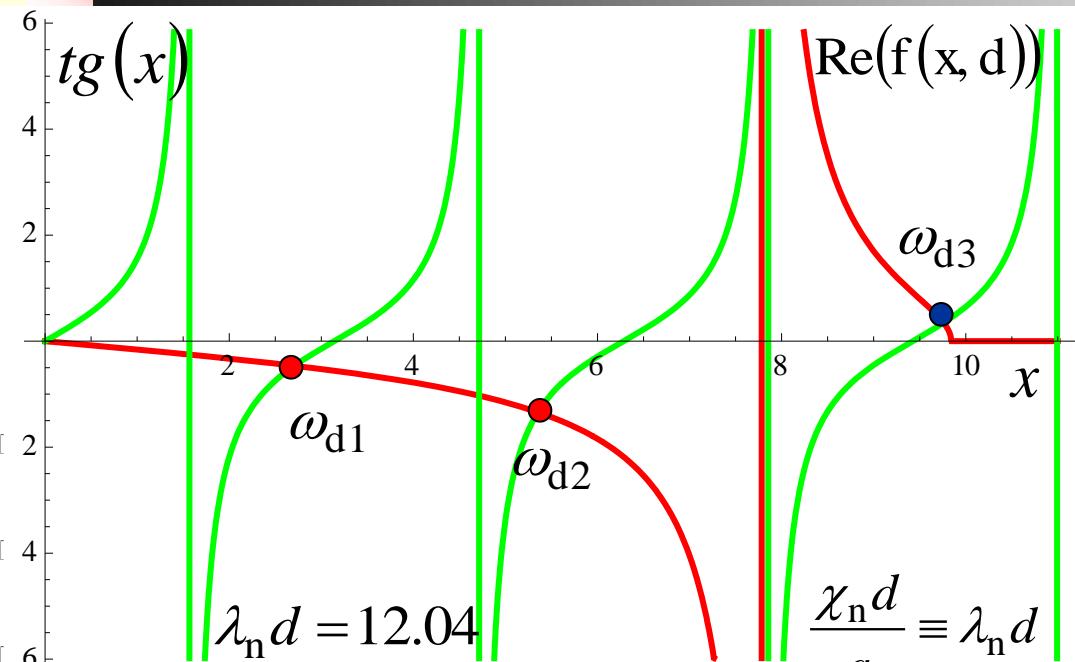
$$\omega_n = \frac{\chi_n c}{a},$$



F. (9)

$$\Delta_n = \operatorname{tg}(x) - f(x, d) \Big|_{\omega_{dj}} = 0, \quad (12a)$$

$$f(x, d) = \frac{2\varepsilon_1 \sqrt{\varepsilon_2} x \sqrt{(\lambda_n d)^2 (\varepsilon_2 - \varepsilon_1) - x^2 \varepsilon_1}}{x^2 \varepsilon_1 (\varepsilon_1 + \varepsilon_2) - \varepsilon_2 (\lambda_n d)^2 (\varepsilon_2 - \varepsilon_1)}, \quad (12b)$$



$$\varepsilon_1 = 1, \varepsilon_2 = 1.67, \quad n = 1, d/a = 5,$$

$$k_{z2}d = \frac{d}{c} \sqrt{\omega^2 \varepsilon_2 - \omega_n^2} \equiv x, \quad (13)$$

$$\pi k < \lambda_n d \sqrt{(\varepsilon_2 - 1)} < \pi \frac{2k+1}{2}, \quad (14a)$$

An upper branch of roots (blue)

$$\pi \frac{2l-1}{2} < \lambda_n d \sqrt{\frac{\varepsilon_2(\varepsilon_2+1)}{(\varepsilon_2-1)}} < \pi l, \quad (14b)$$

A lower branch of roots (red)

$$\frac{\omega_n}{\sqrt{\varepsilon_2}} \leq |\omega_{dj}| < \omega_n, \quad j = 1, \dots, s = 1 + k \quad (15)$$

The poles ω_{dj} are connected with the finite length of the plate. Contributions of ω_{dj} to the field give stationary waves inside the plate and the field exponentially decreasing with the distance from the boundary outside the plate.

The field in medium 3

$$\beta_{C2} = \frac{1}{\sqrt{\varepsilon_2}}, \quad \beta_{CT2} = \frac{1}{\sqrt{\varepsilon_2 - 1}}$$

Contributions of poles $\pm \omega_{0n}^{(2)}$ give **Cherenkov transition radiation (CTR)**. CTR exists at $\beta_{C2} < \beta < \beta_{CT2}$ in the area $z_3 \leq z \leq z_1$, :

$$E_{z3}^{CTR} = -\frac{4q}{a^2} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_n r}{a}\right)}{J_1^2(\chi_n)} \operatorname{Re} \left[\frac{\left(1 + \varepsilon_2 \sqrt{1 - \beta^2(\varepsilon_2 - 1)}\right) \exp\left[i\kappa_n((z-d)\sqrt{1 - \beta^2(\varepsilon_2 - 1)} - \beta ct)\right]}{2\varepsilon_2 \sqrt{1 - \beta^2(\varepsilon_2 - 1)} \cos(\kappa_n d) + i \sin(\kappa_n d) (\beta^2(\varepsilon_2 - 1)\varepsilon_2^2 - \varepsilon_2^2 - 1)} \right], \quad (16)$$

$$\kappa_n = \frac{\lambda_n}{\sqrt{\varepsilon_2 \beta^2 - 1}} = \frac{\chi_n}{a \sqrt{\varepsilon_2 \beta^2 - 1}}$$

$$z_1(t) = d + v_{g1} \left(t - \frac{d}{\beta c} \right), \quad t > \frac{d}{\beta c}, \quad v_{g1} = \frac{c \sqrt{1 - \beta^2(\varepsilon_2 - 1)}}{\beta} \quad (17)$$

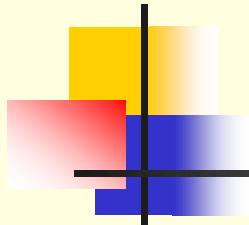
T.Yu. Alekhina and A.V. Tyukhtin, Phys. Rev. ST – AB 15, 091302 (2012).

$$z_3(t) = d + v_{g1} \left(t - \frac{d}{v_{g2}} \right), \quad t > \frac{d}{v_{g2}}, \quad v_{g2} = \frac{c}{\beta \varepsilon_2} \quad (18)$$

T.Yu. Alekhina and A.V. Tyukhtin, Journal of Physics: Conference Series 357, 012010 (2012).



