Lectures on Standard Model Particle Physics

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July 4-10, 2013, CERN Summer Student Lecture Programme

Lectures will discuss:

- Spacetime symmetries of the Standard Model
- Internal (gauge) symmetries of the Standard Model
- Observables and their precision tests of the theory
- Higgs boson theory and its discovery

Special Relativity

We begin with a statement of our most cherished symmetries. Laws of physics should be invariant under special relativity transformations: rotations (3 of them) and velocity boosts (3 of them). This implies that the length ds^2 should be invariant under transformations

$$ds^2 = c^2 dt^2 - dx_i dx_i = c^2 dt'^2 - dx'_i dx'_i \tag{1}$$

where c is the same in all reference frames. Define it to be $c = 1$.

Construct Lorentz four-vector $dx^\mu = (dt, dx^1, dx^2, dx^3)$ where $\mu = 0, 1, 2, 3$. Define metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

Then $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Any "position" 4-vector x^μ contracted with itself is its length and should be invariant.

$$x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x'^\mu x'_\mu \tag{3}$$

What are the transformations on x^μ that leave its length invariant?

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \text{suppressing indices}: \;\; x' = \Lambda x \tag{4}$$

Substituting this into the invariance-requirement equation above gives

$$x^T g x = x'^T g x' = (\Lambda x)^T g (\Lambda x) = x^T (\Lambda^T g \Lambda) x \tag{5}$$

Thus we have to find matrices Λ that satisfy

$$g = \Lambda^T g \, \Lambda \tag{6}$$

If $g = E$ where $E = \text{diag}(1,1,1,1)$ is the identity matrix, it would be much more familiar to you. In that case

$$E = \Lambda^T \Lambda \quad \Longrightarrow \quad \Lambda^T = \Lambda^{-1} \tag{7}$$

This last condition is the definition of special orthogonal matrices, which must have $\det \Lambda = \pm 1$. The set of matrices is then $SO(4)$, 4×4 orthogonal matrices.

Definition of a group

$SO(4)$ is a "group", which has a very precise mathematical meaning.

A group G is a collection of elements $g \in G$ endowed with a multiplication operator that satisfies four axioms:

1. **Closure:** For every $g_1, g_2 \in G$, $g_1 g_2 \in G$
2. **Associativity:** For all $g_1, g_2, g_3 \in G$, $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
3. **Identity:** There exists an $e \in G$ such that for all $g \in G$, $eg = ge = g$
4. **Inverse:** For every $g \in G$ there is a $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$

The mathematics of group theory plays a significant role in the description of symmetries, which includes the symmetries of the Standard Model.

Note, there is no requirement that $g_1 g_2 = g_2 g_1$.

If this equality is satisfied it is an "Abelian group" (e.g., $U(1)$, $SO(2)$); otherwise, it is called a "Non-Abelian group" (e.g., $SU(2)$, $SO(3)$, etc.).

Z_2 discrete group

One of the simplest groups of all is the Z_2 group. It has two elements $\{1, -1\}$ and group multiplication is normal multiplication.

Multiplication table:

	1	-1
1	1	-1
-1	-1	1

This is sometimes called the even/odd group.

Forms a group because the four axioms are respected:
1. Closure: check
2. Associativity: check
3. Identity: check
4. Inverse: check

This is an example of a discrete abelian group.

Example of a Group, $SO(2)$

These are simply two-dimensional rotations that you are used to. Every group element is parameterized by a rotation angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{8}$$

Multiply two elements together and we get

$$\begin{aligned} g(\theta_1)g(\theta_2) &= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} & (9) \\ &= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \cos\theta_1\sin\theta_2 \\ -\cos\theta_1\sin\theta_2 - \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} = g(\theta_3) & (10) \end{aligned}$$

Thus, closure is satisfied by $g(\theta_1)g(\theta_2) = g(\theta_3)$ where $\theta_3 = \theta_1 + \theta_2$.

Associativity obviously works; the identity element is when $\theta = 0$; and, the inverse of $g(\theta)$ is $g(-\theta)$, which is in $SO(2)$. Thus, all the group axioms are satisfied.

Back to the Lorentz Group, $SO(3,1)$

However, in our case our metric tensor g is not the identity matrix but rather has a mixed metric of three -1 entries and one $+1$ entry. Nevertheless the elements Λ that satisfy $g = \Lambda^T g\, \Lambda$ form a group, called $SO(3,1)$.

Here are a few examples of elements in $SO(3,1)$:

$$\Lambda_R = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \quad \text{where } \mathbf{R} \text{ are the } 3 \times 3 \text{ rotation matrices } SO(3). \tag{11}$$

$$\Lambda_{B_x} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{velocity boost in the } x \text{ direction} \tag{12}$$

where

$$\cosh\eta = \gamma \;\text{ and }\; \sinh\eta = \beta\gamma, \;\text{ with }\; \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \;\text{ and }\; \beta = v/c. \tag{13}$$

With a little algebra you can see that $x' = \Lambda_{B_x} x$ is equivalent to what you are used to seeing

$$ct' = \frac{1}{\sqrt{1 - v^2/c^2}}\left(ct - \frac{v}{c}x\right) \tag{14}$$

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt) \tag{15}$$

$$y' = y \tag{16}$$

$$z' = z \tag{17}$$

Summary of this: $SO(3,1)$ is a group and the matrices Λ are its elements, and their transformations on Lorentz vectors are rotations and boosts that we are familiar with.

