Solution of the NLO BFKL Equation and analytic NLO γ^*-γ^*-cross section from High-Energy OPE in Wilson-line operators

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Outline

- Leading-Log-Approximation at high-energy: general case.
- BFKL in $\mathcal{N}$=4 SYM theory.
- Solution of the NLO BFKL equation in QCD.
- General Form of the Solution of Higher-Order BFKL equation.
- DGLAP anomalous dimension from solution of BFKL.
- NLO photon impact factor high-energy OPE
  $\Rightarrow$ γ^*-γ^* cross section.
- Conclusions.

DGLAP evolution equation

- Resum $\alpha_s \ln \frac{Q^2}{\Lambda_{\rm QCD}}$
- Eigenfunctions at any order: Powers of $x_{\rm B}$

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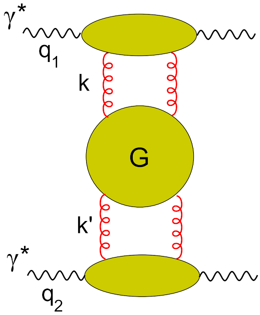
- Eigenfunctions at any order: Powers of $x_{\rm B}$

BFKL evolution equation

- LO: resum $(\alpha_s \ln s)^n$. NLO: resum $\alpha_s(\alpha_s \ln s)^n$
- Eigenfunctions:
  - LO: $(k^2)^{\gamma-1}$
  - NLO and higher order: Perturbative eigenfunctions (G.A.C. and Kovchegov).

Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q \, K(k, q) \, G(q, k', Y)$$

$$G(k, k', Y = 0) = \frac{1}{2\pi k} \delta(k - k')$$

$$k \equiv |\vec{k}_\perp| \text{ and } k' \equiv |\vec{k}'_\perp|$$

$$Y = \ln \frac{s}{k\,k'} \text{ and } s = (q_1 + q_2)^2$$

- Resum $(\alpha_s Y)^n \longrightarrow$ LO BFKL eq.
- Resum $\alpha_s (\alpha_s Y)^n \longrightarrow$ NLO BFKL eq.
- The kernel is real and symmetric: $K(k, k') = K(k', k) \Rightarrow K(k, k')$ is Hermitian and the eigenvalues are real.

LO BFKL equation

$$\frac{\partial}{\partial Y} G(k,k',Y) = \int d^2q\, K^{\rm LO}(k,q)\, G(q,k',Y)$$

$$\int d^2q\, K^{\rm LO}(k,q)\ (q^2)^{1-\gamma} = \bar\alpha_\mu\, \chi_0(\gamma) (k^2)^{1-\gamma} \qquad \bar\alpha_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{1-\gamma}$ are eigenfunctions.
- For $\gamma = \frac{1}{2} + i\nu$ and ν real parameter $\Rightarrow$ $(k^2)^{1-\gamma}$ form a complete set.
- $\Rightarrow$ LO eigenvalues $\chi_0(\nu) = 2\,\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ are real and sym. $\nu \leftrightarrow -\nu$
- LO BFKL is Conformal invariant.

$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2\, k\, k'} \left(\frac{k^2}{k'^2}\right)^{i\nu} e^{\bar\alpha_\mu \chi_0(\nu)\, Y}$$

BFKL equation in the $\mathcal{N}$=4 SYM case

- In $\mathcal{N}=4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

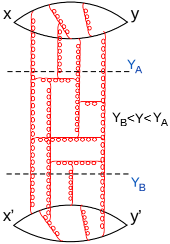
$$K(q,k) = \bar{\alpha}_\mu K^{\rm LO}(q,k) + \bar{\alpha}_\mu^2 K^{\rm NLO}(q,k) + \dots$$

$$\int d^2q\, K(q,k)\, (q^2)^{-\frac{1}{2}+i\nu} = [\alpha_s\chi_0(\nu) + \alpha_s^2\chi_1(\nu)\dots]\, (k^2)^{-\frac{1}{2}+i\nu}$$

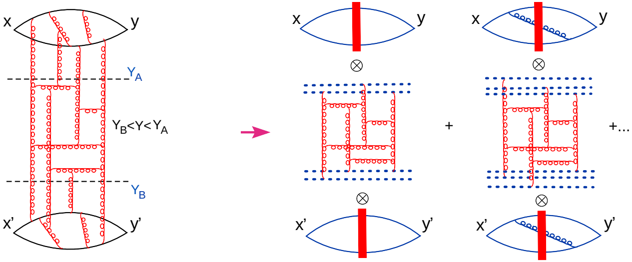
$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2 k\, k'}\, e^{[\alpha_s\chi_0(\nu)+\alpha_s^2\chi_1(\nu)\dots]} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- The eigenvalues $\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu) + \dots$ are real and symmetric for $\nu \leftrightarrow -\nu$.

