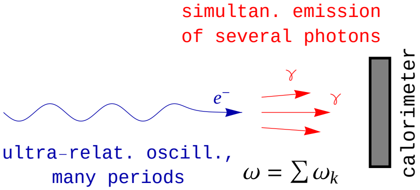
MULTIPHOTON EFFECTS IN COHERENT RADIATION SOURCES

M.V. Bondarenco

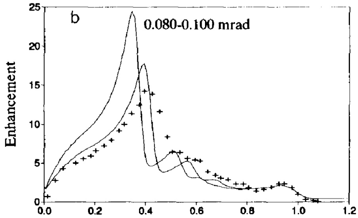
NSC Kharkov Institute of Physics & Technology,
Kharkov, Ukraine

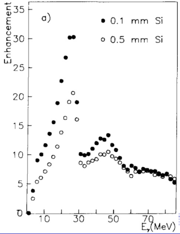
Outline

Motivation



Single-photon spectrum $\omega_1 \frac{dw_1}{d\omega_1}$ is more directly related to predictions of classical electrodynamics, whereas multiphoton spectrum (radiative energy loss) $\omega \frac{dw}{d\omega}$ is measured at practice.
Find the connection.





Basic relations

If $\frac{dw_1}{d\omega_1} = \frac{1}{\omega_1}\frac{dI}{d\omega_1}$ is the single-photon spectrum, the resummation proceeds as

$$\frac{dw}{d\omega} = W_0 \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} d\omega_1 \frac{dw_1}{d\omega_1} \dots \int_0^{\infty} d\omega_n \frac{dw_1}{d\omega_n} \delta\left(\omega - \sum_{k=1}^{n} \omega_k\right) \quad (1a)$$

$$= e^{-\int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{s\omega} \left(e^{\int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1} e^{-s\omega_1}} - 1 \right). \quad (1b)$$

$W_0 = 1 - \int_0^\infty d\omega \frac{dw}{d\omega} = e^{-\int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1}} > 0$ is the photon non-emission probability.

N.B.: The statistical independence of photon emissions holds only for a completely prescribed classical electromagnetic current. So, averaging over initial states is due *after* the resummation.

N.B.: $dw/d\omega$ differs from the distribution function for the radiating electrons, which equals

$$\Pi(E_e - \omega) = \frac{dw}{d\omega} + W_0 \delta(\omega),$$

and is normalized to unity: $\int_0^{E_e} \Pi(E_e - \omega) = 1$.
If $W_0 \neq 0$ (IR-finite), $\Pi(E_e - \omega)$ is singular at $\omega = 0$, whereas $dw/d\omega$ is not.

Kinetic equations

As an alternative to explicit, but oscillatory contour integral (1), the multiphoton spectrum may be sought as a solution to integro-differential equation

$$\frac{\partial}{\partial L}\frac{dw}{d\omega} = \int_0^E d\omega_1 \left(\left.\frac{dw}{d\omega'}\right|_{\omega'=\omega-\omega_1} - \frac{dw}{d\omega} \right) \frac{\partial}{\partial L}\frac{dw_1}{d\omega_1} + W_0(L)\frac{\partial}{\partial L}\frac{dw_1}{d\omega} \tag{2}$$

(inhomogeneous w.r.t. $\frac{dw}{d\omega}$). $\frac{\partial}{\partial L}\frac{dw_1}{d\omega_1}$ plays the role of the radiation cross-section. The initial condition for (2) is

$$\left.\frac{dw}{d\omega}\right|_{L=0} \equiv 0. \tag{3}$$

For the electron distribution function $\Pi(E_e-\omega) = \frac{dw}{d\omega} + W_0\delta(\omega)$, the kinetic equation becomes strictly homogeneous:

$$\frac{\partial}{\partial L}\Pi(E_e-\omega) = \int_0^E d\omega_1 \frac{\partial}{\partial L}\frac{dw_1}{d\omega_1}\left[\Pi(E_e-\omega-\omega_1) - \Pi(E_e-\omega)\right], \tag{4}$$

but with a singular initial condition

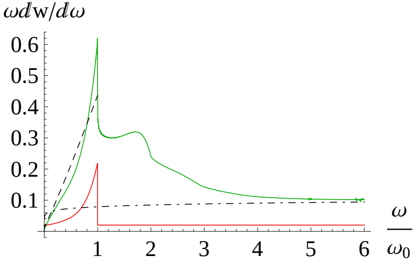
$$\Pi(E_e-\omega)\big|_{L=0} \equiv \delta(\omega). \tag{5}$$