What does all of this have to do with particle physics?

The "Symmetry Invariance Principle"

Symmetry Invariance Principle

When we say that nature is invariant under some symmetry, it means

- All objects in the theory have well defined transformation properties (i.e., well defined "representation" of the symmetry group) under the symmetry, and
- Every interaction is invariant (i.e., a "singlet") under the symmetry transformations

The "objects" of particle physics are particle fields.

The interactions in particle physics are the operators in the lagrangian.

Singlet and Triplet representations of $SO(3)$

Representations of groups can be intuitively understood from tensor analysis from rotations, the group $SO(3)$ with elements R_{ij} satisfying the condition $R^T = R^{-1}$ and $\det R = 1$.

Let's start with a vector. If we rotate the vector v we get

$$v \to v' = R\,v, \quad \text{or equivalently} \quad v'_i = R_{ij}v_j. \tag{18}$$

The vector v is definite transformations properties under $SO(3)$ and it has three independent elements (v_x, v_y, v_z) and so it defines a "three-dimensional representation" or "triplet representation" of $SO(3)$. Or, for short, **3**.

There is always the trivial or "singlet" representation:

$$c \to c' = c \quad \text{singlet representation.} \tag{19}$$

This is the **1** representation, or sometimes called the "scalar representation".

We have just defined rather precisely the **1** and **3** representations of $SO(3)$ from the scalar and vector. What about tensors? Does a tensor form a separate representation of $SO(3)$. Yes, but it's slightly more complicated!

Tensor representations of $SO(3)$

Let us look at the tensor formed from two vectors: $T_{ij} = a_i b_j$. This tensor has 9 elements. However, there are subspaces of these 9 elements that have definite and closed transformation properties under $SO(3)$.

The most obvious is the *trace*: $\tau = \mathrm{Tr}(T) = a_i b_i$. Under rotation it is preserved.

$$\tau' \longrightarrow R_{il} a_l R_{ik} b_k = R^T_{ki} R_{il} a_l b_k = \delta_{kl} a_l b_k = a_l b_l = \tau \tag{20}$$

The trace of the tensor therefore is a singlet $\mathbf{1}$ representation of $SO(3)$.

Now let us look at the *anti-symmetric tensor* $A^T = -A$,

$$A_{ij} = a_i b_j - a_j b_i = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} \quad \text{(3 independent elements)} \tag{21}$$

The anti-symmetric tensor does not change its character under transformations

$$A'_{ml} = R_{mi} R_{lj} A_{ij} = R_{mi} R_{lj} (-A_{ji}) = -R_{lj} R_{mi} A_{ji} = -R_{li} R_{mj} A_{ij} = -A'_{lm}$$

Thus the anti-symmetric tensor forms a $\mathbf{3}_A$ representation of $SO(3)$.

Let's now take the symmetric tensor $S = S^T$,

$$S_{ij} = a_i b_j + a_j b_i = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} \quad \text{(6 distinct entries)} \tag{22}$$

However, we have already used the trace to form a representation, so we need to "subtract out the trace". What we really need is the traceless symmetric tensor $\hat{S}^T = \hat{S}$ with $\mathrm{Tr}(\hat{S}) = 0$:

$$\hat{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & -S_{11} - S_{22} \end{pmatrix} \quad \text{(5 independent elements)} \tag{23}$$

This traceless symmetric tensor preserves its character under transformations

$$\hat{S}_{ml} = R_{mi} R_{lj} \hat{S}_{ij} = R_{mi} R_{lj} \hat{S}_{ji} = R_{li} R_{mj} \hat{S}_{ij} = \hat{S}_{lm} \tag{24}$$

Thus, the traceless symmetric tensor forms a $\mathbf{5}_S$ representation of $SO(3)$.

So, the 9 elements of the tensor form a reducible rep of $SO(3)$, which can be decomposed into three irreps of dimension $\mathbf{1}$, $\mathbf{3}$ and $\mathbf{5}$. In group theory language we we did was show that

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}_A \oplus (\mathbf{1} \oplus \mathbf{5})_S \tag{25}$$

There are an infinite number of representations of $SO(3)$

All representations can found by taking tensor products of the vectors

$$T_{ij} = a_i b_j \tag{26}$$

$$T_{ijk\ldots} = a_i b_j c_k \ldots \tag{27}$$

There are techniques to do this, and tables exist that classify all representations.

Representation of dimension d: Group elements $g \in G$ are mapped to $d \times d$ matrices $M(g)$ that preserve all the group multiplications. I.e., if $g_1 g_2 = g_3$ then $M(g_1)M(g_2) = M(g_3)$.

Warning: Often it is said that

"X is a representation r of the symmetry group G",

whereas what is really meant is

"X is an object such that when a symmetry transformation of G is applied, it transforms under the r representation"

What does this have to do with the Lorentz group?

The relevance is because

1. $SO(3)$ and $SU(2)$ are closely related, and

2. The representations of $SO(3,1)$ can be classified in terms of representations of $SU(2) \times SU(2)$

Next, I will remind you why point 1 is correct.

In order to show you that point 2 is correct, I will need to tell you about "generators" for group elements and the Lie algebras that they form.