Factorization in rapidity and composite conformal operator



Factorization in rapidity and composite conformal operator



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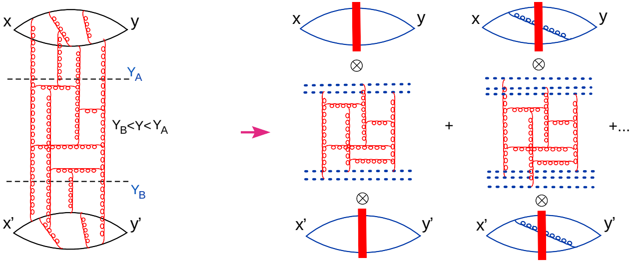

$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)\,[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\,\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x^+y^+}{(x-y)^2}$, $b_0 = \frac{x'^-y'^-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy
$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$

BFKL equation at NLO in QCD

$$K^{\mathrm{LO+NLO}}(k,q) \equiv \bar{\alpha}_\mu K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2 K^{\mathrm{NLO}}(k,q)$$

$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\; q^{2\gamma-2} = \left[\bar{\alpha}_\mu\,\chi_0(\gamma) - \bar{\alpha}_\mu^2\,\beta_2\,\chi_0(\gamma)\,\ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\,\frac{\delta(\gamma)}{4}\right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\,N_c - 2\,N_f}{12\,N_c}$$

- $-\bar{\alpha}_\mu^2\,\beta_2\,\chi_0(\gamma)\,\ln\frac{k^2}{\mu^2}$ 1-loop running coupling.
- $\delta(\gamma) = -2\,\beta_2\,\chi_0'(\gamma) + 4\,\chi_1(\gamma)$ NLO Conformal terms Fadin-Lipatov (1998)
- $\chi_1(\gamma)$ Real and symmetric in $\gamma \leftrightarrow 1-\gamma$ $\gamma = \frac{1}{2} + i\nu$.
- $\frac{d}{d\gamma}\chi_0(\gamma) \equiv \chi_0'(\gamma)$ imaginary and not symmetric.

Perturbative eigenfunctions $H_\gamma(k) = k^{2\gamma-2} + \bar{\alpha}_\mu F_\gamma(k)$

- we have to determine $F_\gamma(k)$ so that

$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\, H_\gamma(q) = \Delta(\gamma)\, H_\gamma(k)$$

- $\Delta(\gamma)$ eigenvalues to be determined.

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- Ansatz

$$F_\gamma(k) = k^{2\gamma-2}\left[c_0(\gamma) + c_1(\gamma)\, \ln\frac{k^2}{\mu^2} + c_2(\gamma)\ln^2\frac{k^2}{\mu^2} + c_3(\gamma)\ln^3\frac{k^2}{\mu^2} + \dots\right]$$

$c_n(\gamma)$ complex valued functions

NLO eigenfunctions

- Truncate the series at $n = 2$ $\Rightarrow$

$$F_\gamma(k) = k^{2\gamma-2}\left[c_0(\gamma) + c_1(\gamma)\,\ln\frac{k^2}{\mu^2} + c_2(\gamma)\ln^2\frac{k^2}{\mu}\right]$$

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- $\Rightarrow$ For $c_2(\gamma) = \frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}$ and for any $c_0(\gamma)$ and $c_1(\gamma)$ the eigenfunction is

$$H_\gamma(k) = k^{2\gamma-2}\,\left[1 + \bar{\alpha}_\mu\left(\frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}\,\ln^2\frac{k^2}{\mu^2} + c_1(\gamma)\ln\frac{k^2}{\mu^2} + c_0(\gamma)\right)\right]$$

and eigenvalues

$$\Delta(\gamma) = \bar{\alpha}_\mu\,\chi_0(\gamma) + \bar{\alpha}_\mu^2\left(-\frac{1}{2}\,\beta_2\,\chi_0'(\gamma) + \chi_1(\gamma) + c_1(\gamma)\chi_0'(\gamma) + \frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}\right)$$

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- Next step is to impose Completeness relation.