A CTR train in medium 3



$$\beta > \frac{1}{\sqrt{\varepsilon_2}}$$

$$\Delta z^{\text{CTR}} = z_1(t) - z_3(t) = \frac{d(\beta^2 \varepsilon_2 - 1)}{\beta^2} \sqrt{1 - \beta^2(\varepsilon_2 - 1)}, \quad (19)$$

$$\Delta z^{\text{CTR}} \Big|_{\beta_0} - \max, \quad \beta_0 = \sqrt{\frac{\sqrt{1 + 8\varepsilon_2(\varepsilon_2 - 1)^{-1}} - 1}{2\varepsilon_2}}, \quad \beta_0 \Big|_{\varepsilon_2 = \varepsilon_{20}} = 1.034,$$

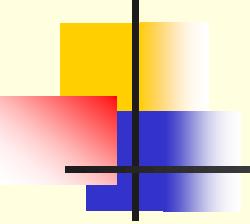
$$\Delta z^{\text{CTR}} \Big|_{\varepsilon_{20}} - \max, \quad \varepsilon_{20} = \frac{3 + 2\beta^2}{3\beta^2}, \quad \varepsilon_{20} \Big|_{\beta \rightarrow 1} \approx 1.67,$$

$$\max \Delta z^{\text{CTR}} \Big|_{\substack{\varepsilon_{20} \\ \beta \rightarrow 1}} = 0.39 d, \quad (20)$$

The CTR train has **finite length** that does not depend on time but only on the parameters of the problem ε_2 and β .

For the optimal parameters the maximum length of the CTR is always less than the length of the dielectric plate.

Numerical approach


$$E_{zj}^b = \frac{2qc^2}{\pi a^4 \varepsilon_j} \sum_{n=1}^{\infty} \frac{\chi_n^2 J_0\left(\frac{\chi_n r}{a}\right)}{J_1^2(\chi_n)} I_{zj}^b, \quad j=1,2,3,$$

$$I_{zjs}^b = \int_{-\infty}^{+\infty} f_{js}(\omega) \exp[F_{js}(\omega)] d\omega,$$

Drifting of the singularities from the integration path in a complex plane of (ω) at $\text{Im } k_{z1,2} > 0$ makes it possible to consider not only a charge but also a bunch moving in a waveguide. Next results presented are obtained for the bunch which characterized by Gaussian distribution with the charge density $\rho = q\delta(x)\delta(y)\exp(-\zeta^2/(2\sigma^2))$ where $\zeta = z - \beta ct$ and σ is a length of the beam. This leads to the fact that the amplitudes of waveguide modes are equal to amplitudes of modes of the point charge multiplied by $\exp\left[-\omega^2\sigma^2(2\beta^2c^2)^{-1}\right]$. This exponential factor results in decrease of significance of modes with large numbers and, as well, in decrease of calculating time.

A dielectric plate in a vacuum

$$\varepsilon_1 = 1, \varepsilon_2 = 1.67$$

Dependence of the first mode of the field \tilde{E}_z on z/a at different moments ct/a and for different β . $\beta_{C2} = 0.78$, $\beta_{CT2} > 1$