Manifestation of multiphoton effects

$$\frac{dw_1}{d\omega_1} = \frac{dw_{1c}}{d\omega_1} + \frac{dw_{1i}}{d\omega_1},$$

$$\frac{dw_{1i}}{d\omega_1} = \frac{a}{\omega_1}\theta(\omega_0 - \omega), \qquad a \ll 1,$$

$\frac{dw_{1c}}{d\omega_1} = bP\left(\frac{\omega_1}{\omega_0}\right)\theta(\omega_0 - \omega_1)$, $P(z) = 1 - 2z + 2z^2$ ('one-point' dipole spectrum).



- Mimicking the second harmonic
- Suppression of discontinuities
- Suppression of low-ω region

Gausian limit

At high intensity, one can approximately evaluate the contour integral by the steepest descent method, getting

$$\frac{dw_c}{d\omega} \simeq \frac{e^{-\rho^2/2}}{\sqrt{2\pi\overline{\omega^2_{1\,c}}}}, \tag{6}$$

with

$$\rho = \frac{\omega - \overline{\omega_{1c}}}{\sqrt{\overline{\omega^2_{1\,c}}}}. \tag{7}$$

If a correction is needed,

$$\frac{dw_c}{d\omega} \simeq \frac{e^{-\rho^2/2}}{\sqrt{2\pi\overline{\omega^2_{1\,c}}}} \left[1 + \frac{\gamma_3}{12\sqrt{2}} H_3\left(\frac{\rho}{\sqrt{2}}\right)\right], \tag{8}$$

where

$$\gamma_3 = \frac{\overline{(\omega - \overline{\omega})^3}}{\overline{(\omega - \overline{\omega})^2}^{3/2}} = \frac{\overline{\omega_1^3}}{\overline{\omega_1^2}^{3/2}} \propto \frac{1}{\sqrt{\bar{n}}}, \tag{9}$$

and $H_3(z) = -e^{z^2} \frac{d^3}{dz^3} e^{-z^2} = 8z^3 - 12z$ is the Hermite polynomial of order 3.

The Chebyshev correction $\propto \gamma_3$ mildly breaks the Gaussian scaling, and accounts for the residual spectral skewness.

Experimental observation of Gaussian limit

Passage through a crystal in axial orientation (e.g., [1]) can be impact parameter independent to a good accuracy due to the dynamical chaos. In that case, the Gaussian shape will not be significantly affected by averaging.

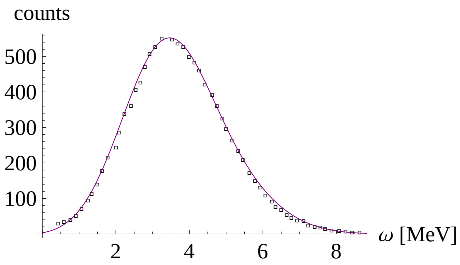


Figure: Radiation from 40 GeV electrons traversing 2.5 cm thick Ge crystal along ⟨110⟩ axis. Points, experimental data [1]. Curve, – fit by Eq. (8) to the data. Determined skewness $\gamma_3 \approx 0.02$ (almost Gaussian).

Parabolic cylinder limit

If, in the presence of an incoherent radiation component, the mean and the variance for the single-photon spectrum diverge, the Gaussian approximation does not apply. Instead,

$$\frac{dw}{d\omega} \approx \frac{1}{\sqrt{2\pi E^a \overline{\omega_{1c}^2}}^{\frac{1-a}{2}}} e^{-\gamma_E a - \rho^2/4} D_{-a}(-\rho), \qquad (10)$$

$D_{-a}(-\rho)$ – parabolic cylinder function.
(An intermediate case between Gaussian and Lévy distributions.)