Completeness of the NLO eigenfunctions

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\gamma}{2\pi i} H_\gamma(k)\, H^*_\gamma(k') = \delta(k^2 - k'^2)$$

- Completeness relation has to be satisfied order-by-order in α_μ.

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- Completeness relation at LO is satisfied with $\sigma = \frac{1}{2}$ and $\gamma = \frac{1}{2} + i\nu$ with ν real parameter.
  $\Rightarrow$ we have to impose α_μ-order to be 0 $\Rightarrow$

$$\mathrm{Re}[c_1(\nu)] = \frac{\beta_2}{2} \frac{\partial}{\partial\nu} \frac{\chi_0(\nu)}{\chi_0'(\nu)} \qquad \frac{\partial}{\partial\nu} \mathrm{Im}[c_1(\nu)] + 2\,\mathrm{Re}[c_0(\nu)] = 0$$

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- If completeness relation is satisfied then orthogonality relation is also satisfied.

Completeness of the NLO H-eigenfunctions

- provided that $\frac{\partial}{\partial\nu}\text{Im}[c_1(\nu)] + 2\,\text{Re}[c_0(\nu)] = 0$ the eigenfunctions are

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\left(i\frac{\beta_2\chi_0(\nu)}{2\chi_0'(\nu)}\ln^2\frac{k^2}{\mu^2} + \frac{\beta_2}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2} + i\text{Im}[c_1(\nu)]\ln\frac{k^2}{\mu^2} + \text{Re}[c_0(\nu)]\right)\right]$$

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- and the eigenvalues are real ($\text{Im}[c_1(\nu)]\,\chi_0'(\nu)$ is real function of ν)

$$\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\left[\chi_1(\nu) + \text{Im}[c_1(\nu)]\,\chi_0'(\nu)\right]$$

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$$\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\left[\chi_1(\nu) + \mathrm{Im}[c_1(\nu)]\,\chi_0'(\nu)\right]$$

- The imaginary term $-2\,\beta_2\,\bar{\alpha}_\mu^2\chi_0'(\gamma)$ has canceled.
- We have freedom to chose $\mathrm{Re}[c_0(\nu)]$ and $\mathrm{Im}[c_1(\nu)]$
- This freedom will not affect the solution. It is just an artifact of the phase and ν-reparametrization freedom of the $H_{\frac{1}{2}+i\nu}$ function.

Phase freedom and ν-reparametrization

- Phase freedom of the H-functions:

$$H_{\frac{1}{2}+i\nu}(k) \to e^{-i\bar{\alpha}_\mu\left(\mathrm{Im}[c_0(\nu)]-\mathrm{Im}[c_1(\nu)]\ln\mu^2\right)} H_{\frac{1}{2}+i\nu}(k)$$

- ν-reparametrization: $\nu' = \nu + \bar{\alpha}_\mu \mathrm{Im}[c_1(\nu)]$

$\Rightarrow$
The Phase freedom and ν-reparametrization allow us to put $c_0(\nu) = 0$ and $\mathrm{Im}[c_1(\nu)] = 0$.

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

- NLO eigenvalues $\quad \Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)$

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Solution of NLO BFKL equation **G.A.C. and Yu. Kovchegov**

$$G(k,k',Y) = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi^2}\,e^{\left[\bar{\alpha}_\mu\,\chi_0(\nu)+\bar{\alpha}_\mu^2\,\chi_1(\nu)\right]Y}\,H_{\frac{1}{2}+i\nu}(k)\,\left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$

- The perturbative expansion is in both the exponent and in the eigenfunctions (contrary to DGLAP case and $\mathcal{N}$=4 BFKL).

- The $H_\gamma(k)$ eigenfunctions diagonalize the LO+NLO BFKL kernel

$$\bar{\alpha}_\mu K^{\rm LO}(k,q) + \bar{\alpha}_\mu^2 K^{\rm NLO}(k,q) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi^2 i} \Delta(\gamma) H_\gamma(k) H^*_\gamma(q)$$

- LO+NLO BFKL kernel is μ-independent up to $\mathcal{O}(\alpha_\mu^3) \Rightarrow$
- So is its diagonalization through $H_\gamma(k)$ eigenfunctions.