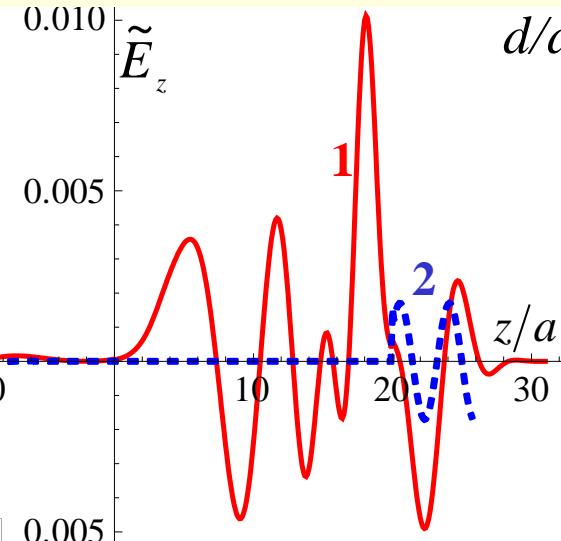
$$n = 1, \sigma/a = 1$$

$$a = 1 \text{ cm},$$

$$r = 0$$

$$\beta = 0.99$$

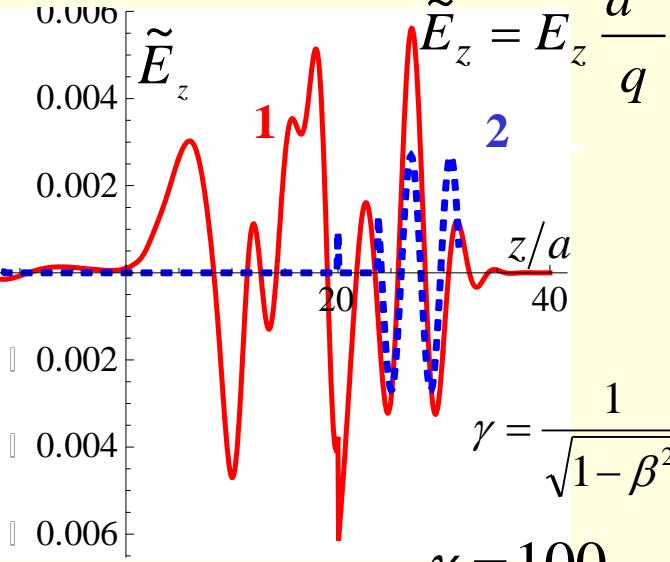
$$ct/a = 30$$



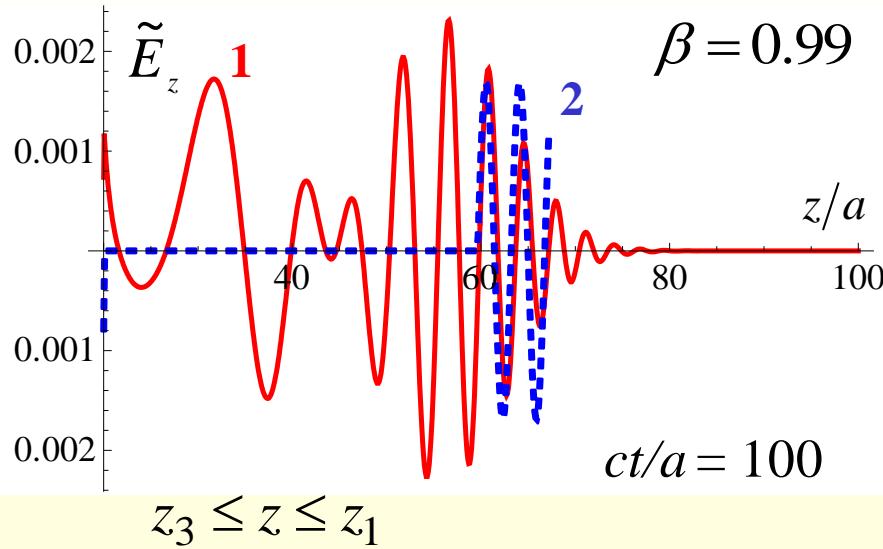
$$d/a = 20$$

$$\beta = 0.99$$

$$ct/a = 40$$



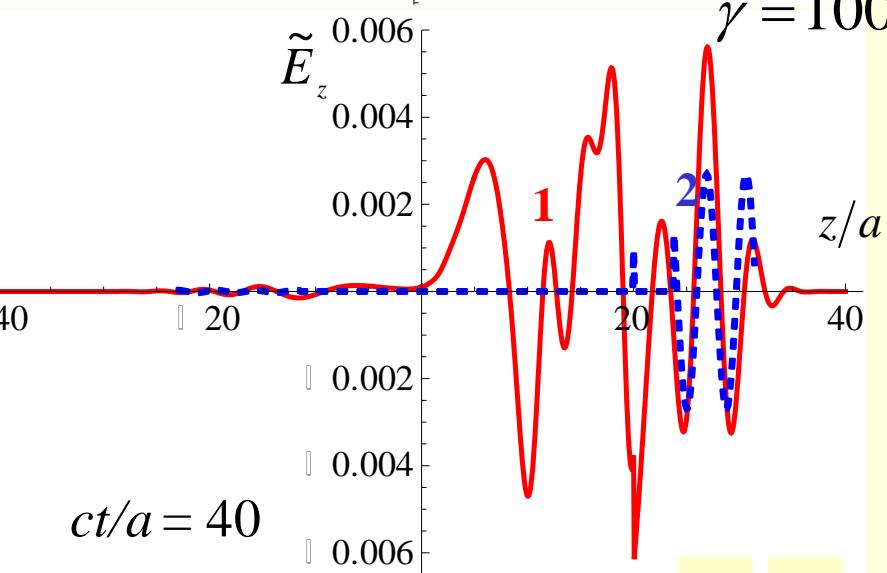
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$\beta = 0.99$$

