Bourses postdoctorales Banting Postdoctoral Fellowships

McGill

Derivation of fluid dynamics from kinetic theory

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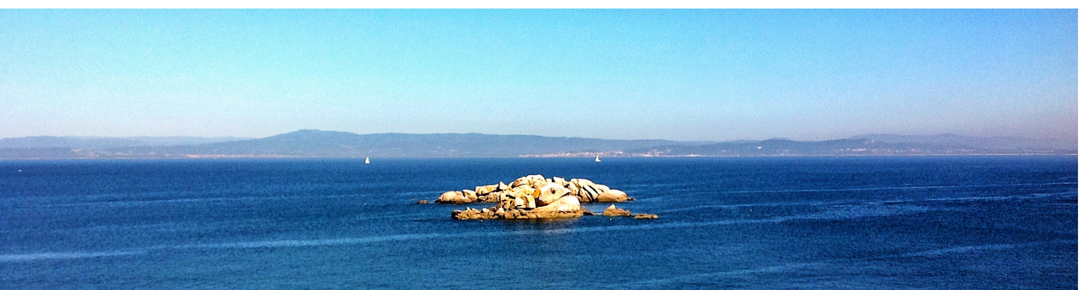


IS2013

Illa de A Toxa, Galicia, Spain

Matching fluid dynamics to kinetic theory

$$\boxed{K^\mu \partial_\mu f_K = C[f]}$$

relativistic Boltzmann equation

$\Downarrow$

Including all non-linear terms, bulk viscous pressure and heat flow.

$$\boxed{\begin{aligned} \partial_\mu N^\mu &= 0, \\ \partial_\mu T^{\mu\nu} &= 0, \\ &\vdots \end{aligned}}$$

relativistic fluid dynamics

Basics of fluid dynamics

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

Charge conservation

$$\partial_\mu N^\mu = 0$$

$$N^\mu = n u^\mu + n^\mu ,$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} (P_0 + \Pi) + \pi^{\mu\nu}$$

Particle diffusion current

Bulk viscous pressure

Shear stress tensor

Always true, regardless of the applicability fluid dynamics

Spatial projector

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$u_\mu u^\mu = 1$$

Transport equations (shear) - two approaches

Navier-Stokes theory

Dissipative currents $\pi_{\mu\nu}$ are proportional to gradients

$$\pi^{\mu\nu} = 2\,\eta\,\sigma^{\mu\nu} + \ldots$$

The equations are **acausal** and **unstable** !!!!

Israel-Stewart theory (transient theory)

Dissipative currents $\pi_{\mu\nu}$ become dynamical variables …

$$\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = 2\,\eta\,\sigma^{\mu\nu} + \ldots$$

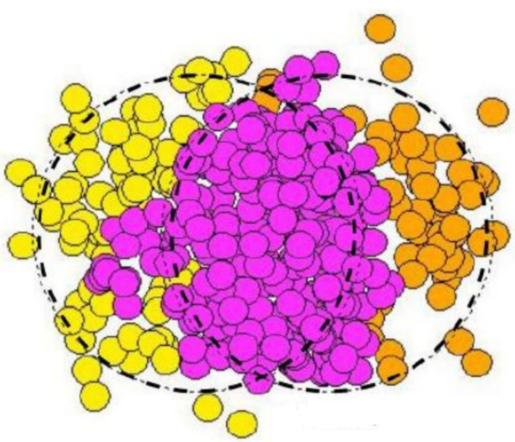
The equations can be **causal** and **stable** !!!!

This is what we want to derive

why bother with a kinetic derivation?

- **Photon** and **dilepton** production & **Jet-medium** interaction
- **Hadron resonance gas** & **freezeout (connecting fluids to particles)**
- **Preequilibrium dynamics**