μ-independence of the NLO solution

$$
\begin{aligned}
G(k,k',Y) &= \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\,k\,k'}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]\,Y}\, H_\gamma(k) H_\gamma^*(q) \\
&= \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\,k\,k'}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]\,Y} \left(\frac{k^2}{k'^2}\right)^{i\,\nu} \left(1-\bar\alpha_\mu^2\,\beta_2\,\chi_0(\nu)\,Y\,\ln\frac{kk'}{\mu^2}\right)
\end{aligned}
$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.

μ-independence of the NLO solution

$$
\begin{aligned}
G(k,k',Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} H_\gamma(k) H_\gamma^*(q) \\
&= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln \frac{kk'}{\mu^2}\right)
\end{aligned}
$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.
- At NLO we may write the solution as

$$
G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\, k')\, \chi_0(\nu) + \bar{\alpha}_s^2(k\, k')\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}
$$

- At this order the scale $\bar{\alpha}_s^{\lambda}(k^2)\, \bar{\alpha}_s^{\lambda}(k'^2) \bar{\alpha}_s^{1-2\lambda}(k\, k')$ (for real λ) works as well.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)+\ldots\right]\,Y}\,\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- $\chi_2(\nu)$ and higher-order coefficients indicated by the ellipsis in the exponent are the scale-invariant (conformal) $(\nu \leftrightarrow -\nu)$-even (real-valued) parts of the prefactor function generated by the action of the next-to-next-to-leading-order (NNLO) (and higher-order) kernels on the LO eigenfunctions.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)+\ldots\right]\,Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- To check the ansatz we have to plug it in the evolution eq. and we need the two-loop beta-function β_3:

$$\mu^2 \frac{d\bar{\alpha}_\mu}{d\mu^2} = -\beta_2 \bar{\alpha}_\mu^2 + \beta_3 \bar{\alpha}_\mu^3$$

and

$$\int d^2q\, K^{\mathrm{LO+NLO+NNLO}}(k,q)\; q^{-1+2i\nu} = \Big\{ \bar{\alpha}_\mu\, \chi_0(\nu) \left[1 - \bar{\alpha}_\mu\, \beta_2\, \ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2 \beta_2^2\, \ln^2\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\, \beta_3\, \ln\frac{k^2}{\mu^2}\right]$$
$$+\bar{\alpha}_\mu^2 \left[\frac{i}{2}\,\beta_2\, \chi_0'(\nu) + \chi_1(\nu)\right] \left[1 - 2\,\bar{\alpha}_\mu\, \beta_2\, \ln\frac{k^2}{\mu^2}\right] + \bar{\alpha}_\mu^3\, \left[\chi_2(\nu) + i\,\delta_2(\nu)\right] \Big\}\; k^{-1+2i\nu}$$

- The ansatz does not work, but it allows us to recover the structure of the NNLO solution

$$G(k,k',Y) = \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k\,k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

$$\times \left\{1 + (\bar{\alpha}_\mu\beta_2)^2 \left[-\frac{1}{24}(\bar{\alpha}_\mu Y)^3\chi_0(\nu)^2\chi_0''(\nu) + \frac{1}{4}(\bar{\alpha}_\mu Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu)\right) + \bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

provided that the imaginary part $i\delta_2(\gamma)$ is

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

This solution satisfies also the initial condition: the solution is unique so it is the right one.

General Form of the Solution of All-Order BFKL equation

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$$\times\left\{1+(\bar{\alpha}_\mu\beta_2)^2\left[-\frac{1}{24}(\bar{\alpha}_\mu Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar{\alpha}_\mu Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

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This solution satisfies also the initial condition: the solution is unique so it is the right one.

- The imaginary part $i\delta_2(\nu)$ has to be confirmed from the explicit calculation of the NNLO eigenfunction.

- So, let us calculate explicitly the NNLO eigenfunction.

