The connection between $SO(3)$ and $SU(2)$

$SU(2)$ is the set of all 2×2 complex matrices of $\det = +1$ that satisfies $A^\dagger A = I$,

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1. \tag{28}$$

There are three independent parameters required to specify an element here of $SU(2)$. Here is one

$$A = \begin{pmatrix} \cos(\theta/2)\exp\{\frac{i}{2}(\psi+\phi)\} & \sin(\theta/2)\exp\{\frac{i}{2}(\psi-\phi)\} \\ -\sin(\theta/2)\exp\{-\frac{i}{2}(\psi-\phi)\} & \cos(\theta/2)\exp\{-\frac{i}{2}(\psi+\phi)\} \end{pmatrix} \tag{29}$$

$$\text{where } \; 0 \le \theta \le \pi, \quad 0 \le \psi \le 4\pi, \quad 0 \le \phi \le 2\pi. \tag{30}$$

The objects that transform under $SU(2)$ elements like this are called "spinors" χ_α, where $\alpha = 1, 2$. They are the analogy of vectors in three dimensional rotation group ($SO(3)$):

$$\chi'_\alpha = A_{\alpha\beta}\,\chi_\beta, \quad \alpha, \beta = 1, 2. \tag{31}$$

2-to-1 homomorphic mapping of $SU(2)$ onto $SO(3)$

Theorem (Cornwell 1984): "There exists a two-to-one homomorphic mapping of the group $SU(2)$ onto the group $SO(3)$. If $A \in SU(2)$ maps onto $R(A) \in SO(3)$, then $R(A) = R(-A)$ and the mapping may be chosen so that

$$R(A)_{jk} = \frac{1}{2}\mathrm{Tr}\left(\sigma_j A \sigma_k A^{-1}\right), \quad j,k = 1,2,3 \tag{32}$$

$$\text{where} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{33}$$

are the Pauli spin matrices."

This implies that the representations of $SO(3)$ are also representations of $SU(2)$. In addition there are "spinor" representations of $SU(2)$ that have no analog in $SO(3)$.

The "defining representation" of $SU(2)$ with 2×2 elements $A_{\alpha\beta}$ is the lowest dimension spinor representation.

Representations of $SU(2)$

You already know the $SU(2)$ representations well from quantum mechanics!

In QM you analyzed spin carefully and found basis functions of the form

$$|\ell m\rangle \quad \text{where } \ell = 0,\ 1/2,\ 1,\ 3/2,\ 2, \ldots, \ \text{and } m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

Each ℓ labels a distinct irreducible representation of $SU(2)$, and the number of m's ($= 2\ell + 1$) is the dimensionality of the representation.

The representations of integer ℓ are equivalent to $SO(3)$ reps (bosonic reps), and the half-integer representations are the spinor representations (fermionic reps).

Ok, now we understand that $SU(2)$ and $SO(3)$ have lots of different representations of different dimensions.

Before we show how this relates to the Lorentz group and the labels we give to elementary particles, we must introduce two more concepts: generators and Lie algebras. We will do this first through the more intuitive $SO(3)$ and then show them for $SU(2)$ and then for the Lorentz group $SO(3,1)$.

Generators of $SO(3)$ group elements

Let's consider the matrix

$$s_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{34}$$

And now compute $e^{i\phi s_x}$:

$$e^{i\phi s_x} = 1 + i\phi s_x + \frac{1}{2!} i^2 \phi^2 s_x^2 + + \frac{1}{3!} i^3 \phi^3 s_x^3 + \cdots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \tag{35}$$

which is a rotation about the x axis. In this sense s_x is a "generator" of the rotations.

The word "generator" comes from the fact that an infinitesimal rotation about the x-axis is $\propto \phi$, and so one can build up or "generate" the full rotation by adding an infinite number of infinitesimal rotations:

$$R_x(\phi) = \lim_{N\to\infty} \left(1 + i\frac{\phi}{N} s_x\right)^N \longrightarrow e^{i\phi s_x}. \tag{36}$$

Similarly, rotations about the y and z axis are generated by

$$s_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad s_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

The set of three generators s_x, s_y and s_z of $SO(3)$ forms a real Lie algebra.

What is a Lie algebra?

Reminder: Algebra

(Szekeres 2004) "An algebra consists of a vector space A over a field K together with a law of composition or product of vectors, $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ denoted

$$(A, B) \to AB \in \mathcal{A} \quad (A, B \in \mathcal{A}), \tag{38}$$

which satisfies a pair of distributive laws:

$$A(aB + bC) = aAB + bAC, \quad (aA + bB)C = aAC + bBC \tag{39}$$

for all scalars $a, b \in K$ and vectors A, B and C."

