Fluctuations in an integrated dynamical approach for heavy-ion collisions

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[arXiv:1304.3243] Relativistic fluctuating hydrodynamics with memory functions and colored noises

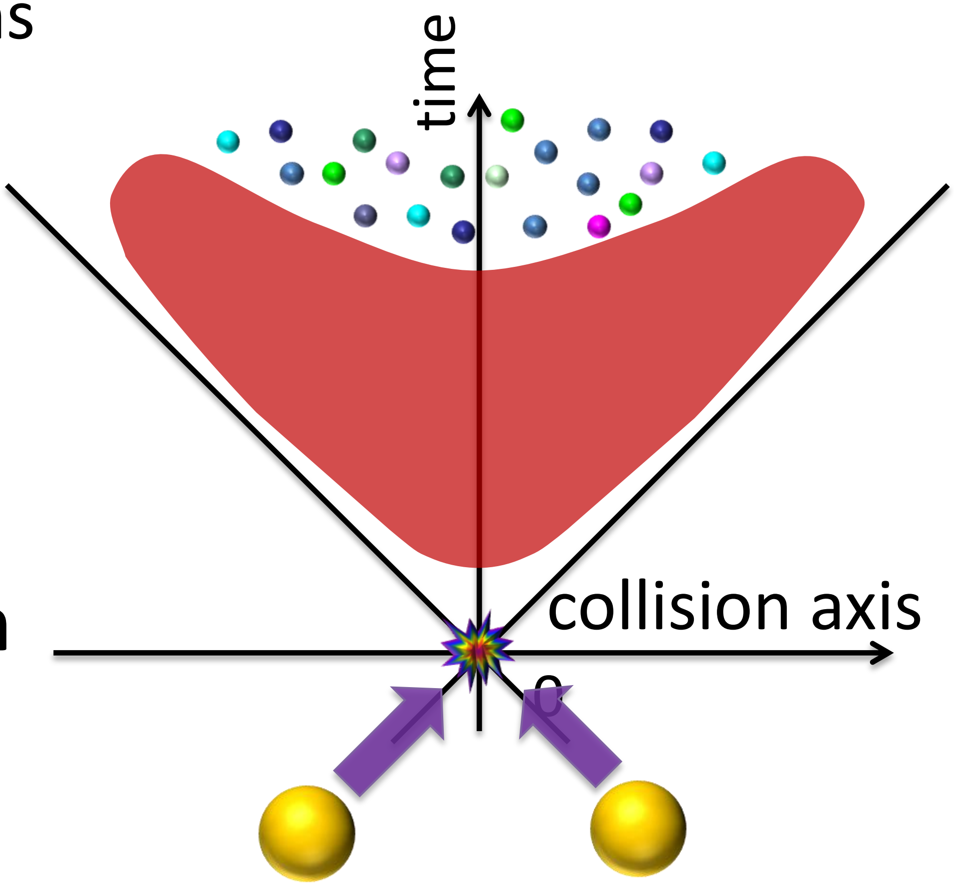
Fluctuations in heavy-ion collisions

- **Final state observables**
  - flow harmonics
  - two-particle correlations

Hadronic cascade

Viscous hydro

- **Initial state fluctuation**
  - nucleon distribution
  - quantum fluctuations