Also: Extreme time and spatial **scales**



~10 fm

~10 fm

→**very small** system

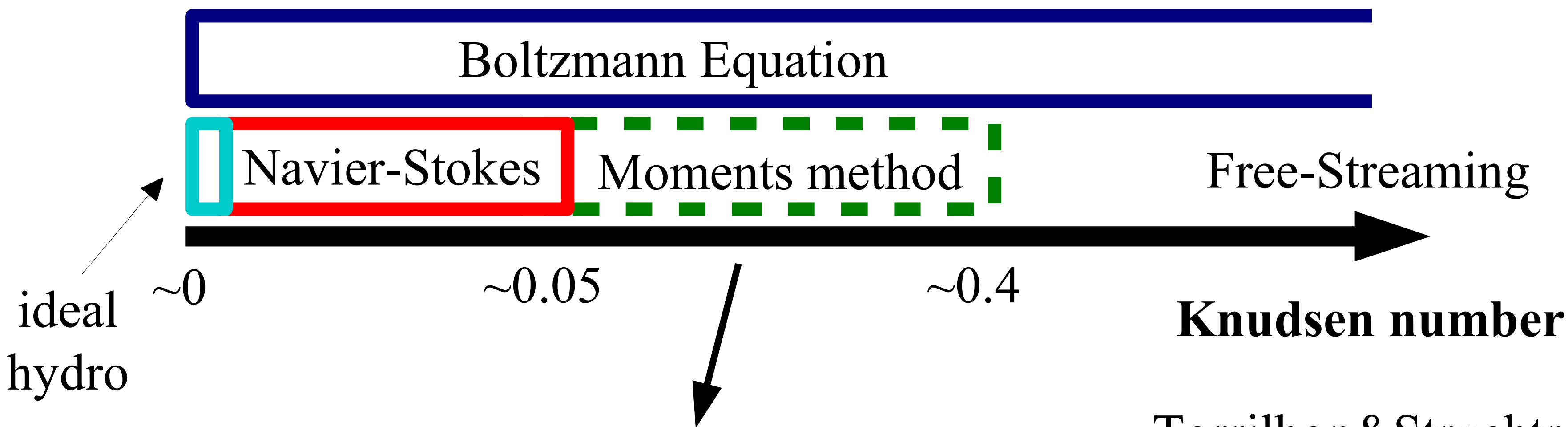
→**very large** gradients

→**very large** expansion rate

- From the fluid-dynamical point of view, challenging to describe

For a dilute gas

In terms of Knudsen number $\mathrm{Kn} = \frac{\ell_{\mathrm{micro}}}{L_{\mathrm{macro}}}$



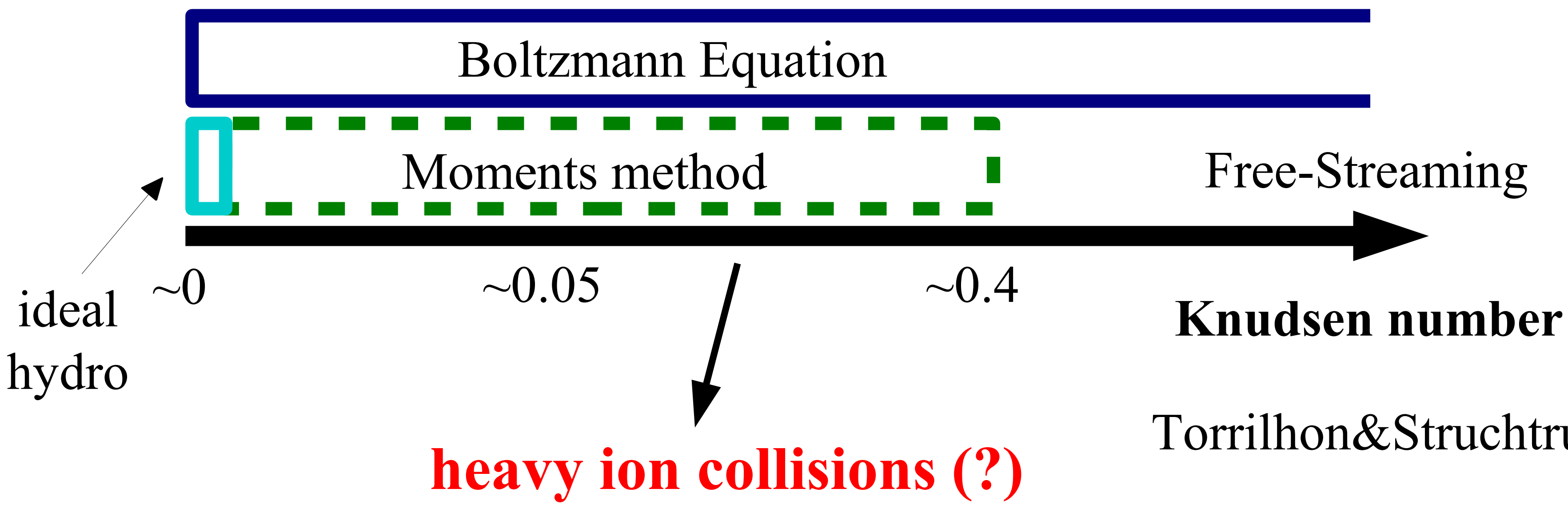
Torrilhon&Struchtrup

heavy ion collisions (?)