$$ct/a = 100$$

$$z_3 \leq z \leq z_1$$



$$\gamma = 100$$

$$\omega_0 = 2\pi \cdot 10 \text{ GHz}$$

1 – the total field, 2 – the forced field + CTR



A dielectric plate in a vacuum

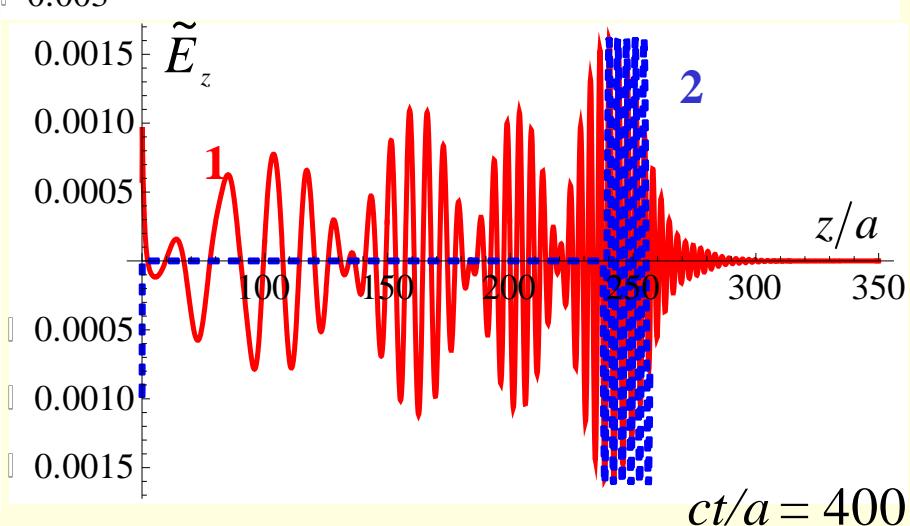
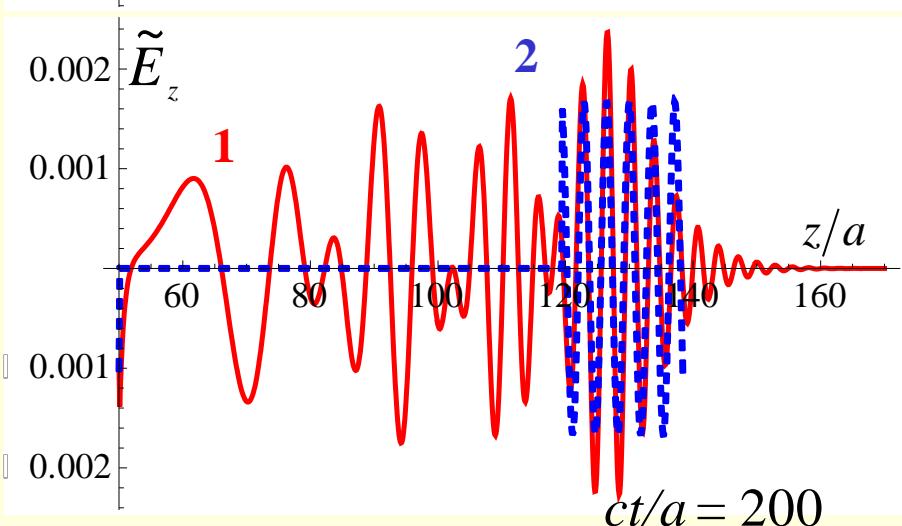
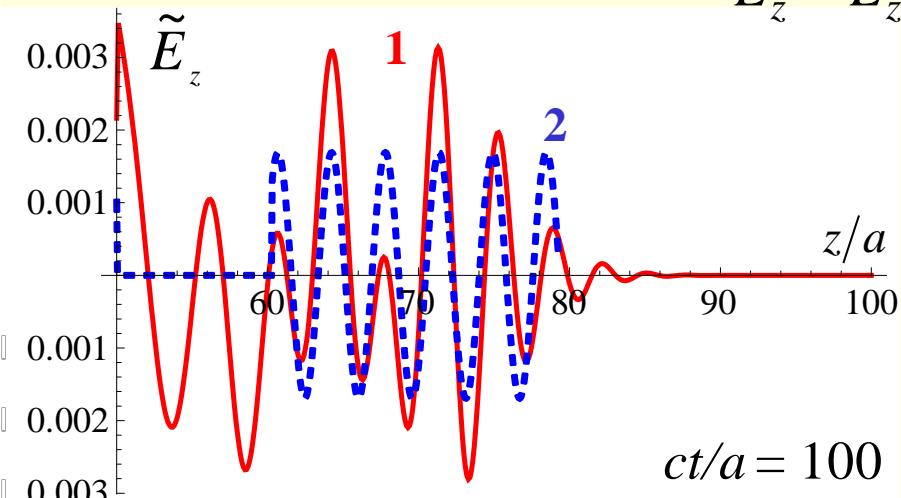
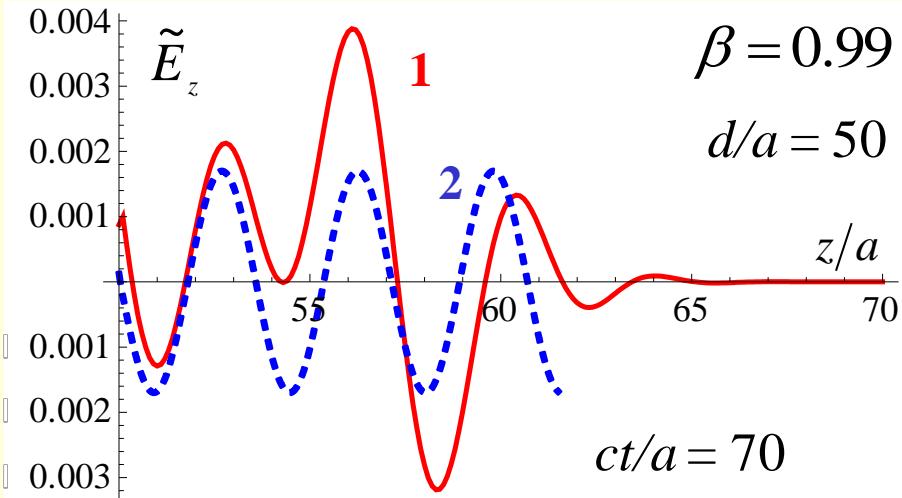
$$\varepsilon_1 = 1, \varepsilon_2 = 1.67$$

Dependence of the first mode of the field \tilde{E}_z on z/a

$$n = 1, \sigma/a = 1$$

in medium 3 at different moments ct/a .