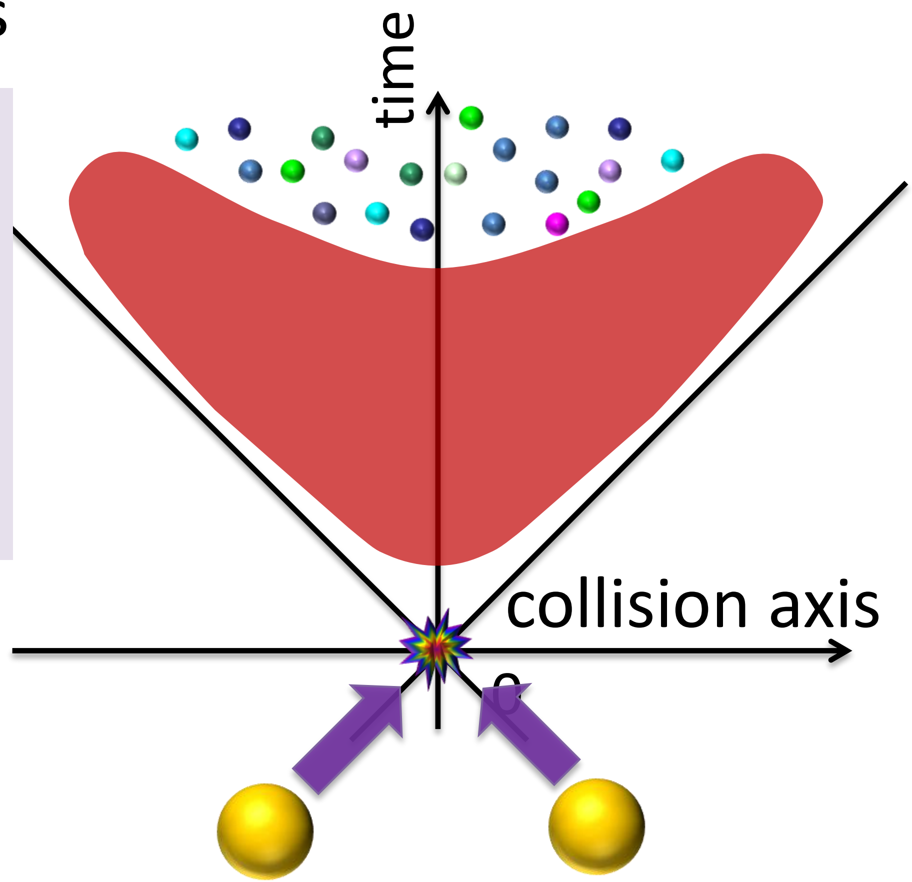


Fluctuations in heavy-ion collisions

- **Final state observables**
  - flow harmonics
  - two-particle correlations

Other fluctuations
- hydro fluctuation
- jets/mini-jets
- critical phenomena
- …

- **Initial state fluctuation**
  - nucleon distribution
  - quantum fluctuations



Fluctuations in heavy-ion collisions

- **Final state observables**
  - flow harmonics
  - two-particle correlations

Experimental data

Other fluctuations
- hydro fluctuation
- jets/mini-jets
- critical phenomena
- …

Quantitative comparison between dynamical model and data

- **Initial state fluctuation**
  - nucleon distribution
  - quantum fluctuations

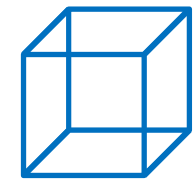
- shear viscosity
- Initial state model
- etc.

Hydrodynamic fluctuation

= thermal fluctuation of the dissipative current field

$$\pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle} + \delta\pi^{\mu\nu},$$

$$\langle\delta\pi^{ij}\delta\pi^{ij}\rangle \sim 4T\eta/V$$



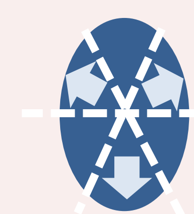
V: 3+1 dim. volume

- **shear viscosity** decreases spatial anisotropy
  → measurable effect in higher harmonics

- **hydrodynamic fluctuations** increases spatial anisotropy
  – $\delta\pi^{\mu\nu}/2\eta\partial^{\langle\mu}u^{\nu\rangle} \sim 1/\sqrt{(\eta/s)(V/\mathrm{fm}^4)}$

  T ~ 200 MeV
  length ~ 1fm

  – important in small η/s and small V
  ← peripheral, pA?, higher harmonics v_n
  – Non-linear evolution



V ~ 1/n

Causal hydrodynamics

- First-order constitutive equation

$$\pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle}$$

  – instability, and acausal modes
- Second-order constitutive equation

$$\pi^{\mu\nu} = -\tau_\pi D\pi^{\langle\mu\nu\rangle} + 2\eta\partial^{\langle\mu}u^{\nu\rangle} + \cdots$$

  – common in recent dynamical calculations

⇒ **hydrodynamic fluctuations in a causal dissipative hydrodynamics**

Constitutive equation (CE)