General Form of the Solution of All-Order BFKL equation

The eigenfunction of the NNLO BFKL equation is

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2\,i\,\nu}\left[1 \quad + \quad \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2}+\frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right.$$
$$\left. + \quad \bar{\alpha}_\mu^2\,f_2\left(\frac{k}{\mu},\nu\right)+\dots\right]$$

- The function $f_2(k/\mu,\nu)$ denotes the NNLO corrections to the eigenfunctions.
- Ansatz for $f_2(k/\mu,\nu)$:

$$f_2(k/\mu,\nu) = c_0^{(2)}(\nu) + c_1^{(2)}(\nu)\,\ln\frac{k^2}{\mu^2} + c_2^{(2)}(\nu)\ln^2\frac{k^2}{\mu^2} + c_3^{(2)}(\nu)\ln^3\frac{k^2}{\mu^2} + \dots$$

NNLO BFKL Solution: the NNLO eigenfunction

- This time we truncated the series up to $c_4^{(2)}$ and proceeding similarly to the NLO case and we get

$$\mathrm{Re}[c_1^{(2)}] = \beta_2\left(\frac{\chi_1'(\nu))}{2\chi_0'(\nu)} - \frac{\chi_1(\nu)\chi_0''(\nu)}{\chi_0'^2(\nu)} - \frac{\chi_1''(\nu)\chi_0(\nu)}{2\chi_0'^2(\nu)} + \frac{\chi_0(\nu)\chi_1'(\nu)\chi_0''(\nu)}{\chi_0'^3(\nu)}\right) - \beta_3\left(-\frac{1}{2} - \frac{\chi_0(\nu)\chi_0''(\nu)}{2\chi_0'^2(\nu)}\right)$$

$$c_2^{(2)} = \beta_2^2\left(\frac{5}{8}\frac{\chi_0''^2(\nu)\chi_0^2(\nu)}{\chi_0'^4(\nu)} - \frac{\chi_0(\nu)\chi_0''(\nu)}{4\chi_0'^2(\nu)} - \frac{\chi_0^2(\nu)\chi_0''(\nu)}{4\chi_0'^3(\nu)} - \frac{1}{8}\right) - i\beta_2\left(\frac{\chi_1(\nu)}{\chi_0'(\nu)} - \frac{\chi_0(\nu)\chi_1'(\nu)}{2\chi_0'(\nu)}\right) - i\beta_3\frac{\chi_0(\nu)}{\chi_0'(\nu)}$$

$$c_3^{(2)} = i\beta_2^2\left(-\frac{5}{12}\frac{\chi_0''(\nu)\chi_0^2(\nu)}{\chi_0'^3(\nu)} + \frac{\chi_0(\nu)}{4\chi_0'(\nu)}\right)$$

$$c_4^{(2)} = -\beta_2^2\frac{\chi_0^2(\nu)}{8\chi'^2(\nu)}$$

The $\mathrm{Im}[c_1^{(2)}(\nu)]$ is fixed to be 0 using again the ν-reparametrization.

Structure of the NNLO BFKL Solution

- From explicit calculation of $f_2(\gamma)$ we not only confirm the imaginary part

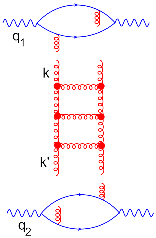
$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

- but also confirm the structure of the NNLO BFKL solution obtained above in an indirect way

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$
$$\times\left\{1+(\bar{\alpha}_\mu\beta_2)^2\left[-\frac{1}{24}(\bar{\alpha}_\mu\, Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar{\alpha}_\mu\, Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

- It looks like QCD is not just conformal part and running coupling contributions.

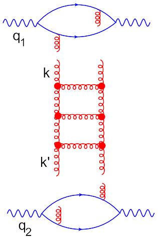
BFKL equation in DIS case



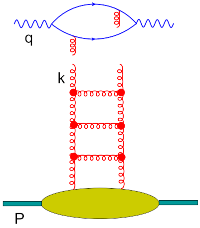
K_{BFKL} is $k \leftrightarrow k'$ symmetric

$$Y^{\text{sym}} = \ln \frac{s}{k\,k'}$$

BFKL equation in DIS case



vs.



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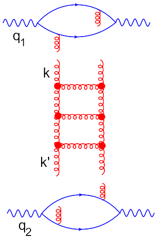
NLO $K^{\mathrm{DIS}}_{\mathrm{BFKL}}$ is not Symmetric

$$Y^{\mathrm{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_{\mathrm{B}}}$$

BFKL equation in DIS case



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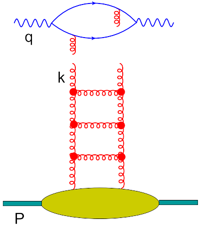

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NLO $K^{\mathrm{DIS}}_{\mathrm{BFKL}}$ is not Symmetric

