Interactions preserving Lorentz invariance

From mathematical point of view

particle: an object that has a well-defined transformation property under Lorentz symmetry.

If nature is to be invariant under the Lorentz symmetry then the only interactions allowed among particles are those that are singlets under both left and right spin groups.

You know how to do this!

There are two facts you learned from early days that help:

1. Tensor product of spin s_1 and s_2 give spin $|s_1 - s_2|$ and $s_1 + s_2$ results, and
2. If all tensor indices are contracted, the result is a scalar – invariant!

We can manipulate and understand invariants using these facts, and build up all Lorentz invariants of the theory.

Products of spin – Majorana fermion mass

Just as in QM we realized that

$$|1/2\rangle \otimes |1/2\rangle = |0\rangle \oplus |1\rangle \tag{49}$$

we can do the same thing with particles, with the slight complication that we need to keep track of the left and right $SU(2)$'s separately.

Consider the spinor $f_R = (0, 1/2)$, and let's ask if $f_R \cdot f_R$ interaction is ok:

$$\text{left}: \quad 0 \otimes 0 = 0 \quad \text{contains singlet} \tag{50}$$

$$\text{right}: \quad 1/2 \otimes 1/2 = 0 \oplus 1 \quad \text{contains singlet} \tag{51}$$

so this is an invariant. It is the mass operator: $m f_R \cdot f_R$.

To be more precise, there is a spinor-metric on the contraction which is $i\sigma^2 = \varepsilon$. The mass operator is

$$m\, f_R^T i\sigma^2 f_R \quad \text{(Majorana mass)} \tag{52}$$

If f_R has charge (e.g., electric charge) this term is not invariant, and not allowed. In the Standard Model only right-handed neutrinos qualify for this type of mass:

$$M_R\, \nu_R^T i\sigma^2 \nu_R. \tag{53}$$

Dirac fermion mass

What if we have $f_L = (1/2, 0)$ and $f_R = (0, 1/2)$.

We learned early that a Lorentz invariant is $\chi_R^T i\sigma^2 f_R$, and we also learned that $i\sigma^2 f_L^*$ transforms like a RH-fermion. Thus, identifying $\chi_R = i\sigma^2 f_L^*$, we have

$$\chi_R^T i\sigma^2 f_R \; = \; (i\sigma^2 f_L^*)^T i\sigma^2 f_R = f_L^\dagger i(\sigma^2)^T i\sigma^2 f_R = f_L^\dagger f_R \tag{54}$$

which used the facts that $(\sigma^2)^T = -\sigma^2$ and $\sigma^2\sigma^2 = 1$.

Likewise $f_R^\dagger f_L$ is an invariant, which is just the conjugate of $f_L^\dagger f_R$.

Therefore, we have identified a new fermion bilinear invariant (i.e., mass term):

$$m_f\,(f_L^\dagger f_R + f_R^\dagger f_L). \tag{55}$$

This is often called Dirac mass.

Four component spinor representation

We have been talking about f_L and f_R, and making mass terms that connect the two. But you have more commonly heard only labels like "electron", "muon", "quarks." We can put the two-component f_L and f_R spinors, Weyl spinors, into a four-component spinor, the Dirac spinor as this:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{56}$$

We can construct the projection operators $P_L\Psi = (\psi_L\ 0)$ and $P_R\Psi = (0\ \psi_R)$ from

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma^5) \ \text{ where } \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{57}$$

The four dimensional analogy to the σ^μ matrices are the γ^μ matrices, where in the Weyl representation they are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{58}$$

The γ^0 acting on Ψ interchanges $\psi_L \leftrightarrow \psi_R$ (parity operation).

Four-component spinor invariants

In this four component notation, we can write the Majorana and Dirac mass terms.

For the Majorana mass, let us define the built-up four-component spinor of χ_L (which transforms under $(1/2, 0)$ representation) to be

$$\Psi_M = \begin{pmatrix} \chi_L \\ i\sigma^2 \chi_L^* \end{pmatrix} \quad \text{and} \quad \Psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{59}$$

The Majorana and Dirac mass terms are then

$$m\, \Psi_M^T (-i\gamma^0\gamma^2) \Psi_M \quad \text{(Majorana mass)} \tag{60}$$

$$m_D\, \bar{\Psi}_D \Psi_D \quad \text{where } \bar{\Psi}_D = \Psi_D^\dagger \gamma^0 \quad \text{(Dirac mass)} \tag{61}$$

Vector particle invariants

A vector particle A_μ (e.g., photon) has many invariants. Easy: just contract all the Lorentz indices.