Constitutive eq. in linear response regime

- integral form: $\Gamma = \Pi, \pi^{\mu\nu}, \nu_i^{\mu}$

$$\Gamma = \int d^4x' \underbrace{G(x-x')}_{\text{memory function}} \underbrace{\kappa F(x')}_{\text{1st order term}}$$

memory function

1st order term

- differential form:

$$\underline{\mathscr{L}\Gamma} = \underline{\kappa F}$$

1st order

Γ + (higher order terms)

finite-order in derivatives

e.g.

$$\mathscr{L} = 1 + \tau_\Pi D$$

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta,$$

Fluctuation-dissipation relation

- Hydro fluctuations appearing in constitutive eq.

$$\Gamma(x) = \int d^4x' \underline{G(x-x')}\underline{\kappa F(x')} + \underline{\delta\Gamma(x)}.$$

hydro fluctuation

Fluctuating hydrodynamics

- **Fluctuation-dissipation relation (FDR)**
  in Generalized Langevin Equation

$$\begin{aligned}\langle \delta\Gamma(x)\delta\Gamma(x')\rangle &= T\kappa G'(x-x') \\ &= T\kappa[G(x-x') + G(x'-x)]\end{aligned}$$

G is not delta function in general → **colored noise**

In differential form

- Fluctuation in the differential form

$$\underline{\mathscr{L}\Gamma} = \underline{\kappa F} + \underline{\xi}, \qquad \xi = \mathscr{L}\delta\Gamma$$

1st order term — hydro fluctuation

- FDR

$$\langle \delta\Gamma(x)\delta\Gamma(x')\rangle = TG'(x-x')$$

colored noise

$$\langle \xi(x)\xi(x')\rangle = \langle \mathscr{L}\delta\Gamma(x)\mathscr{L}\delta\Gamma(x')\rangle = ?$$

white/colored?

- Condition to be white

$$\xi = \mathscr{L}\delta\Gamma: \text{ white} \Leftrightarrow \Re\mathscr{L}_{\omega,\boldsymbol{k}} = L_{\boldsymbol{k}}$$

General properties of memory function G

A) Retarded

$G(x) = 0$ if $t<0$

B) Relaxing in a finite time

$G(x) \to 0$ as $t \to \infty$

C) Timelike

$G(x) = 0$ if $x^2 < 0$

D) Positive-semidefinite (from FDR)

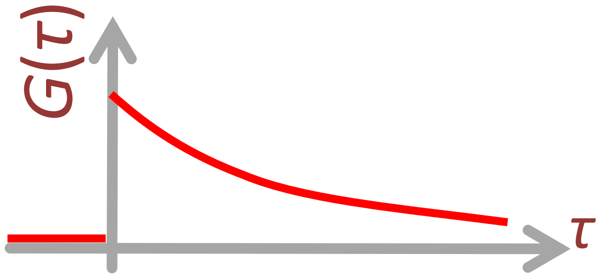
$\langle \delta\Gamma_x \delta\Gamma_{x'} \rangle = TG'_{xx'}$:

variance-covariance matrix of $\delta\Gamma$

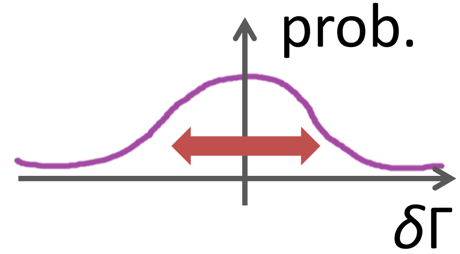
Property of fluctuation in differential form

$$\mathscr{L} = 1 + D^1 + \cdots + D^N$$

L is expanded to ***finite order*** in derivatives



retarded: G=0 if t<0
relaxation: $G \to 0$ as $t \to \infty$
timelike: G=0 if x^2<0



Covariance matrix is ***positive-semidefinite***

proof in backup

$$\mathscr{L}_{\omega,\boldsymbol{k}} = 1 + i\omega A_{\boldsymbol{k}}.$$

$$\langle \xi(x)\xi(x') \rangle = 2T\kappa\delta^{(4)}(x - x').$$

Fluctuation in the differential form is ***white***

Summary

- There are sources of event-by-event fluctuations *during the dynamical evolution* as well as in the initial state.
- ***Hydrodynamic fluctuations***
  - substantial effects on observables, v_n
  - ***colored noise*** in the integral form of CE in a causal dissipative hydrodynamics
  - ***white noise*** in the differential form of CE
    ← constraints on the linear operator of CE, $\mathscr{L}$
    ← the properties of the memory function