$$\tilde{E}_z = E_z \frac{a^2}{q}$$



$$a = 1 \text{ cm}, r = 0$$

$$\omega_0 = 2\pi \cdot 10 \text{ GHz}$$

1 – the total field, 2 – the forced field + CTR

A dielectric plate in a vacuum

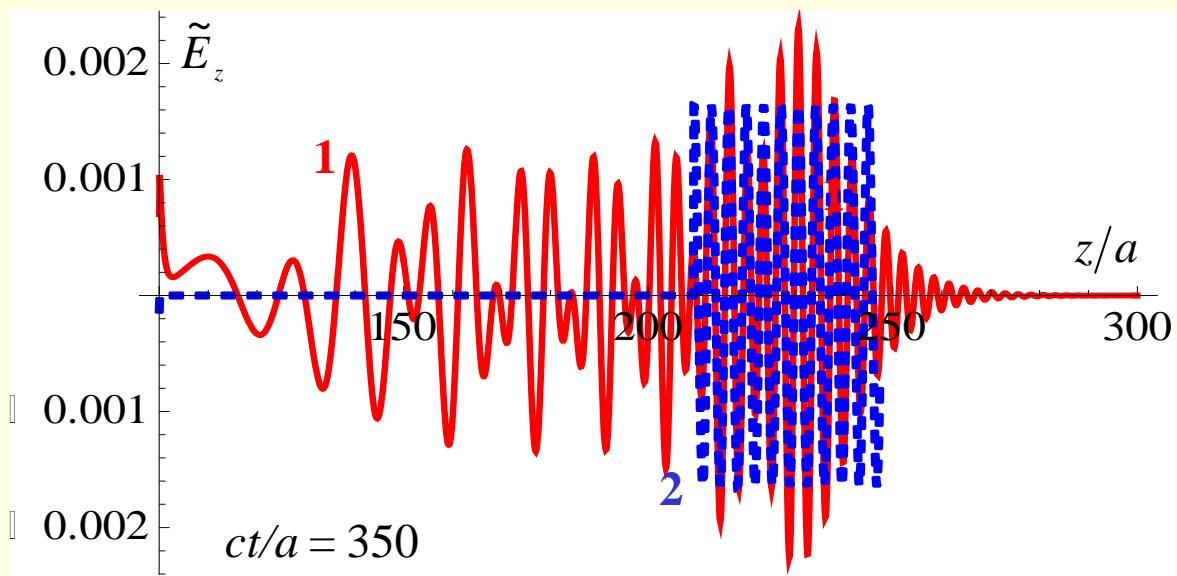
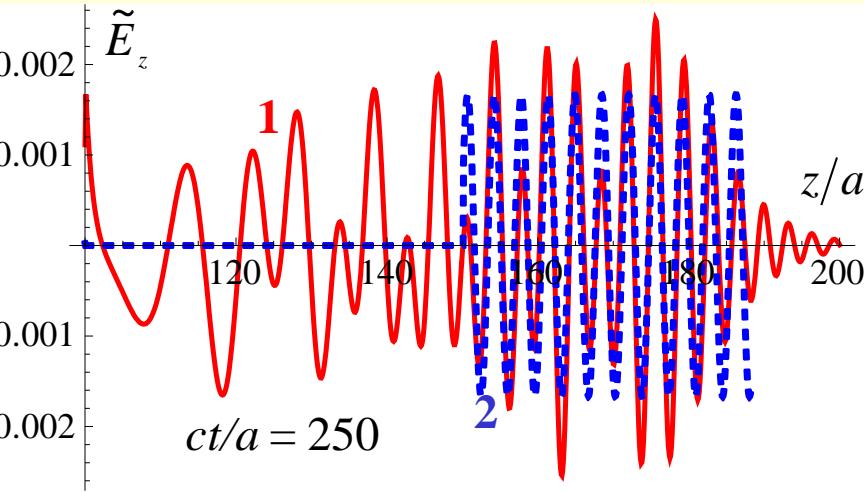
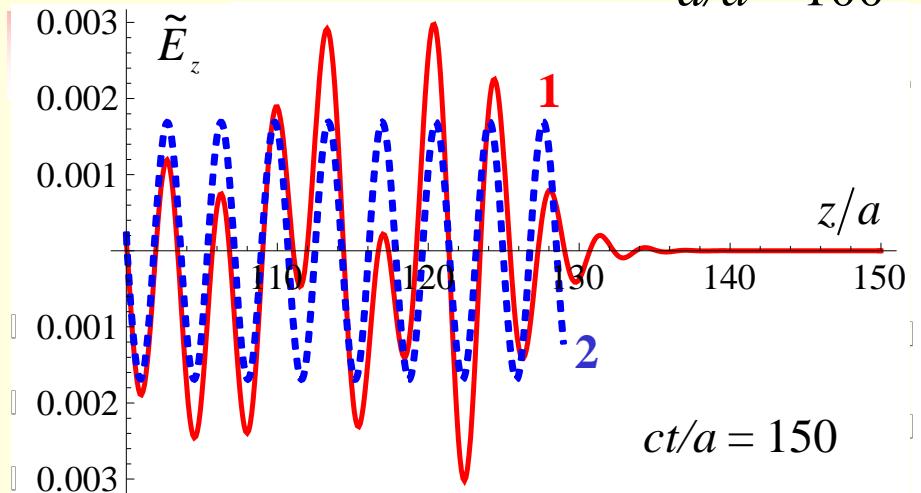
$$\varepsilon_1 = 1, \varepsilon_2 = 1.67$$

Dependence of the first mode of the field \tilde{E}_z on z/a
in medium 3 at different moments ct/a .

$$\beta = 0.99$$

$$d/a = 100$$

$$\tilde{E}_z = E_z \frac{a^2}{q}$$



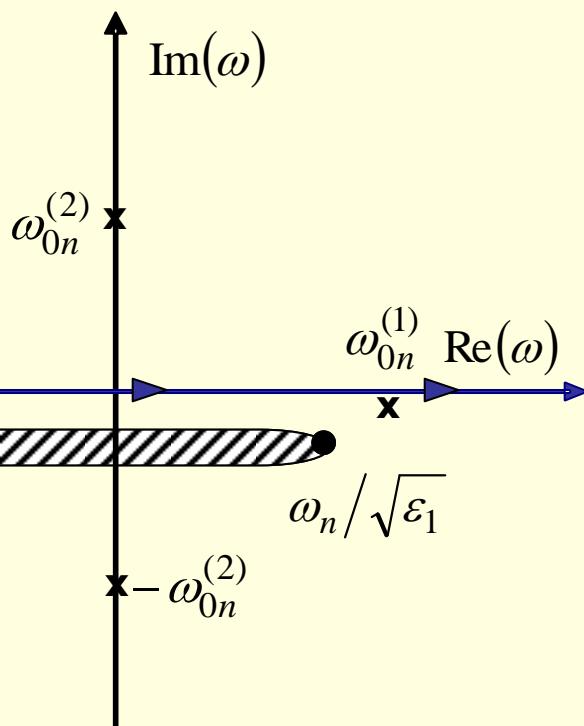
$$n = 1, \sigma/a = 1$$

$$a = 1 \text{ cm}, r = 0$$

1 – the total field,
2 – the forced field + CTR

$$\omega_0 = 2\pi \cdot 10 \text{ GHz}$$

Medium 3: analysis of A_{n3} in a complex plane of (ω) at $\beta > \beta_{c2}$



Branch points : $\pm \tilde{\omega}_n^{(1)} = \pm \omega_n / \sqrt{\varepsilon_1} - i\delta_1$