In a heavy ion collision ... $\mathrm{Kn} \sim \tau_\pi \nabla_\mu u^\mu \sim \frac{\eta}{s}\frac{1}{T\tau} \sim 0.2 - 1$

For a dilute gas

In terms of Knudsen number $\mathrm{Kn} = \frac{\ell_{\mathrm{micro}}}{L_{\mathrm{macro}}}$

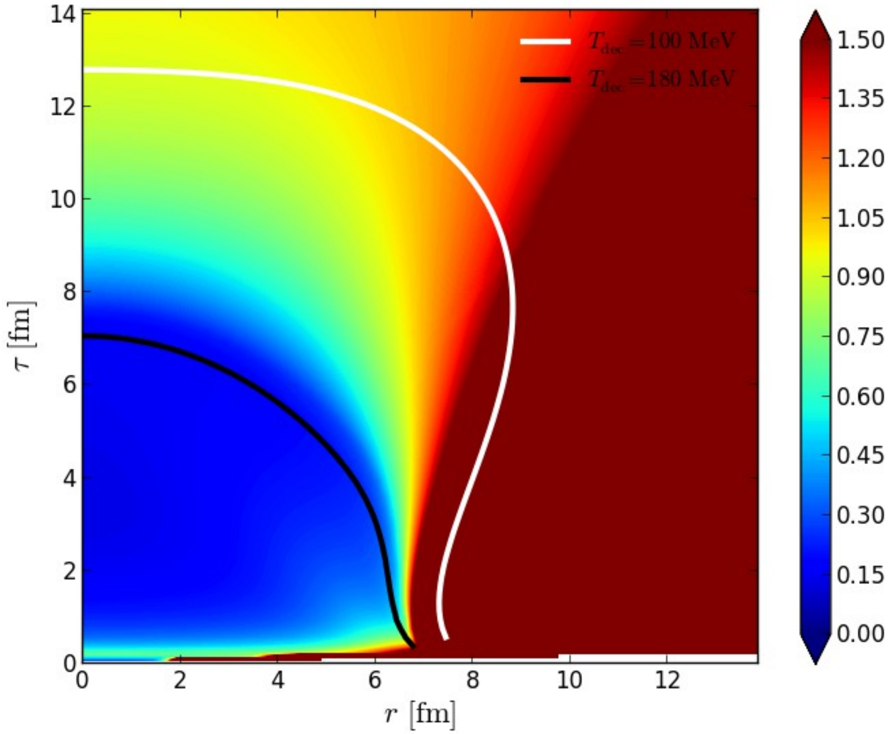


Torrilhon&Struchtrup

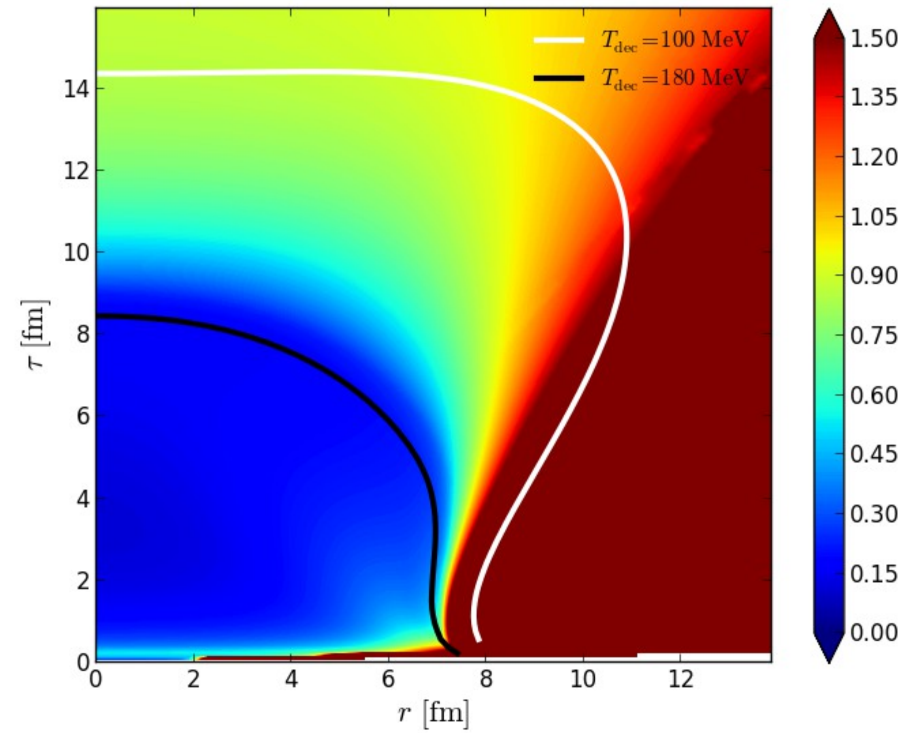
In a heavy ion collision ... $\mathrm{Kn} \sim \tau_\pi \nabla_\mu u^\mu \sim \frac{\eta}{s}\frac{1}{T\tau} \sim 0.2 - 1$