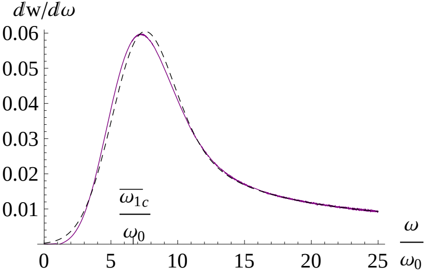


Figure: Multiphoton spectrum for $b\omega_0 = 20$, $a = 0.3$. Solid purple curve, exact distribution; dashed black curve, parabolic cylinder approximation (10).

Averaging over electron impacts for multiphoton channeling radiation

$b \propto x_0^2$
Planar: $\langle ... \rangle = \frac{1}{2x_{0\,\max}} \int_{-x_{0\,\max}}^{x_{0\,\max}} dx_0 ....$
Axial: $\langle ... \rangle = \frac{1}{x_{0\,\max}^2} \int_0^{x_{0\,\max}} dx_0^2 ....$
$\langle b \rangle = \frac{1}{r} b_{\max}$,
$r = 3$ for planar channeling, $r = 2$ for axial channeling, $r = 1$ for fixed oscillation amplitude.

$$\left\langle \frac{dw}{d\omega} \right\rangle = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{s\omega} \left\{ F_r \left[r \int_0^\infty d\omega_1 \left\langle \frac{dw_1}{d\omega_1} \right\rangle \left(1 - e^{-s\omega_1}\right) \right] - F_r(r \langle w_1 \rangle) \right\}, \quad (11)$$

with

$$F_r(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!(rk+1)}.$$

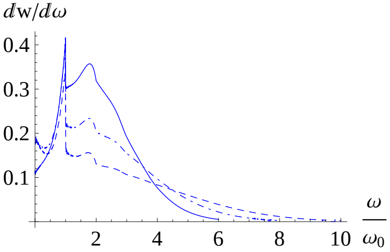
Noye: Axial channeling radiation case is intermediate between planar channeling and fixed-amplitude radiation.

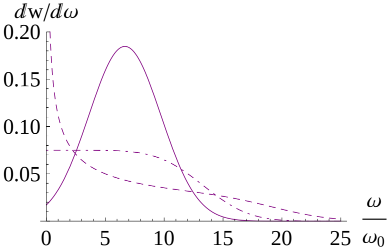
High-multiplicity limits for planar and axial channeling radiation

Not Gaussians, but integrals thereof:

$$\left\langle \frac{dw_c}{d\omega} \right\rangle_{ax} \approx \frac{1}{4\left\langle \overline{\omega_1} \right\rangle_c} \mathrm{erfc}\left(\frac{\omega - 2\left\langle \overline{\omega_1} \right\rangle_c}{2\sqrt{\left\langle \overline{\omega_1^2} \right\rangle_c}} \right). \tag{12}$$

$$\left\langle \frac{dw_c}{d\omega} \right\rangle_{pl} \approx \frac{1}{4\sqrt{3\left\langle \overline{\omega_1} \right\rangle_c \omega}} \mathrm{erfc}\left(\sqrt{\frac{\left\langle \overline{\omega_1} \right\rangle_c \omega}{2\left\langle \overline{\omega_1^2} \right\rangle_c}} \ln \frac{\omega}{3\left\langle \overline{\omega_1} \right\rangle_c} \right). \tag{13}$$





Observation of high-multiplicity limit for planar channeling radiation

Was high-multiplicity limit for planar channeling radiation ever observed?