Branch cut : $\text{Re} \sqrt{\omega^2 - (\tilde{\omega}_n^{(1)})^2} = 0$

Poles: $q_n|_{\omega_{0n}^{(1),(2)}} = 0 \quad F.(10)$

$$\pm \omega_{0n}^{(1)} = \pm \frac{i\beta\omega_n}{\sqrt{1-\varepsilon_1\beta^2}}, \quad \beta < \frac{1}{\sqrt{\varepsilon_1}}$$

$$\pm \omega_{0n}^{(1)} = \pm \frac{\beta\omega_n}{\sqrt{\varepsilon_1\beta^2 - 1}} - i\delta_2, \quad \beta > \frac{1}{\sqrt{\varepsilon_1}}$$

$$\pm \omega_{0n}^{(2)} = \pm \frac{i\beta\omega_n}{\sqrt{1-\beta^2}},$$



The field in medium 3

Contributions of poles $\pm \omega_{0n}^{(1)}$ give the transmitted wave of CR (CTR) at $\beta_{c1} < \beta < \beta_{CT1}$ in the area $d \leq z \leq z_2$

$$z_2 = d + v_{g1} \left(t - \frac{d}{v_{g2}} \right), \quad t > \frac{d}{v_{g2}}, \quad v_{g2} = \frac{c\sqrt{1-\beta^2(\varepsilon_1-1)}}{\beta}, \quad v_{g1} = \frac{c}{\beta\varepsilon_1} \quad (20)$$

T.Yu. Alekhina and A.V. Tyukhtin, Journal of Physics: Conference Series 357, 012010 (2012).

$$\kappa_n = \frac{\chi_n}{a\sqrt{\varepsilon_1\beta^2-1}}$$

$$E_{z3}^{CTR} = -\frac{1}{2} E_{z3}^{CR} - \frac{-8q}{a^2} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_n r}{a}\right)}{J_1^2(\chi_n)} \operatorname{Re} \left[\frac{\sqrt{1-\beta^2(\varepsilon_1-1)} \exp[i\kappa_n(z-d-\beta ct)]}{2\varepsilon_1\sqrt{1-\beta^2(\varepsilon_1-1)} \cos(\kappa_n d) + i \sin(\kappa_n d) (\beta^2(\varepsilon_1-1)\varepsilon_1^2 - \varepsilon_1^2 - 1)} \right], \quad (21)$$

$$E_{z3}^{CR} = \frac{-4q}{\pi a^2 \varepsilon_1} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_n r}{a}\right)}{J_1^2(\chi_n)} \cos[\kappa_n(z - \beta ct)] \theta(\beta ct - z) \quad (22)$$

For the case of the vacuum-dielectric boundary there is the total compensation of wakefield with CTR in some domain.

The field in medium 3 (after the cavity)

Correlation between E_{z3}^{CTR} and E_{z3}^{CR} can take place:
in frequency at

$$d_s = \frac{2\pi s a \sqrt{\beta^2 \varepsilon_1 - 1}}{\left(1 - \sqrt{1 - \beta^2(\varepsilon_1 - 1)}\right) \chi_n}, \text{ or } d_s = \frac{2\pi s a \sqrt{\beta^2 \varepsilon_1 - 1}}{\left(1 + \sqrt{1 - \beta^2(\varepsilon_1 - 1)}\right) \chi_n}, \quad (23)$$

$s = 1, 2, \dots$

In amplitude at $\varepsilon_1 = \varepsilon_{10} = \left. \frac{1 + \sqrt{1 + 4\beta^2}}{2\beta^2} \right|_{\beta \rightarrow 1} \approx 1.67$

$$\left. 2\varepsilon_1 \sqrt{1 - \beta^2(\varepsilon_1 - 1)} \right|_{\varepsilon_{10}} = \left. \beta^2(\varepsilon_1 - 1)\varepsilon_1^2 - \varepsilon_1^2 - 1 \right|_{\varepsilon_{10}},$$

$$d_s = \frac{2\pi s a \sqrt{\beta^2 \varepsilon_1 - 1}}{\left(1 - \sqrt{1 - \beta^2(\varepsilon_1 - 1)}\right) \chi_n}, \quad \Rightarrow \quad E_{z3}^{CTR} = \frac{1}{2} E_{z3}^{CR}$$

$$n = 1, \quad \varepsilon_1 = 1.67,$$

$$E_{z3}^{CR} + E_{z3}^{CTR} = \frac{3}{2} E_{z3}^{CR}$$

$$d_1 = 5.035a, d_2 = 10.07a, d_3 = 15.1a, d_4 = 20.14a, \dots$$

An analysis of CTR shows that at some optimal parameters of the problem we can have some amplification of wakefield.