Kn of the QGP $\mathrm{Kn} \sim \tau_\pi \nabla_\mu u^\mu$

RHIC



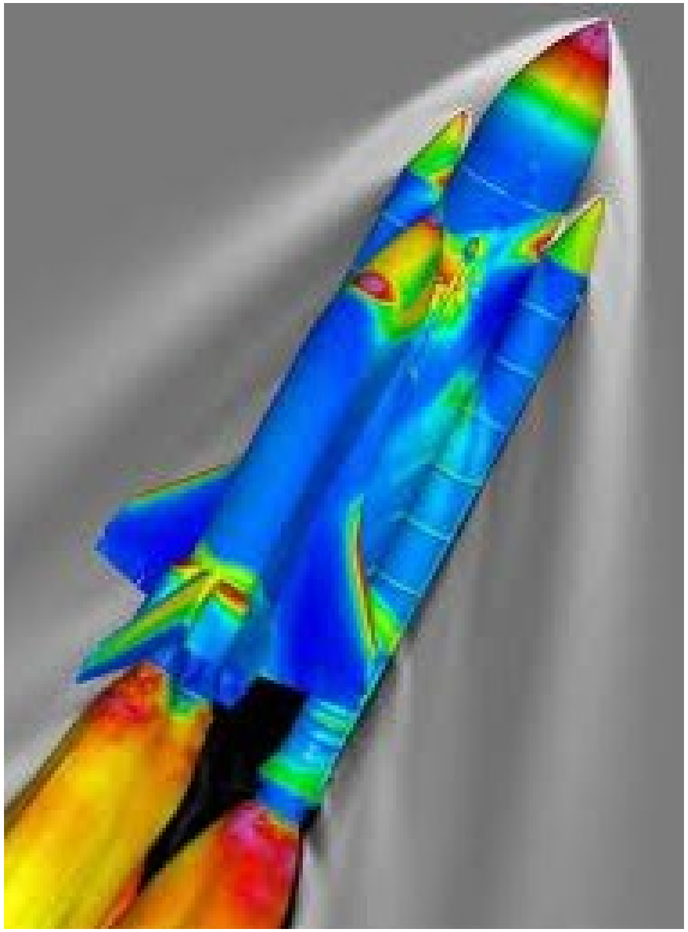
LHC



plots courtesy of H. Niemi

It is important to go beyond Navier-Stokes theory

We should not complain



Aerodynamics of space shuttles

Kn=0.01--20

we have it easy ...

Several Approaches

- Chapman-Enskog-Hilbert

Hilbert (1912), Chapman (1916) and Enskog (1917)
Marle (1964), Samojeden&Kremer(2000)

- Method of moments

Maxwell (1867), Grad (1949) , Mintzner(1965)
Chernikov (1963), Stewart (1969), Marle (1966/69)
Israel&Stewart(1978), Prakash *et al* (1993)
Torrilhon&Struchtrup (2005/08), GSD *et al*(2009/12)
Amaresh *et al*(2012)

- Cumulant expansion

Mao&Eu(1993), Myong(2000)

- Anisotropic hydrodynamics

Strickland&Martinez(2010), Florkowski&Ryblewski(2011)

Several Approaches

- Chapman-Enskog-Hilbert $\longrightarrow$ unstable

Hilbert (1912), Chapman (1916) and Enskog (1917)
Marle (1964), Samojeden&Kremer(2000)

- Method of moments

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Torrilhon&Struchtrup (2005/08), GSD *et al*(2009/12)
Amaresh *et al*(2012)

- Cumulant expansion $\longrightarrow$ not going to discuss here

Mao&Eu(1993), Myong(2000)

- Anisotropic hydrodynamics $\longrightarrow$ already discussed

Strickland&Martinez(2010), Florkowski&Ryblewski(2011)

Israel and Stewart's 14-moment approximation

$$\boxed{f_{\mathbf{k}} \;=\; f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}}$$

Israel-Stewart's theory:

1) **Truncated** moment expansion

$$\phi_{\mathbf{k}} \;=\; \varepsilon + \varepsilon_{\mu} k^{\mu} + \varepsilon_{\mu\nu} k^{\mu} k^{\nu}$$

→ dof's of the Boltzmann are reduced by the **explicit truncation** of the moment expansion!

