A state of the art l-QCD + HRG EOS:
Full vs Partial Chemical Equilibrium

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IS2013, Illa da Toxa, 8-14 September 2013
A state of the art l-QCD + HRG EOS:

Outline

■ Motivation
■ The conceptual setup
■ Results

Motivations

The QCD Equation of State affects soft-physics observables

- *directly*, entering into the hydrodynamic equations describing the medium evolution

\[ \partial_{\mu} T^{\mu\nu} = 0 \quad \text{with} \quad P = P(\epsilon) \]

- *indirectly*, entering into the description of the transition from fluid to particles through the Cooper-Frey formula

\[ E \frac{dN_i}{dp} = \frac{g_i}{(2\pi)^3} \int_{\Sigma_{fo}} p_{\mu} d\Sigma^\mu f_i(x, p), \]

where (in the ideal case) \( f_i^{eq}(x, p) = \frac{1}{e[p \cdot u(x) - \mu_i(x)]/T(x) \pm 1} \)
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where (in the ideal case) \( f_i^{eq}(x, p) = \frac{1}{e^{[p \cdot u(x) - \mu_i(x)]/T(x)} + 1} \)

- The formula fixes unambiguously both the shape of the spectrum and its **normalization**;
- The particle distributions must be consistent with the ones entering into the hadronic EOS.
A state of the art l-QCD + HRG EOS:

Motivation

**Hadron-Resonance Gas EOS**

- A system of stable hadrons interacting through the exchange of resonances (e.g. $\pi + \pi \rightarrow \rho \rightarrow \pi + \pi$) can be described as an ideal gas of hadrons (e.g. $\pi$) and resonances (e.g. $\rho$).

- The partition function factorizes:

  $$Z_{GC} = \prod_i Z_{GC}^{(i)},$$

  where $i$ is taken to run over all hadrons/resonances up to $\sim 2$ GeV.

- The hadronic phase is usually described by such an EOS, where

  $$P = \sum_i g_i \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{E} \frac{p^2}{3} f_{eq}^{(i)}(p),$$

  the other quantities following from thermodynamics:

  $$s = (\partial P/\partial T)_\mu \quad \text{and} \quad \epsilon = Ts - P + \mu n$$
HRG-EOS: chemical composition

At least three different scenarios can be conceived:

- **Full chemical equilibrium**: the multiplicity of each H/R species follows from $f_i^{eq}(p)$. In the absence of stopping (LHC conditions!) $\langle B \rangle = \langle S \rangle = \langle l_3 \rangle = 0$ and all chemical potentials vanish. A unique temperature $T_{fo}$ would describe the *slope of the spectra* and the *particle abundances*;
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- **Early chemical freeze-out**: the multiplicity of *all* H/R species is fixed at *chemical freeze-out* ($T_{chem} \approx 160$ MeV). In order to keep the chemical composition fixed until *kinetic freeze-out* for each hadron/resonance an independent chemical potential is introduced:

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- **Partial chemical equilibrium**: H/R abundances are initially fixed at \( T_{chem} \), but reactions mediated by short-living resonances (e.g. \( \pi + p \rightarrow \Delta \rightarrow \pi + p \)) are allowed.
Chemical vs kinetic freeze-out: experimental evidence

Particle yields consistent with thermal production at $T_{\text{chem}}^{fo} \approx 155$ MeV

Blast-wave fits of the $p_T$-spectra require $T_{\text{kin}}^{fo} \sim 100$ MeV
HRG: partial chemical equilibrium

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- **Scattering** processes mediated by resonances
  
  $\pi + \pi \rightarrow \rho \rightarrow \pi + \pi$
  $\pi + K \rightarrow K^* \rightarrow \pi + K$
  $\pi + p \rightarrow \Delta \rightarrow \pi + p$

  viewed as “chemical reactions” allowed to proceed in both directions, maintaining *partial chemical equilibrium*
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- The chemical potentials of all resonances can then be expressed in terms of the ones of the stable hadrons!
Stable hadrons