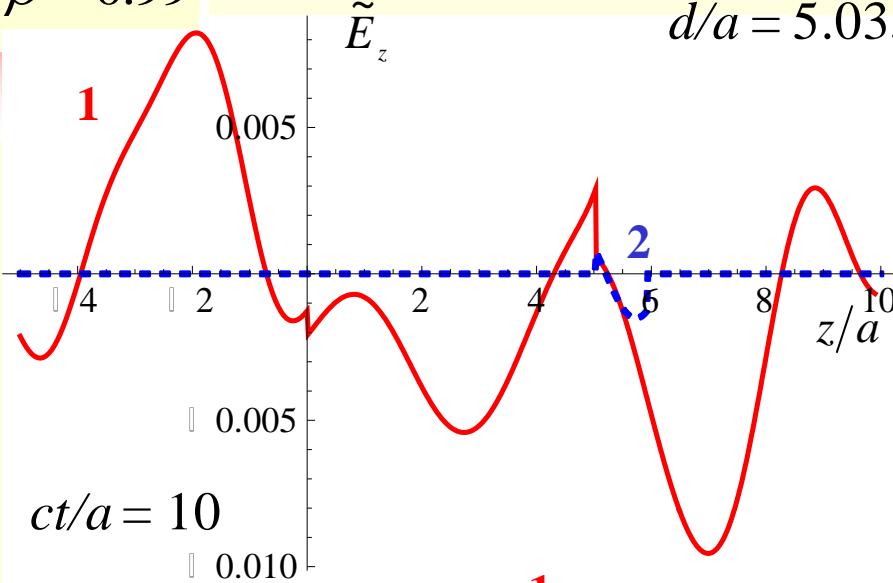
A vacuum cavity in a dielectric

$$\varepsilon_1 = 1.67, \varepsilon_2 = 1$$

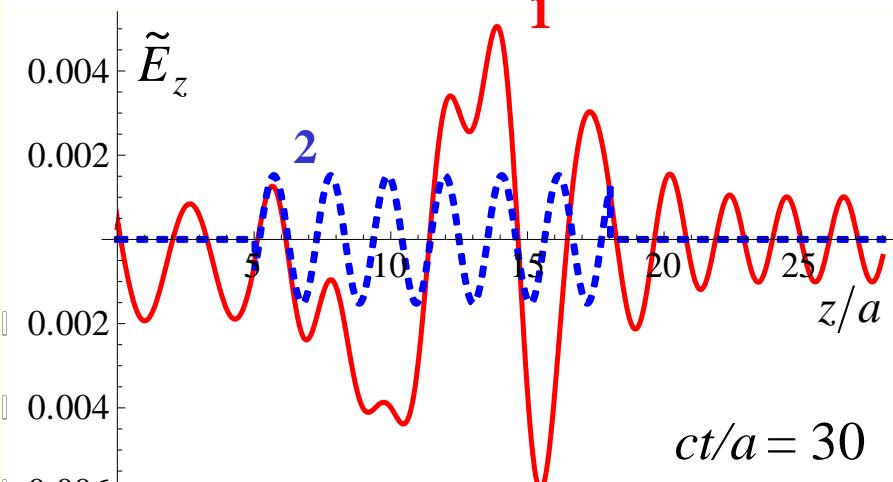
Dependence of the first mode of the field \tilde{E}_z on z/a
at different moments ct/a