→ The expansion coefficients are mapped to the conserved currents via the so-called **matching conditions**.

4 eqs. ⬅
$$\begin{aligned} u_\mu N^\mu &= n_0 \\ u_\mu T^{\mu\nu} &= \varepsilon_0 u^\nu \end{aligned}$$

$$\begin{aligned} \pi^{\mu\nu} &= \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \\ n^\mu &= \Delta^\mu_\alpha N^\alpha \\ \Pi &= -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \end{aligned}$$
➡ 10 eqs.

Israel and Stewart's 14-moment approximation

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}}$$

Israel-Stewart's theory:

1) **Truncated** moment expansion

$$\phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu}k^{\mu} + \varepsilon_{\mu\nu}k^{\mu}k^{\nu}$$

2) Equations of motion taken from the **second** moment of the Boltzmann equation

$$\Delta^{\lambda\rho}_{\mu\nu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C\left[f\right] k^{\mu}k^{\nu}\right) \quad \Rightarrow \quad \text{shear}$$

$$u_{\nu}\Delta^{\lambda}_{\mu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C\left[f\right] k^{\mu}k^{\nu}\right) \quad \Rightarrow \quad \text{heat flow}$$

$$u_{\mu}u_{\nu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C\left[f\right] k^{\mu}k^{\nu}\right) \quad \Rightarrow \quad \text{bulk}$$

→ because of 1), Eqs. of motion are **automatically** closed.

Annals Phys. 118 (1979) 341-372

First equations

Israel and Stewart (1978) originally derived

$$\Pi = -\tfrac{1}{3}\zeta_V(u^{\mu}_{E|\mu} + \beta_0\dot{\Pi} - \alpha_0 q^{\mu}{}_{|\mu} - a_0' q^{\mu}\dot{u}_{\mu})$$

$$(\kappa T)^{-1}\, q_\lambda = \Delta^{\mu}_{\lambda}(\alpha_{|\mu}/\eta\beta - \beta_1\dot{q}_{\mu} + \alpha_0\Pi_{|\mu} + \alpha_1\pi^{\nu}_{\mu|\nu})$$
$$+ a_0\Pi\dot{u}_\lambda + a_1\pi_\lambda{}^{\mu}\dot{u}_\mu + \beta_1\omega_{\lambda\mu}q^{\mu}$$

$$\pi_{\lambda\mu} = -2\zeta_S(\Delta^{\alpha}_{\langle\lambda}(u_E)\,\Delta^{\beta}_{\mu\rangle}(u_E)\,u^{E}_{\alpha|\beta} - a_1 q_{\langle\lambda|\mu\rangle} + \beta_2(\dot{\pi})_{\langle\lambda\mu\rangle}$$
$$- a_1' q_{\langle\lambda}\dot{u}_{\mu\rangle} - 2\beta_2\pi^{\alpha}_{\langle\lambda}\omega_{\mu\rangle\alpha}).$$

- Equations were **incomplete**, **but not because of formalism**.
- Goal was to apply in **cosmology**, where **gradients are small**

In a heavy ion collision, gradients are large

A. Muronga ➡ first to derive complete phen. eqs.

D. Rischke ➡ first to derive complete eqs. from BE

J.Phys. G36 (2009) 064029

First complete equations

Rischke *et al* showed (linearized collision term)

$$\Pi = \Pi_{NS} - \tau_\Pi \dot{\Pi} + \tau_{\Pi q} q \cdot \dot{u} - \ell_{\Pi q} \partial \cdot q - \zeta \hat{\delta}_0 \Pi \theta + \lambda_{\Pi q} q \cdot \nabla \alpha + \lambda_{\Pi \pi} \pi^{\mu\nu} \sigma_{\mu\nu},$$

$$q^\mu = q^\mu_{NS} - \tau_q \Delta^{\mu\nu} \dot{q}_\nu - \tau_{q\Pi} \Pi \dot{u}^\mu - \tau_{q\pi} \pi^{\mu\nu} \dot{u}_\nu + \ell_{q\Pi} \nabla^\mu \Pi - \ell_{q\pi} \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda} + \tau_q \omega^{\mu\nu} q_\nu \frac{\kappa}{\beta} \hat{\delta}_1 q^\mu \theta - \lambda_{qq} \sigma^{\mu\nu} q_\nu + \lambda_{q\Pi} \Pi \nabla^\mu \alpha + \lambda_{q\pi} \pi^{\mu\nu} \nabla_\nu \alpha,$$