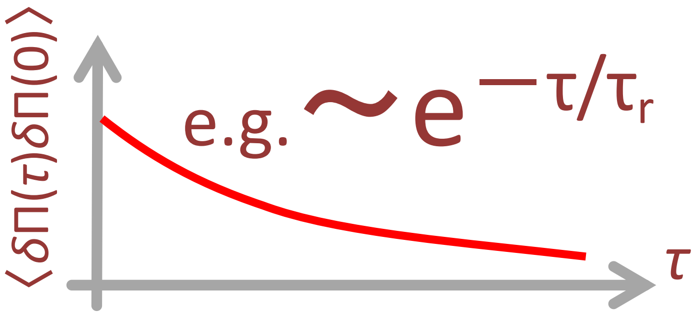
BACKUPS

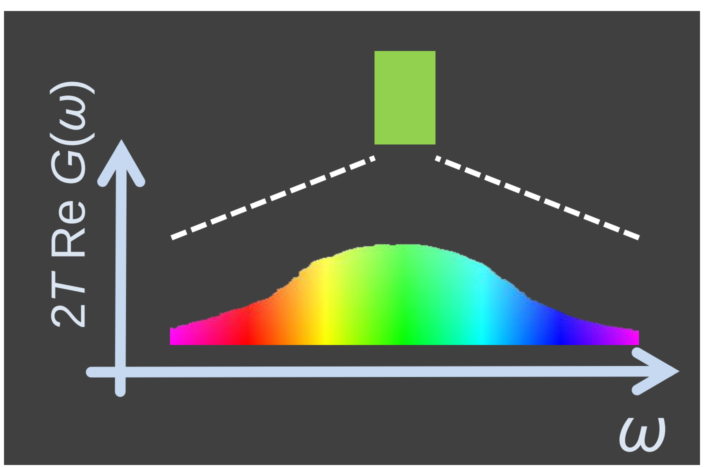
Colored noises

Causal hydro

→time correlation

$\delta\Pi$: colored noise

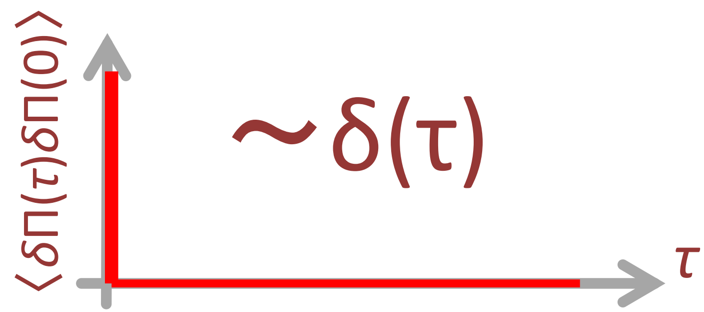


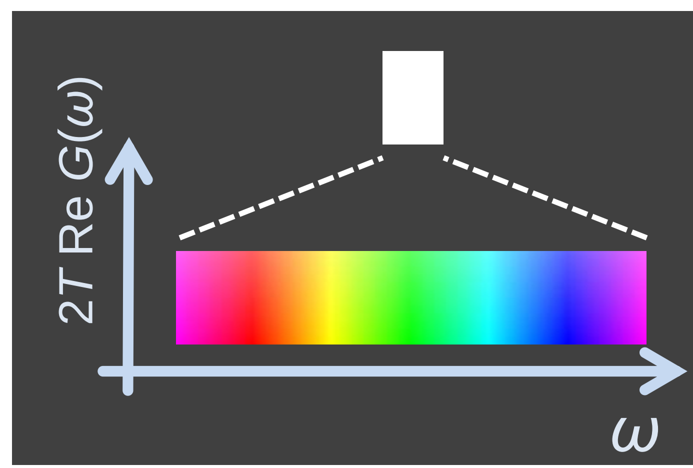


1st order hydro

→No correlation with a finite time difference

$\delta\Pi$: white noise





Backup: condition of ξ to be white

- FDR

$$\langle \delta\Pi^*_{\omega,\boldsymbol{k}} \delta\Pi_{\omega',\boldsymbol{k}'} \rangle = \underline{4\pi\kappa T \Re G_{\omega,\boldsymbol{k}} \delta^{(4)}(k-k')}$$