$$\sigma/a = 1$$

$$\beta = 0.99$$



$$d/a = 5.035$$



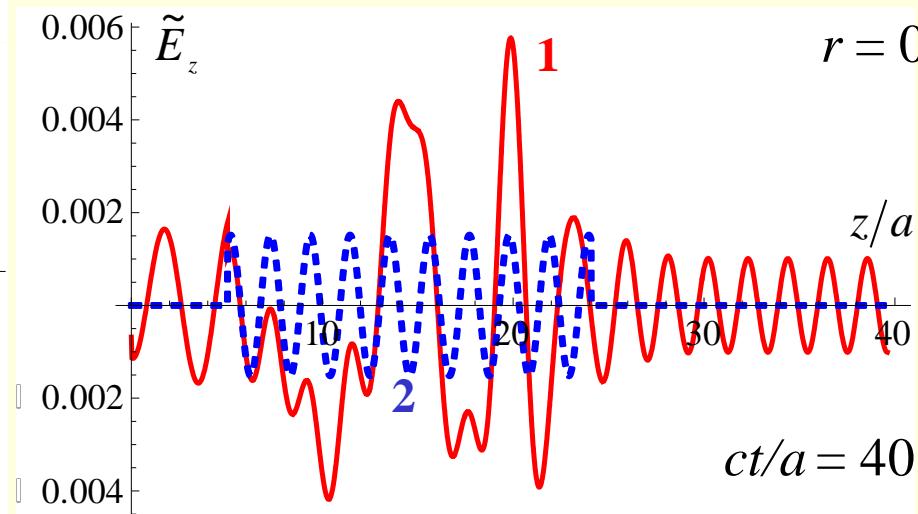
$$\tilde{E}_z = E_z \frac{a^2}{q}$$

$$z/a$$

$$n = 1,$$

$a = 1 \text{ cm},$

$$r = 0$$



$$d \leq z \leq z_2$$

1 – the total field, 2 – the forced field + CTR

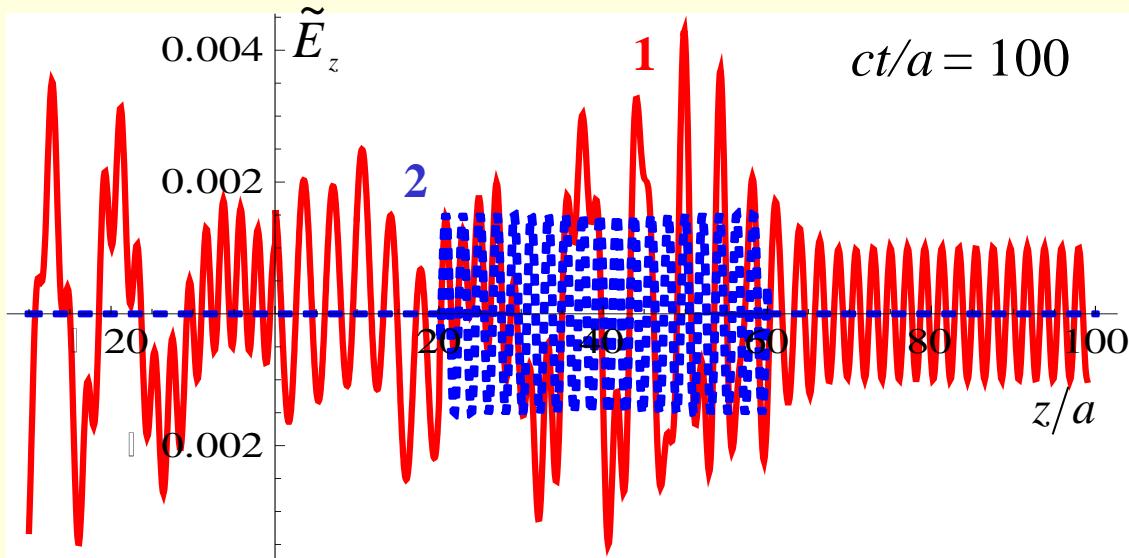
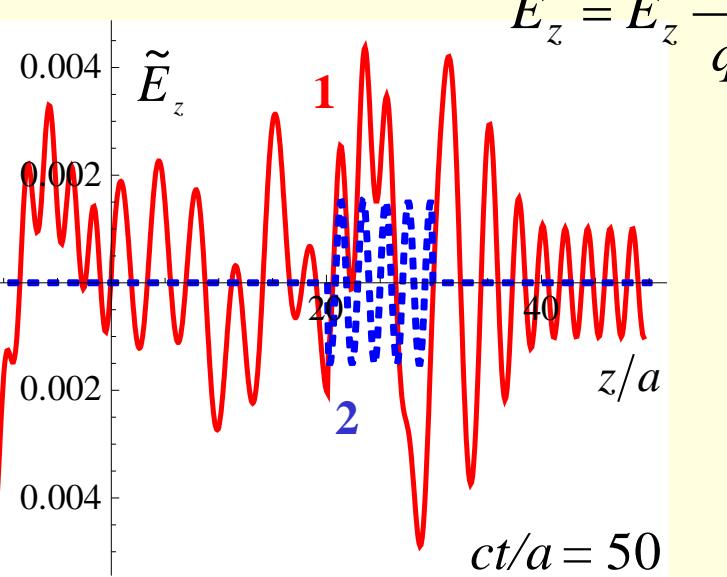
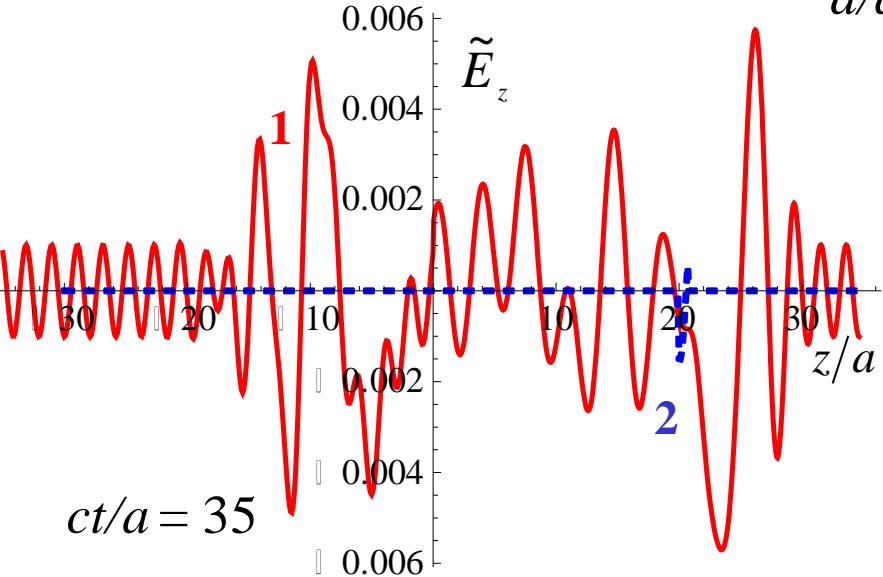


$\beta = 0.99$

$\sigma/a = 1$

Dependence of the first mode of the field \tilde{E}_z on z/a
at different moments ct/a

$d/a = 20.14$



1 – the total field,
2 – the forced field + CTR

$n = 1,$
 $a = 1 \text{ cm},$
 $r = 0$

Conclusions :

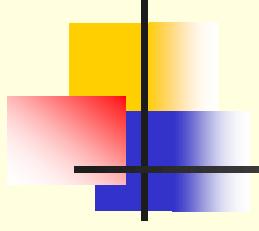
1. The solution to the problem has been found by computational method using exact formulas. Analytical investigation of CTR has been made with methods of the complex variable function theory as well.
2. The case of a dielectric plate situated in a vacuum $\varepsilon_1 = 1, \varepsilon_2 > 1$

There is a CTR train propagating in medium 3 (after the plate) at $1/\sqrt{\varepsilon_2} < \beta < 1/\sqrt{\varepsilon_2 - 1}$. This train has finite length which does not depend on time but only on the parameters of the problem ε_2 and β . For the optimal parameters the maximum length of the CTR is always less than the length of the dielectric plate.

3. The case of a vacuum gap situated in a dielectric $\varepsilon_1 > 1, \varepsilon_2 = 1$

The CTR exists in medium 3 (after the gap) at $1/\sqrt{\varepsilon_1} < \beta < 1/\sqrt{\varepsilon_1 - 1}$. Investigation of the CTR shows that at some optimal parameters of the problem we can have some amplification of wakefield.





Thanks for your attention!