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{NS} - \tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + 2\tau_{\pi q} q^{\langle\mu} \dot{u}^{\nu\rangle} + 2\ell_{\pi q} \nabla^{\langle\mu} q^{\nu\rangle} + 2\tau_\pi \pi_\lambda{}^{\langle\mu} \omega^{\nu\rangle\lambda} - 2\eta \hat{\delta}_2 \pi^{\mu\nu} \theta - 2\tau_\pi \pi_\lambda{}^{\langle\mu} \sigma^{\nu\rangle\lambda} - 2\lambda_{\pi q} q^{\langle\mu} \nabla^{\nu\rangle} \alpha + 2\lambda_{\pi\Pi} \Pi \sigma^{\mu\nu},$$

- Equations are **complete! 23 coefficients** **(no vorticity term)**

Lack of power-counting in Kn makes it difficult to determine the range of applicability

Phys.Rev.Lett. 105 (2010) 162501
Phys.Rev. D85 (2012) 114047

More robust approach

(1) Expansion of distribution function in terms of its moments

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$$

(2) Obtain the exact equations of motion for these moments

$$\dot{\rho}_{(r)}^{\langle \mu_1 \dots \mu_\ell \rangle} = \Delta^{\mu_1 \dots \mu_\ell}_{\nu_1 \dots \nu_\ell} \frac{d}{d\tau} \int dK \left(E_{\mathbf{k}}\right)^r k^{\langle \nu_1} \dots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}},$$

(3) Obtain the long-time limit of these equations $X_{(i)}^{\mu_1 \dots \mu_\ell} \equiv \sum_{j=0} \left(\Omega_\ell^{-1}\right)^{ij} \rho_{(j)}^{\mu_1 \dots \mu_\ell}$

slowest modes are in the transient regime

$$\begin{aligned} \tau_\Pi \dot{X}_{(0)} + X_{(0)} &\sim \theta + \dots, \\ \tau_n \dot{X}^\mu_{(0)} + X^\mu_{(0)} &\sim \nabla^\mu \alpha_0 + \dots, \\ \tau_\pi \dot{X}^{\mu\nu}_{(0)} + X^{\mu\nu}_{(0)} &\sim \sigma^{\mu\nu} + \dots, \end{aligned}$$

other modes are in the asymptotic regime

$$\begin{aligned} X_{(i)} &\sim \theta + \dots, \\ X^\mu_{(i)} &\sim \nabla^\mu \alpha_0 + \dots, \\ X^{\mu\nu}_{(i)} &\sim \sigma^{\mu\nu} + \dots, \end{aligned}$$

i>0

Final equations

The result is,

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{\zeta}{\tau_\Pi}\theta + \underline{\mathcal{J}} + \underline{\mathcal{R}} + \underline{\mathcal{K}},$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \frac{\kappa_n}{\tau_n} I^\mu + \underline{\mathcal{J}^\mu} + \underline{\mathcal{R}^\mu} + \underline{\mathcal{K}^\mu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} + \underline{\mathcal{J}^{\mu\nu}} + \underline{\mathcal{R}^{\mu\nu}} + \underline{\mathcal{K}^{\mu\nu}}.$$

Terms

- *O(1)* in R^{-1},Kn
- *O(2)* in Kn
- *O(2)* in R^{-1}

1) **Knudsen number**

$$\mathrm{Kn} = \frac{\ell_{\mathrm{micr}}}{L_{\mathrm{macr}}},$$

2) **Inverse Reynolds number**

$$\mathrm{R}_\Pi^{-1} \sim \frac{|\Pi|}{P_0}, \quad \mathrm{R}_n^{-1} \sim \frac{|n^\mu|}{n_0}, \quad \mathrm{R}_\pi^{-1} \sim \frac{|\pi^{\mu\nu}|}{P_0}.$$