$Y^{\mathrm{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_B}$

$$K^{\mathrm{DIS}}_{\mathrm{NLO}} = K^{\mathrm{sym}}_{\mathrm{NLO}} - \frac{1}{2}\int d^{D-2}q'\, K_{\mathrm{LO}}(q_1, q')\ln\frac{q'^2}{q^2}\, K_{\mathrm{LO}}(q, q_2) \qquad \text{Fadin} - \text{Lipatov}(1998)$$

- K^{DIS}_{BFKL} is not symmetric $\Rightarrow$ eigenvalues not $\gamma \leftrightarrow 1-\gamma$ symmetric.
- $\Rightarrow$ Eigenvalues get an extra term: $\Delta^{\mathrm{DIS}}(\gamma) = \Delta^{\mathrm{sym}}(\gamma) - \frac{1}{2}\,\bar\alpha_\mu^2\,\chi_0(\gamma)\chi'(\gamma)$
- Reproduced lower order and predicted (and later confirmed) the 3-loop DGLAP anomalous dimension (Fadin-Lipatov (1998))

DGLAP anomalous dimension from NLO BFKL solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\, \chi_0(\nu)+\bar\alpha_\mu^2\, \chi_1(\nu)\right]\,(Y^{\rm DIS}+\ln\frac{k}{k'})}\, H_{\frac12+i\nu}(k)\,\left[H_{\frac12+i\nu}(k')\right]^*$$

$$\simeq \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\, \chi_0(\nu)+\bar\alpha_\mu^2\, \chi_1(\nu)\right]\,Y^{\rm DIS}}\, H_{\frac12+i\nu}(k)\,\left[H_{\frac12+i\nu}(k')\right]^*\left(1+\bar\alpha_\mu\chi_0(\nu)\ln\frac{k}{k'}\right)$$

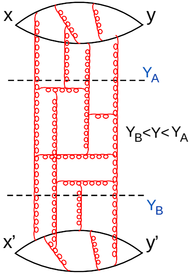
$\Rightarrow$ perform partial integration and exponentiate the $Y^{\rm DIS}$-dependent terms $\Rightarrow$

$$G(k,k',Y^{\rm DIS}) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_\mu\, \chi_0(\nu)+\bar\alpha_\mu^2\, \left(\chi_1(\nu)+2\,i\,\chi_0(\nu)\chi_0'(\nu)\right)\right]\,Y^{\rm DIS}} H_{\frac12+i\nu}(k)\,\left[H_{\frac12+i\nu}(k')\right]^*$$

$$\times\left(1+\frac{i}{2}\,\bar\alpha_\mu\,\chi_0'(\nu)\right)$$

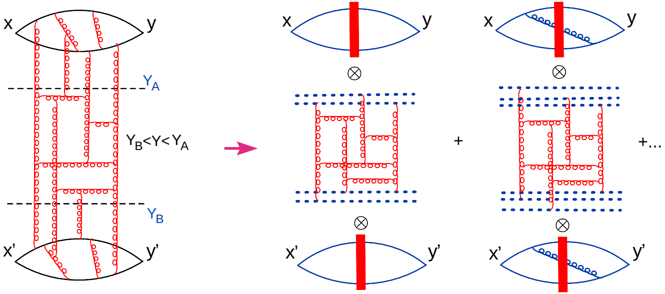
- $\Rightarrow$ $\Delta^{\rm DIS}(\gamma) = \bar\alpha_\mu\,\chi_0(\gamma) + \bar\alpha_\mu^2\,\chi_1(\gamma) - \frac12\,\bar\alpha_\mu^2\,\chi_0(\gamma)\chi_0'(\gamma)$
- Agrees with DGLAP 3-loop anomalous dimension.

γ^*-γ^* scattering cross-section at NLO: work in progress



Using high-energy Operator Product Expansion in composite Wilson line operator we get NLO Impact Factor that does non scale with energy.

γ^*-γ^* scattering cross-section at NLO: work in progress



Using high-energy Operator Product Expansion in composite Wilson line operator we get NLO Impact Factor that does non scale with energy.