- Correlation of ξ (= *L*δΠ)

$$\begin{aligned}\langle \xi^*_{\omega,\boldsymbol{k}} \xi_{\omega',\boldsymbol{k}'} \rangle &= \mathscr{L}^*_{\omega,\boldsymbol{k}} \mathscr{L}_{\omega',\boldsymbol{k}'} \langle \delta\Pi^*_{\omega,\boldsymbol{k}} \delta\Pi_{\omega',\boldsymbol{k}'} \rangle \\ &= \mathscr{L}^*_{\omega,\boldsymbol{k}} \mathscr{L}_{\omega',\boldsymbol{k}'} \underline{4\pi\kappa T \Re[1/\mathscr{L}_{\omega,\boldsymbol{k}}] \delta^{(4)}(k-k')} \\ &= \underline{4\pi\kappa T \Re \mathscr{L}_{\omega,\boldsymbol{k}}} \delta^{(4)}(k-k').\end{aligned}$$

white ⇔ const

ξ: *white noise* ⇔ $\Re \mathscr{L}_\omega = \underset{\omega}{\mathrm{const}}\,.$

Backup: Structure of *L* (1)

- *G* is positive-semidefinite

$$G_{kk'} = (2\pi)^4 \delta^{(4)}(k-k')G'(k)$$
$$= 2(2\pi)^4 \delta^{(4)}(k-k')\Re G(k)$$

Eigenvalues $\Re G(\omega) \geq 0$

- *G*=1/*L*

$$\Re \mathscr{L}_k = \Re G(k)/|G(k)|^2 \geq 0$$

$$\Rightarrow \Re \mathscr{L}_{\omega,\boldsymbol{k}} \geq 0$$

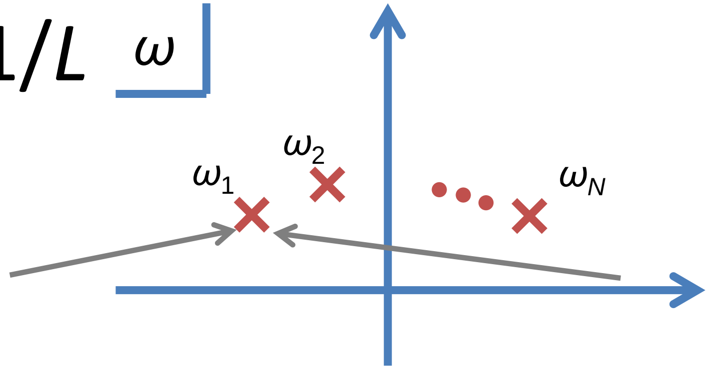
Backup: Structure of *L* (2)

- *L* : up to N^{th} order in *D* (=i*ω*)

$$\mathscr{L}_{\omega,\boldsymbol{k}} \propto a_{\boldsymbol{k}} \prod_{n=1}^{N} i(\omega - \omega_{n,\boldsymbol{k}})$$

N zeroes in *ω*-plane

- zeroes of *L* $=$ poles of *G*=1/*L*
  - *G*: Retarded → $\Im\omega_n \geq 0$
  - *G*: Relaxing → $\Im\omega_n \neq 0$



- complex argument of *L*

$$\arg \mathscr{L} = -\frac{N}{2} + \sum \arg(\omega_{n,\boldsymbol{k}} - \omega)$$
$$\in (-\frac{N}{2}, \frac{N}{2}).$$

➡ $\Re\mathscr{L}_{\omega,\boldsymbol{k}} > 0$
therefore ***N*=0 or 1**

Backup: Structure of *L* (3)

- *N*=0 or 1: $\mathscr{L}_{\omega,\boldsymbol{k}} = i\omega A_{\boldsymbol{k}} + B_{\boldsymbol{k}}$
- FDR

$$\langle \xi(x)\xi(x')\rangle = 2T\kappa\delta(\tau)\left|\frac{\partial\sigma^\alpha}{\partial x^\mu}\right| \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{\sigma}} B_{\boldsymbol{k}}.$$