Definition of a Lie algebra

Real Lie algebra: Real lie algebra $\mathcal{L}$ of dimension $n\,(\geq 1)$ is a real vector space of dimension n equipped with a "Lie product" or "commutator" $[a, b]$ defined for every a and b of $\mathcal{L}$ such that (Cornwell 1984)

1. $[a, b] \in \mathcal{L}$ for all $a, b \in \mathcal{L}$
2. $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$ for all $a, b, c \in \mathcal{L}$ and all real numbers α and β
3. $[a, b] = -[b, a]$ for all $a, b \in \mathcal{L}$
4. for all $a, b, c \in \mathcal{L}$ the Jacobi identity holds

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

The commutator algebra of the $SO(3)$ generators is

$$[s_x, s_y] = i s_z \;\text{ et cyclic} \tag{40}$$

and s_x, s_y and s_z are the "basis vectors" of the vector space.

These commutator relationships are very familiar! They are exactly the same as $SU(2)$, whose generators are $\sigma_i/2$:

$$\left[\frac{\sigma_x}{2}, \frac{\sigma_y}{2}\right] = i\frac{\sigma_z}{2} \;\text{ et cyclic} \tag{41}$$

Thus, we see that although the $SU(2)$ group is different than the $SO(3)$ group, the Lie algebras are isomorphic. This is another way in which $SO(3)$ and $SU(2)$ are very similar.

Now, all this will help in understanding why $SO(3,1) \simeq SU(2) \times SU(2)$.

Generator algebra of $SO(3,1)$

The Lorentz group has three rotations and three boosts, and thus there are six total generators acting on the four-vectors.

The three generators of rotation are

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

The three generators of boosts are

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

The algebra of Lorentz group generators

The generators form an algebra with commutation relations

$$[J_x, J_y] = iJ_z \text{ et cyclic}; \qquad [K_x, K_y] = -iJ_z \text{ et cyclic}; \tag{44}$$

$$[J_x, K_y] = iK_z \text{ et cyclic}; \qquad [J_x, K_x] = 0 \text{ et cetera.} \tag{45}$$

The algebra can be recast more simply be redefining

$$A = \frac{1}{2}(J + iK) \text{ and } B = \frac{1}{2}(J - iK) \tag{46}$$

which leads to

$$[A_x, A_y] = iA_z \text{ et cyclic}; \quad [B_x, B_y] = iB_z \text{ et cyclic}; \tag{47}$$

$$\text{and } [A_i, B_j] = 0, \text{ for all } i, j = x, y, z. \tag{48}$$

This appears to be the same algebra as that of two independent $SU(2)$ algebras.

This is why $SO(3,1)$ representations can be classified as $SU(2) \times SU(2)$ representations.

This is also why $SU(2)$ spin shows up everywhere in quantum mechanics. It's because of Lorentz symmetry!

$SU(2) \times SU(2)$ Representations of the Lorentz Group

Lorentz Rep	Total Spin	Elementary Particle Quantum Field
$(0,0)$	0	"scalars" (Higgs boson)
$\left(\frac{1}{2},0\right)^\sharp$	$\frac{1}{2}$	"left spinor" (leptons, quarks, neutrinos)*
$\left(0,\frac{1}{2}\right)^\sharp$	$\frac{1}{2}$	"right spinor" (leptons, quarks, neutrinos)*
$\left(\frac{1}{2},\frac{1}{2}\right)$	1	"vector gauge field" (γ, Z, $W^\pm$, gluons)†
$\left(1,\frac{1}{2}\right)$	$\frac{3}{2}$	"Rarita-Schwinger field" (no SM particle)
$(1,1)$	2	"spin-2 field" (graviton)

♯ The Lorentz group has an operation that allows a spinor ξ that transforms under $SU(2)_L$ to transform as a spinor under $SU(2)_R$: $\chi = i\sigma_2\xi^*$. Related by parity transformation $K \to -K$ (i.e., $A \to B$ generators.)

* The fermions (leptons, quarks, and neutrinos) are often treated as a four-dimensional representation $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$.

† The gauge fields A^μ take $(1/2,1/2)$ representation form by $\sigma_\mu^{\alpha\dot{\alpha}}A^\mu$, where $\alpha = 1,2$ is spinor index for $SU(2)_L$ and $\dot{\alpha} = 1,2$ is spinor index for $SU(2)_R$. We'll call this σA for short.