Final equations

$\boxed{\textit{O(1)} \text{ in } R^{-1}, Kn}$

$$\mathcal{J} = -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot\nabla P_0 - \lambda_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n\cdot\nabla\alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

$$\mathcal{J}^{\mu} = -n_{\nu}\omega^{\nu\mu} - \delta_{nn}n^{\mu}\theta - \ell_{n\Pi}\nabla^{\mu}\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_{\lambda}\pi^{\lambda}_{\nu} + \tau_{n\Pi}\Pi\nabla^{\mu}P_0$$
$$- \tau_{n\pi}\pi^{\mu\nu}\nabla_{\nu}P_0 - \lambda_{nn}n_{\nu}\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi\nabla^{\mu}\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_{\nu}\alpha_0,$$

$$\mathcal{J}^{\mu\nu} = 2\pi^{\langle\mu}_{\alpha}\,\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}_{\alpha}\,\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$
$$- \tau_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}P_0 + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0.$$

The above transport coefficients depend on **all** the moments of the distribution function.

Final equations

O(2) in R^{-1}

$$\mathcal{R} = \varphi_1 \Pi^2 + \varphi_2 n \cdot n + \varphi_3 \pi_{\mu\nu}\pi^{\mu\nu},$$
$$\mathcal{R}^\mu = \varphi_4 n_\nu \pi^{\mu\nu} + \varphi_5 \Pi n^\mu,$$
$$\mathcal{R}^{\mu\nu} = \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} + \varphi_8 n^{\langle\mu} n^{\nu\rangle}.$$

$$F^\mu = \nabla^\mu P_0$$
$$I^\mu = \nabla^\mu \alpha_0$$

O(2) in Kn

$$\mathcal{K} = \omega_{\mu\nu}\omega^{\mu\nu} + \zeta_1 \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_2 \theta^2 + \zeta_3 I \cdot I + \zeta_4 F \cdot F + \zeta_5 I \cdot F + \zeta_6 \nabla_\mu I^\mu + \zeta_7 \nabla_\mu F^\mu,$$
$$\mathcal{K}^\mu = \kappa_1 \sigma^{\mu\nu} I_\nu + \kappa_2 \sigma^{\mu\nu} F_\nu + \kappa_3 \theta I^\mu + \kappa_4 \theta F^\mu + \kappa_5 \omega^{\mu\beta} I_\beta + \kappa_6 \Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu} + \kappa_7 \nabla^\mu \theta,$$
$$\mathcal{K}^{\mu\nu} = \eta_1 \omega_\lambda^{\ \langle\mu}\omega^{\nu\rangle\lambda} + \eta_2 \theta \sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \eta_4 \sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu} I^{\nu\rangle} + \eta_6 F^{\langle\mu} F^{\nu\rangle} + \eta_7 I^{\langle\mu} F^{\nu\rangle} + \eta_8 \nabla^{\langle\mu} I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu} F^{\nu\rangle}$$

If Kn~R^{-1}

Similar equations (to IS theory) are recovered

$$\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi &= -\zeta\theta - \ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot F - \delta_{\Pi\Pi}\Pi\theta \\
&\quad - \lambda_{\Pi n} n\cdot I + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}\ , \\
\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu &= \kappa_n I^\mu \ - n_\nu\omega^{\nu\mu} - \delta_{nn}n^\mu\theta - \ell_{n\Pi}\nabla^\mu\Pi \\
&\quad + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda_\nu + \tau_{n\Pi}\Pi F^\mu - \tau_{n\pi}\pi^{\mu\nu}F_\nu \\
&\quad - \lambda_{nn}n_\nu\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^\mu - \lambda_{n\pi}\pi^{\mu\nu}I_\nu, \\
\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + 2\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} \\
&\quad + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n}n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} \\
&\quad + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle} + \eta_1\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda}\ .
\end{aligned}$$

All higher-order corrections can be incorporated in transport coefficients

What most people solve

- Most simulations neglect nonlinear terms
- Most simulations neglect bulk viscous pressure
- All simulations neglect heat flow

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \frac{4}{3}\tau_\pi\pi^{\mu\nu}\theta$$

Majority of conclusions of our field are based on these equations

(oldMUSIC, Ohio Group)

H. Niemi → includes all nonlinear terms

P.Bozek → includes bulk

Is there any point in improving this?