- *G*: timelike

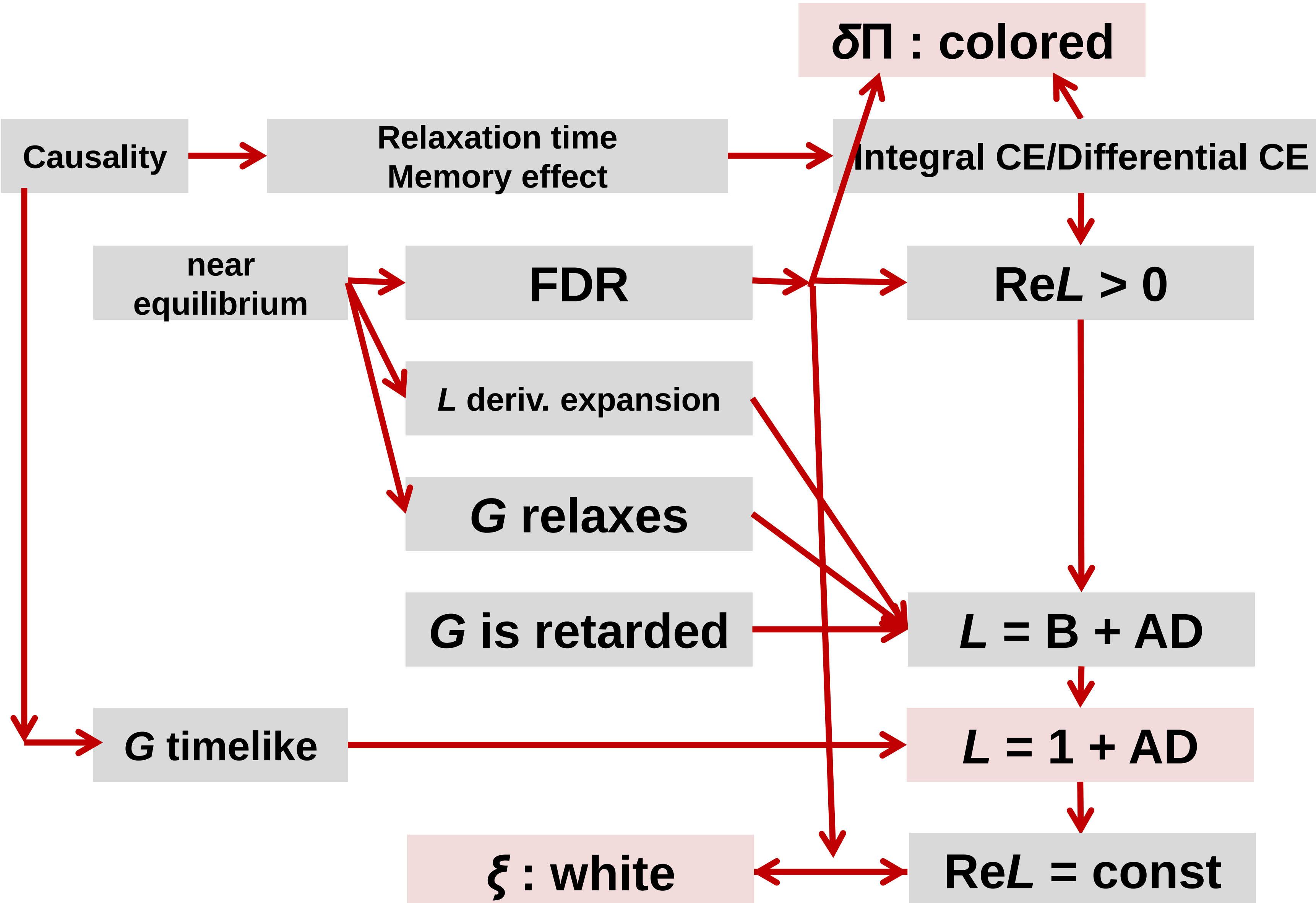
same time → δ-fn in spatial part

$$B_{\boldsymbol{k}} = \text{const} = 1.$$

$$\mathscr{L}_{\omega,\boldsymbol{k}} = 1 + i\omega A_{\boldsymbol{k}}.$$

$$\langle \xi(x)\xi(x')\rangle = 2T\kappa\delta^{(4)}(x - x').$$

Backup: Relations



Example: 1st order dissipative hydro

- Memory function

$$G^*_\Pi(x) = 2\zeta\delta^{(4)}(x)$$

$$G^*_\pi(x)^{\mu\nu\alpha\beta} = 4\eta\delta^{(4)}(x)\Delta^{\mu\nu\alpha\beta}$$