- Pseudoscalar meson-octet;
Stable hadrons

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- Spin 1/2 baryon-octet;
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- $\Omega^-$ from spin 3/2 baryon-decuplet.
Equilibrium relations for chemical potentials: algorithm

- Order the particles $P_i$ contributing to the HRG according to their mass ($M_{i+1} > M_i$);
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- Adding at each step one particle, this either is stable or decays into lighter particles, with the channel $c$ weighted by the BR $\Gamma_c$:

$$P_n \to \sum_c \Gamma_c \sum_{i=1}^{n-1} N^c_i P_i$$
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$$P_n \rightarrow \sum_c \Gamma_c \sum_{i=1}^{n-1} N_i^c P_i$$

At equilibrium the Gibbs free-energy $\Phi \equiv E - TS + PV$ is stationary:

$$\frac{\partial \Phi}{\partial N_n} + \sum_{i=1}^{n-1} \left( \frac{\partial \Phi}{\partial N_i} \right) \left( \frac{\Delta N_i}{\Delta N_n} \right) \mu_i = 0$$
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- In the decay of $P_n$ one has $\Delta N_n = -1 \rightarrow \Delta N_i = \sum_c \Gamma_c N_i^c$, so that

$$\mu_n = \sum_c \Gamma_c \sum_{i=1}^{n-1} N_i^c \mu_i$$
Equilibrium relations for chemical potentials: outcomes

- At the end the chemical potential of each resonance \( r \) is fixed by 8 independent chemical potentials of stable hadrons

\[
\mu_r = \sum_h \langle N_h^{(r)} \rangle \mu_h, \quad \text{with} \quad h = \pi, K, \eta, N, \Lambda, \Sigma, \Xi, \Omega
\]

- One obtains a table like (code, mass, spin-deg, baryon-num, \( \{\mu_h\} \))
EOS with partial chemical freeze-out (I)

Thermodynamic relations

\[ Ts = \epsilon + P - \sum_r \mu_r n_r \quad \text{and} \quad dP = s \, dT + \sum_r n_r \, d\mu_r \]

\( r \) running over all the resonances in the cocktail.

Entropy and particle density:

\[ s = \left( \frac{\partial P}{\partial T} \right)_{\mu_r} \quad \text{and} \quad n_r = \left( \frac{\partial P}{\partial \mu_r} \right)_T \]

Chemical potentials of the resonances \( r \) expressed in terms of the ones of the “stable” hadrons \( h \):

\[ \mu_r = \sum_h \langle N_{h}^{(r)} \rangle \mu_h \]

Density of hadrons \( h \)

\[ \bar{n}_h = \left( \frac{\partial P}{\partial \mu_h} \right)_T = \sum_r \left( \frac{\partial P}{\partial \mu_r} \right)_T \frac{d\mu_r}{d\mu_h} = \sum_r n_r \langle N_{h}^{(r)} \rangle : \]

“primary” \((n_h)\) + “feed-down” \((\sum_{r \neq h} n_r \langle N_{h}^{(r)} \rangle)\) contributions.
EOS with PCE (II): entropy conservation

The conservation of $\overline{N}_h$ entails the conservation of entropy! In fact...
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$$\partial_\mu T^{\mu\nu} = \partial_\mu [(\epsilon + P)u^\mu u^\nu - g^{\mu\nu} P] = 0$$

together with the thermodynamic relations

$$Ts = \epsilon + P - \sum_h \mu_h \bar{n}_h \quad \text{and} \quad dP = s \, dT + \sum_h \bar{n}_h \, d\mu_h$$
A state of the art l-QCD + HRG EOS:

The conceptual setup

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$$Ts = \epsilon + P - \sum_h \mu_h \overline{n}_h \quad \text{and} \quad dP = s dT + \sum_h \overline{n}_h d\mu_h$$

lead to

$$\partial_\mu [su^\mu] = - \sum_h \frac{\mu_h}{T} \partial_\mu [\overline{n}_h u^\mu]$$