CERN, 2012:

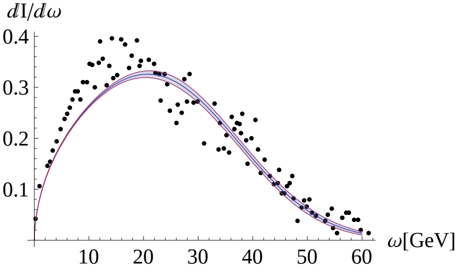
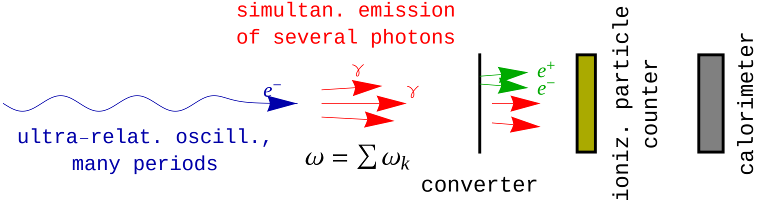


Figure: Energy spectrum of radiation from 120 GeV positrons channeling in a 2mm thick (110) Si crystal (large bending radius, $R = 11$ m). Points, experimental data [4]. Blue band, fit to the data by Eq. (13).

Photon multiplicity spectrum



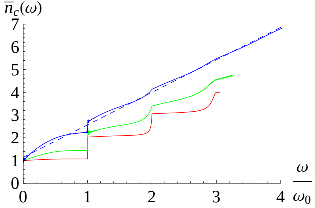
$$\bar{n}(\omega)\frac{dw}{d\omega} = W_0 \sum_{n=1}^{\infty} \frac{n}{n!} \int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1} \cdots \int_0^\infty d\omega_n \frac{dw_1}{d\omega_n} \delta\left(\omega - \sum_{k=1}^{n} \omega_k\right) \quad (14a)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{s\omega + \int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1}\left(e^{-s\omega_1}-1\right)} \int_0^\infty d\omega_1' \frac{dw_1}{d\omega_1'} e^{-s\omega_1'}. \quad (14b)$$

$\bar{n}(\omega) \geq 1$, $\bar{n}(0) = 1$.

If $\frac{dw}{d\omega}$ is obtained from a kinetic equation, $\bar{n}(\omega)$ can be extracted via relation

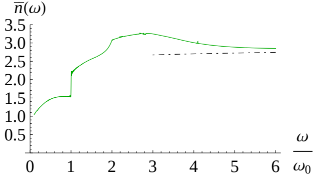
$$\bar{n}(\omega) = w_1 + L\frac{\partial}{\partial L} \ln \frac{dw}{d\omega}, \qquad w_1 = \int_0^\infty d\omega_1 \frac{dw_1}{d\omega_1}. \quad (15)$$



Photon multiplicity spectrum for pure coherent radiation ('one-point' spectrum).
Red, $\bar{n} = 0.2$; green, $\bar{n} = 1.3$; blue, $\bar{n} = 4$.

At high intensity (dashed blue line),

$$\bar{n}_{\rm c}(\omega) \approx w_{1\rm c} + \frac{\overline{\omega_{1\rm c}}}{\overline{\omega^2_{1\,\rm c}}}\left(\omega - \overline{\omega_{1\rm c}}\right).$$

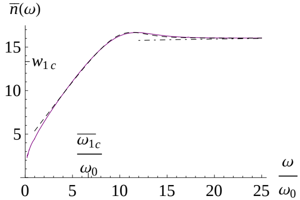


Coherent + incoherent radiation,
for $a = 0.1$, $b\omega_0 = 2$ (green curve).

At large ω (dot-dashed black curve),

$$\bar{n}(\omega) \simeq \bar{n}_{\rm c} + \bar{n}_{\rm i}(\omega) = w_{1\rm c} + 1 + a\ln\omega/\epsilon. \quad (16)$$

ϵ – infrared cutoff.



Limiting shape of photon multiplicity spectrum in the presence of incoherent radiation component. (Expressible through parabolic cylinder functions – dashed black curve).

Summary and outlook

- Resummation over multiple photon emissions is necessary for description of gamma-radiation in crystals $L \gtrsim 1$ mm thick.
- Some manifestations of multiphoton effects are likewise to those of secondary harmonics in the intra-crystal potential, non-dipole effects, or LPM-like suppression.
- Proper averaging is required for channeling radiation *after* the resummation.
- At high photon multiplicity, the limiting spectrum may differ from Gaussian due to the long-ranged incoherent radiation component. For multiphoton channeling radiation, the limiting spectra are strongly non-Gaussian.
- Photon multiplicity spectrum is an important observable complementary to the radiative energy loss spectrum.
- Reconstruction of the single-photon spectrum from the multiphoton one is manageable (see poster).