$$G^*_{ij}(x)^{\mu\alpha} = -2\kappa_{ij}\delta^{(4)}(x)\Delta^{\mu\alpha}$$

- CE

$$\Pi = -\zeta\theta + \delta\Pi,$$

$$\pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\mu\rangle} + \delta\pi^{\mu\nu},$$

$$\nu_i^\mu = \kappa_{ij}T\nabla^\mu\frac{\mu_j}{T} + \delta\nu_i^\alpha$$

- FDR

$$\langle\delta\Pi(x)\delta\Pi(x')\rangle = 2T\zeta\ \delta^{(4)}(x-x'),$$

$$\langle\delta\pi^{\mu\nu}(x)\delta\pi_{\alpha\beta}(x')\rangle = 4T\eta\ \delta^{(4)}(x-x')\Delta^{\mu\nu}{}_{\alpha\beta},$$

$$\langle\delta\nu_i^\mu(x)\delta\nu_j^\alpha(x')\rangle = 2T\kappa_{ij}\,\delta^{(4)}(x-x')\cdot(-\Delta^{\mu\alpha})$$

Example: 2nd order dissipative hydro (1)

1. Differential form of CE

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta,$$

$$\tau_\pi \Delta^{\mu\nu}{}_{\alpha\beta} D\pi^{\alpha\beta} + \pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle},$$

$$\tau_{ij} \Delta^{\mu}{}_{\alpha} D\nu_j^\alpha + \nu_i^\mu = \kappa_{ij} T\nabla^\mu \frac{\mu_j}{T},$$

- $D = u^\alpha \partial_\alpha$ time derivative
- projectors for $\pi^{\mu\nu}$, ν^μ

Example: 2nd order dissipative hydro (2)

2. Memory function *G*

$$G_\Pi(x-x') = \zeta \frac{1}{\tau_\Pi} \exp\left(-\frac{\tau-\tau'}{\tau_\Pi}\right) \theta^{(4)}(x-x'),$$

$$G_\pi(x-x')^{\mu\nu\alpha\beta} = 2\eta \frac{1}{\tau_\pi} \exp\left(-\frac{\tau-\tau'}{\tau_\pi}\right) \times \Delta(\tau;\tau')^{\mu\nu\alpha\beta} \theta^{(4)}(x-x'),$$

$$G_{ij}(x-x')^{\mu\alpha} = \tau_{ij}^{-1} \left[\mathrm{T} \exp\left(-\int_{\tau'}^{\tau} d\tau'' \tau_{jk}^{-1}|_{\tau''} \right) \right]_{jk} \kappa_{kl} \times \Delta(\tau;\tau')^{\mu\alpha} \theta^{(4)}(x-x'),$$

$$\theta^{(4)}(x-x') = \left| \frac{\partial \sigma^\mu}{\partial x^\nu} \right| \delta^{(3)}(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\Theta(\tau-\tau')$$

$$\sigma^\mu = (\tau(x), \boldsymbol{\sigma}(x))$$ proper time/comoving coord.