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C.

$$\int d^4x\, e^{iq\cdot x}\langle p|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|p\rangle = \frac{s}{2}\int \frac{d^2k_\perp}{k_\perp^2} I_{\mu\nu}(q,k_\perp)\mathcal{V}_{a=x_B}(k_\perp)$$

where the evolution of the dipole gluon distribution at NLO is

$$\begin{aligned}
2a\frac{d}{da}\mathcal{V}_a(k) \;&=\; \frac{\alpha_s N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2} - \frac{(k,k')\mathcal{V}_a(k)}{k'^2(k-k')^2}\Big] \\
&\times\Big(1+\frac{\alpha_s b}{4\pi}\Big[\ln\frac{\mu^2}{k^2}+\frac{N_c}{b}\big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\big)\Big]\Big) - \frac{b\alpha_s}{4\pi} \\
&\times\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2}\ln\frac{(k-k')^2}{k'^2} - \frac{k^2\mathcal{V}_a(k)}{k'^2(k-k')^2}\ln\frac{(k-k')^2}{k^2}\Big] \\
&+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2} + F(k,k')+\Phi(k,k')\Big]\,\mathcal{V}_a(k')\Big\} + 3\frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathcal{V}_a(k)
\end{aligned}$$

$\mathcal{V}(k_\perp) \equiv \int d^2z_\perp e^{-i(k,z)_\perp}\mathcal{V}(z_\perp)$ $\qquad$ $\mathcal{V}(z_\perp) = -\partial_\perp^2[1-\frac{1}{N_c}\mathrm{Tr}\{U(z_\perp)U^\dagger(0)\}]$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C.

$$
\begin{aligned}
& I^{\mu\nu}(q,k_\perp) \\
&= \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu}\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} \\
&+\Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big] \\
&+\frac{\frac{1}{4}+\nu^2}{2k_\perp^2}\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_3(\nu)\Big)\left[\tilde{P}^{\mu\nu}\bar{k}^2+\bar{P}^{\mu\nu}\tilde{k}^2\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
P_1^{\mu\nu} &= g^{\mu\nu}-\frac{q_\mu q_\nu}{q^2} \\
P_2^{\mu\nu} &= \frac{1}{q^2}\Big(q^\mu-\frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu-\frac{p_2^\nu q^2}{q\cdot p_2}\Big) \\
\bar{P}^{\mu\nu} &= \Big(g^{\mu 1}-ig^{\mu 2}-p_2^\mu\frac{\bar{q}}{q\cdot p_2}\Big)\Big(g^{\nu 1}-ig^{\nu 2}-p_2^\nu\frac{\bar{q}}{q\cdot p_2}\Big) \\
\tilde{P}^{\mu\nu} &= \Big(g^{\mu 1}+ig^{\mu 2}-p_2^\mu\frac{\tilde{q}}{q\cdot p_2}\Big)\Big(g^{\nu 1}+ig^{\nu 2}-p_2^\nu\frac{\tilde{q}}{q\cdot p_2}\Big)
\end{aligned}
$$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C.

$$\begin{aligned}
\mathcal{F}_{1(2)}(\nu) &= \Phi_{1(2)}(\nu) + \chi_\gamma \Psi(\nu), \quad \mathcal{F}_3(\nu) = F_6(\nu) + \left(\chi_\gamma - \frac{1}{\bar\gamma\gamma}\right)\Psi(\nu), \\
\Psi(\nu) &\equiv \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma), \\
F_6(\gamma) &= F(\gamma) - \frac{2C}{\bar\gamma\gamma} - 1 - \frac{2}{\gamma^2} - \frac{2}{\bar\gamma^2} - 3\frac{1+\chi_\gamma - \frac{1}{\gamma\bar\gamma}}{2+\bar\gamma\gamma}, \\
\Phi_1(\nu) &= F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2} \\
\Phi_2(\nu) &= F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma}, \\
F(\gamma) &= \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}
\end{aligned}$$

$\gamma = \frac{1}{2} + i\nu$

Conclusions

- The NLO BFKL eigenfunctions have been constructed: they satisfy completeness and orthogonality condition $\Rightarrow$ NLO BFKL solution.
- NNLO Eigenfunctions has also been presented $\Rightarrow$ The structure of the NNLO solution has been found up to the still unknown conformal contribution $\chi_2(\nu)$.
- Procedure to construct the solution of the BFKL equation to any order is now available.
- With the shift $Y^{\text{sym}} \to Y^{\text{DIS}} + \ln\frac{k}{k'}$ one can obtain the solution with non-symmetric kernel for DIS from the symmetric one.
- Using the High-Energy Product Expansion we have calculated the NLO photon impact factor for pomeron exchange in k_T-factorized form (Linear case) and for DIS off a large nucleus (non-linear case for Electron Ion Collider).