*In spite of partial deviations from full CE, the conservation of the “effective charges” $\overline{N}_h$ leads to the conservation of the entropy $S$ (thanks to P. Huovinen for discussions)*
...Hence the recipe to impose partial chemical freeze-out at $T_c$:

$$\frac{\overline{n}_h(T, \{\mu_{h'}\})}{s(T, \{\mu_{h'}\})} = \frac{\overline{n}_h(T_c, \{\mu_{h'} = 0\})}{s(T_c, \{\mu_{h'} = 0\})}$$

expressing the conservation of entropy and of the “charges” $\overline{N}_h$;
A state of the art l-QCD + HRG EOS: The conceptual setup

EOS with PCE (III)

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\]

expressing the conservation of entropy and of the “charges” $\bar{N}_h$;

- Particle ratios are fixed at $T_c$:

\[
\frac{\bar{n}_{h1}(T, \{\mu_{h'}\})}{\bar{n}_{h2}(T, \{\mu_{h'}\})} = \frac{\bar{n}_{h1}(T_c, \{\mu_{h'} = 0\})}{\bar{n}_{h2}(T_c, \{\mu_{h'} = 0\})}
\]

- Bibliography:
  - T. Hirano and K. Tsuda, PRC 66 (2002), 054905;
  - H. Bebie et al., NPB 378 (1992), 95;
  - R. Rapp, PRC 96 (2002), 017901;
A state of the art I-QCD + HRG EOS:

- The conceptual setup

HRG with PCE: some results

![Graph showing the chemical potentials as a function of temperature.](image)

- for the chemical potentials (here $T_{ch} = 155$ MeV);
A state of the art l-QCD + HRG EOS:

The conceptual setup

HRG with PCE: some results

for the chemical potentials (here $T_{ch} = 155$ MeV);

for the hadronic densities.
The full setup: l-QCD + HRG

In order to have an EOS covering the full range of temperatures of interest for the experiment and for hydro simulations...

- For high temperatures \((T > 172 \text{ MeV})\) continuum-extrapolated lattice-QCD results with realistic quark masses by the Wuppertal-Budapest collaboration\(^1\) were employed;

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- At low temperature ($T < 172$ MeV) a HRG-EOS was adopted, implementing both full (CE) and partial chemical equilibrium (PCE), in this case exploring only $T_{\text{ch}} < 160$ MeV;

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- An interpolation procedure was employed in the range $165 < T < 180$ MeV, ensuring the continuity of the pressure $P$ and of its first two derivatives with respect to $T$.

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A state of the art l-QCD + HRG EOS:
The full lQCD+HRG setup

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NB in view of a generalization to the finite-density case we have treated each isospin state (e.g. $\pi^+, \pi^0, \pi^-$...) separately.

The full setup: results
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![Graph showing results of the full IQCD+HRG setup. The graph plots $\mu_i$ [GeV] against $T$ [GeV].]
The full setup: results

\[ p \text{ [GeV/fm}^3\text{] vs } \epsilon \text{ [GeV/fm}^3\text{]} \]
A state of the art l-QCD + HRG EOS:
The full lQCD+HRG setup

The full setup: results
A state of the art l-QCD + HRG EOS:
The full lQCD+HRG setup

The full setup: results

![Graph showing the results of the full lQCD+HRG setup](image)
Summary and outlook

- We have built a realistic QCD-EOS exploiting the state of the art lattice-QCD data and supplementing them with a HRG model at low temperature, with the possibility of implementing PCE, so that the description of the hadronic phase includes the correct chemical composition;
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Our lQCD+HRG EOS is public, all necessary data-table can be downloaded from http://personalpages.to.infn.it/~ratti/EoS/Equation_of_State/Home.html: feel free to implement it into your hydro codes (and in case of doubts to contact us)!
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For the future we plan to extend our approach to the finite-density case, of interest for the low-energy scan at RHIC and for FAIR.