Compare

wo/ nonlinear terms

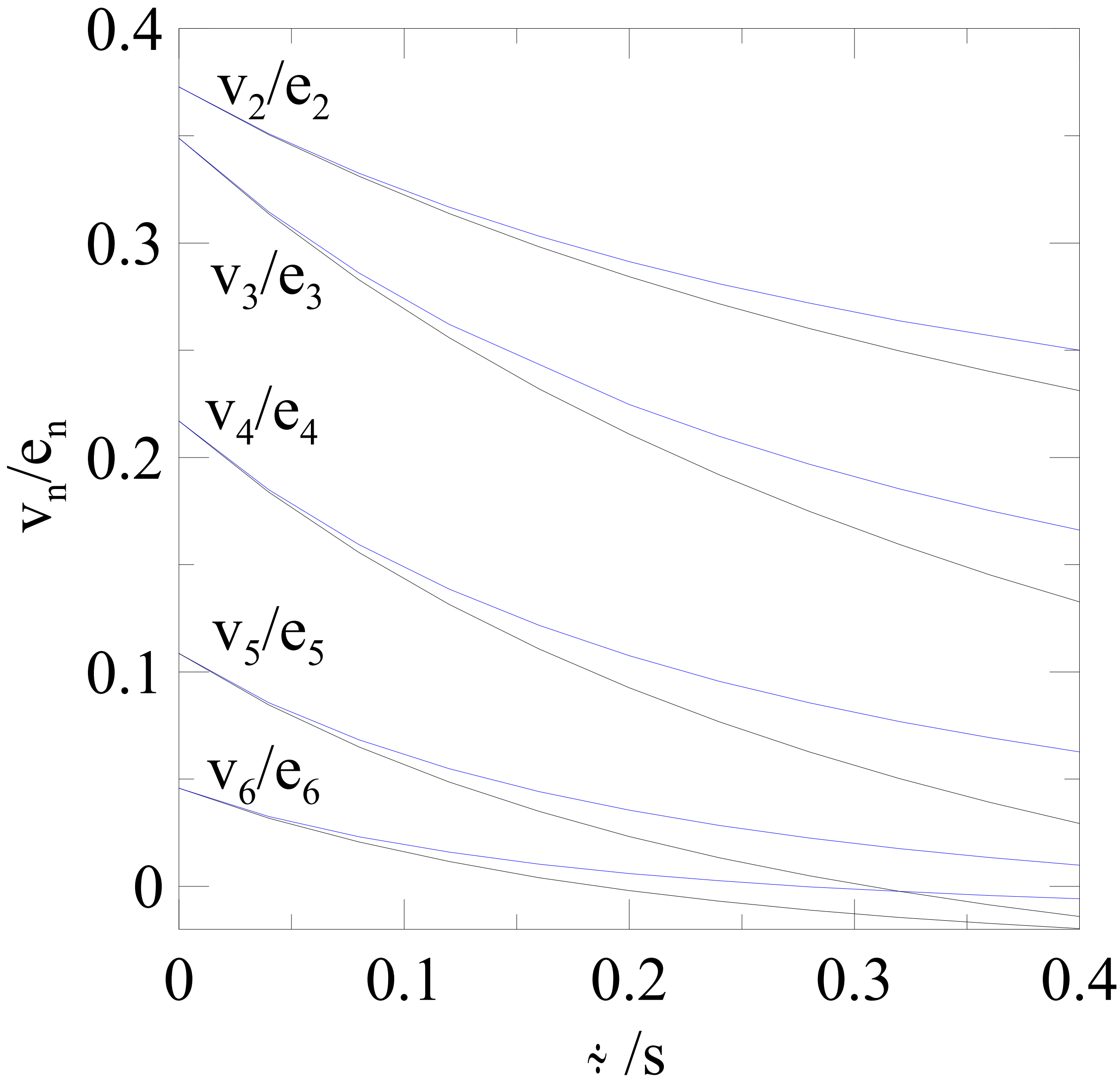
$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \frac{4}{3}\tau_\pi \pi^{\mu\nu}\theta$$

w/ nonlinear terms

$$\begin{aligned}\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} \; &= \; 2\eta\sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \frac{4}{3}\tau_\pi \pi^{\mu\nu}\theta \\ &\quad + \frac{18}{35}\tau_\pi \frac{\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha}}{\varepsilon_0 + P_0} - \frac{10}{7}\tau_\pi \pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}.\end{aligned}$$

what is the difference?

Effect of nonlinear terms



Ultracentral collisions - LHC

wo/ nonlinear terms

w/ nonlinear terms

MUSIC

Summary/conclusions

We derived hydro from kinetic theory using the method of moments

- When viscosity is large, nonlinear terms become relevant

- Effects on dileptons and photons can be high since they probe the early phases

Ongoing

- Effects of bulk viscosity and coupling between bulk and shear
- Heat flow
- δf