- Exponential relaxation

Example: 2nd order dissipative hydro (3)

- $\Delta(\tau;\tau')$ in memory function: time-by-time projector

$$\Delta(\tau_{\rm f};\tau_{\rm i})^{\mu\nu}{}_{\alpha\beta} = \lim_{N\to\infty} \Delta(\tau_{\rm f})^{\mu\nu}{}_{\alpha_0\beta_0} \times \left[\prod_{k=0}^{N-1} \Delta(\tau_{\rm f} + \tfrac{\tau_{\rm i}-\tau_{\rm f}}{N}k)^{\alpha_k\beta_k}{}_{\alpha_{k+1}\beta_{k+1}}\right] \Delta(\tau_{\rm i})^{\alpha_N\beta_N}{}_{\alpha\beta},$$

$$\Delta(\tau_{\rm f};\tau_{\rm i})^{\mu}{}_{\alpha} = \lim_{N\to\infty} \Delta(\tau_{\rm f})^{\mu}{}_{\alpha_0} \times \left[\prod_{k=0}^{N-1} \Delta(\tau_{\rm f} + \tfrac{\tau_{\rm i}-\tau_{\rm f}}{N}k)^{\alpha_k}{}_{\alpha_{k+1}}\right] \Delta(\tau_{\rm i})^{\alpha_N}{}_{\alpha}.$$

- $\pi^{\mu\nu}$, ν^{μ} is confined in its representation of spatial rotation
  → time-by-time projectors naturally appears in CE.

Example: 2nd order dissipative hydro (4)

3. Integral form of CE

$$\Pi = -\zeta \int_{-\infty}^{\tau} d\tau' \frac{1}{\tau_\Pi} \exp\left(-\frac{\tau - \tau'}{\tau_\Pi}\right) \theta(x') + \delta\Pi,$$

$$\pi^{\mu\nu} = 2\eta \int_{-\infty}^{\tau} d\tau' \frac{1}{\tau_\pi} \exp\left(-\frac{\tau - \tau'}{\tau_\pi}\right) \times \Delta(\tau;\tau')^{\mu\nu\alpha\beta} (\partial_{\langle\alpha} u_{\beta\rangle}|_{x'}) + \delta\pi^{\mu\nu},$$

$$\nu_i^\mu = -\int_{-\infty}^{\tau} d\tau' \tau_{ij}^{-1} \left[\mathrm{T}\exp\left(-\int_{\tau'}^{\tau} d\tau'' \tau_{jk}^{-1}|_{\tau''}\right)\right]_{jk} \kappa_{kl} \times \Delta(\tau;\tau')^{\mu\alpha} (T\nabla_\alpha \tfrac{\mu_j}{T}|_{x'}) + \delta\nu_i^\mu.$$

Example: 2nd order dissipative hydro (5)

4. FDR

$$\langle \delta\Pi(x)\delta\Pi(x')\rangle = T\zeta\frac{1}{\tau_\Pi}\exp\left(-\frac{|\tau-\tau'|}{\tau_\Pi}\right)\theta^{*(4)}(x-x'),$$

$$\langle \delta\pi^{\mu\nu}(x)\delta\pi_{\alpha\beta}(x')\rangle = 2T\eta\frac{1}{\tau_\pi}\exp\left(-\frac{|\tau-\tau'|}{\tau_\pi}\right) \times \Delta(\tau;\tau')^{\mu\nu\alpha\beta}\theta^{*(4)}(x-x'),$$

$$\langle \delta\nu_i^\mu(x)\delta\nu_j^\alpha(x')\rangle = T\tau_{ij}^{-1}\left[\mathrm{T}\exp\left(-\left|\int_{\tau'}^{\tau} d\tau''\tau_{jk}^{-1}|_{\tau''}\right|\right)\right]_{jk}\kappa_{kl} \times \Delta(\tau;\tau')^{\mu\alpha}\theta^{*(4)}(x-x'),$$

In numerical simulations

- The integral form of CE and FDR

e.g.
$$\Pi = -\int_{x^0 > x'^0} d^4x' G_\Pi(x-x') \underline{\theta(x')} + \delta\Pi,$$
in the past

$$\langle \delta\Pi(x)\delta\Pi(x')\rangle = T\zeta \frac{1}{\tau_\Pi} \exp\left(-\frac{-|\tau-\tau'|}{\tau_\Pi}\right) \theta^{*(4)}(x-x'),$$
colored

- The differential form of CE and FDR

e.g.
$$[\hat{A}_\nabla D + 1]\Pi = -\zeta\underline{\theta} + \xi_\Pi,$$
at the moment

$$\langle \xi_\Pi(x)\xi_\Pi(x')\rangle = 2T\zeta\delta^{(4)}(x-x').$$
white

- The differential form of the CE is useful even with the hydrodynamic fluctuations