References

R. Medenwaldt et al.
Phys. Lett. B **227** (1989) 483.

R. Medenwaldt *et al.*
Phys. Lett. B **260** (1991) 235.

J. Bak *et al.*,
Nucl. Phys. B **254** (1985) 491.

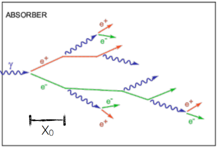
D. Lietti *et al.*,
NIM B **283** (2012) 84.

M.V. Bondarenco,
Phys. Rev. A **81** (2010) 052903.

Backup slides

Calorimeters

Narrow beaming of radiation from ultra-relativistic electrons begets a pileup problem, when different photons emitted by the same electron fly nearly along the same ray, and thus can hit the same detector cell. At practice, the objective of photon counting is commonly abandoned, and electromagnetic calorimeters are utilized for measuring only the total energy deposited by γ-quanta per electron passed through the radiator.



Lateral shower spread:

Main contribution must come from low energy electrons as $\langle\theta\rangle \sim 1/E_e$, i.e. for electrons with $E = E_c$ ...
Assuming the approximate range of electrons to be X_0 yields $\langle\theta\rangle \approx 21$ MeV/E_e ➛ lateral extension: $R = \langle\theta\rangle \cdot X_0$ ...

Molière Radius:

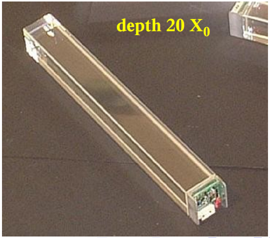
$$\boxed{R_M = \frac{21\ \mathrm{MeV}}{E_c} X_0}$$

Lateral shower spread characterized by R_M !

On average 90% of the shower energy contained in cylinder with radius R_M around shower axis ...

examples:

	X_0 (cm)	E_c (MeV)	R_M (cm)
Pb	0.56	7.4	1.6
plastic scint	34.7	80	9.1
Fe	1.76	21	1.8
Ar (liquid)	14	35	9.5
BGO	1.12	10.5	2.3
Pb glass (SF5)	2.4	11.8	4.3



Energetically ordered form of multiphoton spectrum

Our definition was

$$\frac{dw}{d\omega} = W_0 \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} d\omega_1 \frac{dw_1}{d\omega_1} \dots \int_0^{\infty} d\omega_n \frac{dw_1}{d\omega_n} \delta\left(\omega - \sum_{k=1}^{n} \omega_k\right). \tag{17}$$

But since all $\omega_k > 0$,

$$\begin{aligned} \frac{dw}{d\omega} &= W_0 \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\omega+0} d\omega_1 \frac{dw_1}{d\omega_1} \dots \int_0^{\omega+0} d\omega_n \frac{dw_1}{d\omega_n} \delta\left(\omega - \sum_{k=1}^{n} \omega_k\right) & \text{(18a)} \\ &= e^{-\int_{\omega}^{\infty} d\omega_1 \frac{dw_1}{d\omega_1}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{s\omega + \int_0^{\omega+0} d\omega_1 \frac{dw_1}{d\omega_1} \left(e^{-s\omega_1} - 1\right)} - W_0 \delta(\omega). & \text{(18b)} \end{aligned}$$

Eqs. (18a) compared to Eq. (17) are an *energetically ordered* form. This means that apart from the 'non-dynamical' suppressing factor

$$e^{-\int_{\omega}^{\infty} d\omega_1 \frac{dw_1}{d\omega_1}} = W_0(\omega) \tag{19}$$

(the probability of non-emission of any photon with energy greater than ω), the multiphoton spectrum in the energetically ordered form involves only contributions from the single-photon spectrum with $\omega_1 < \omega$.