The invariants up to dimension four are

$$A_\mu A^\mu, \quad \partial_\mu A^\mu, \quad \partial_\mu A^\mu \partial^\nu A_\nu, \quad \partial_\mu A^\nu \partial^\mu A_\nu, \quad A^\mu A^\nu A_\mu A_\nu. \tag{62}$$

If A_μ is the gauge field of a $U(1)$ invariant theory, such as QED, interactions must be invariant *also* under gauge transformations

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \Lambda \tag{63}$$

The only interaction that is invariant under both Lorentz symmetry and gauge symmetry is

$$F_{\mu\nu}F^{\mu\nu}, \quad \text{where } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{64}$$

This is the well-known kinetic energy term in the QED lagrangian

$$\mathcal{L}_{QED}^{KE} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{65}$$

Vector particles interacting with fermions

The vector A_μ interaction with fermions requires us to consider its $(1/2, 1/2)$ representation: $\sigma_\mu A^\mu$. A general interaction

$$(1/2, 0) \otimes (1/2, 1/2) \otimes (0, 1/2) = \text{ contains } (0,0) \text{ contains singlet} \tag{66}$$

If $\chi_R = (0, 1/2)$ and $f_L = (1/2, 0)$ then we can have interaction

$$\chi_R^T \cdot \sigma_\mu A^\mu \cdot f_L \tag{67}$$

where the first (second) $\cdot$ refers to $SU(2)_R$ ($SU(2)_L$) contraction.

Consider $\chi_R = i\sigma^2 f_L^*$. Interaction becomes

$$(i\sigma^2 f_L^*)^T \cdot \sigma_\mu A^\mu \cdot f_L = -i f_L^\dagger (\sigma^2)^T \sigma^2 \sigma_\mu A^\mu \cdot f_L = i f_L^\dagger \sigma_\mu A^\mu \cdot f_L \tag{68}$$

In four component language we see this interaction as

$$i\bar{\Psi}\gamma^\mu A_\mu \Psi. \tag{69}$$

Scalar interactions with itself, fermions and vector bosons

The invariant interactions of vector bosons with scalars is also easy. If we assume real scalar ϕ, we have

$$\partial_\mu\phi\partial^\mu\phi, \quad \phi, \quad \phi^2, \quad \partial^\mu\partial_\mu\phi, \quad \phi^3, \quad \phi^4, \quad \text{etc.} \tag{70}$$

For charged complex scalar Φ (like the Higgs boson doublet) invariance under Lorentz and charge symmetry allow

$$(\partial_\mu\Phi^*)(\partial^\mu\Phi), \quad \Phi^*\Phi, \quad (\Phi^*\Phi)^2, \quad \text{etc.} \tag{71}$$

Interactions with the vector bosons are

$$A^\mu A_\nu\Phi^*\Phi, \quad A^\mu\Phi^*\,\partial_\mu\Phi, \quad \text{etc.} \tag{72}$$

Interactions with fermions include

$$\phi\nu_R^T i\sigma^2\nu_R, \quad \Phi\nu_R^T i\sigma^2\nu_R, \quad \phi f_R^\dagger f_L, \quad \Phi f_R^\dagger f_L, \quad \text{etc.} \tag{73}$$

Lorentz invariance is too general for what is witnessed in nature

Lorentz invariance alone allows us to classify particles and gives strong constraints on what particles are allowed to interact. For example, one cannot have the interactions

$$A^\mu f_R, \quad f_L^\dagger f_L f_R, \quad \Phi f_L A_\mu, \quad \text{etc.} \quad \text{(Lorentz forbidden)} \tag{74}$$

But there are many other interactions forbidden that Lorentz invariance alone does not preclude. These include

$$A_\mu A^\mu, \quad e_L^\dagger \sigma^\mu A_\mu \cdot u_L, \quad \tau_R^T \sigma^2 \tau_R, \quad \mu_L^\dagger \Phi\, t_R, \quad \text{etc.} \tag{75}$$

These are forbidden by "internal" gauge symmetries. The Standard Model particles are charged not only under $SU(2)_L \times SU(2)_R$ Lorentz symmetry, but also under $SU(3)_c \times SU(2)_W \times U(1)_Y$ gauge symmetries.

Interactions must be invariant under the transformations of every symmetry.

We discuss next the gauge symmetries of the Standard Model.

Strong, weak and hypercharge forces

The Standard Model particles also transform as representations of the strong, weak and hypercharge forces, which in group theory language is

$$SU(3)_c \times SU(2)_W \times U(1)_Y \quad \text{(Standard Model gauge groups).} \tag{76}$$

If a particle φ transforms as d dimensional representation R of group G, then

$$\varphi \to \varphi' = e^{i\theta_k T_k^R} \varphi \tag{77}$$

where T^R are $d \times d$ dimensional generator matrices associated with the representation R, and θ_k are the parameters of the group, analogous to the angle of rotation in $SO(2)$.

Global symmetries mean θ_k do not depend on spacetime, whereas with local symmetries they do, $\theta_k(x)$.

Gauge symmetries are local internal symmetries.

Hypercharge gauge symmetry

Hypercharge is a $U(1)$ gauge symmetry, and its generator is the hypercharge operator Y, and the parameter we can define as α.

Under gauge transformation

$$\psi \to \psi' = e^{i\alpha(x)Y}\psi \tag{78}$$

Let's look at the transformation of the kinetic operator

$$\begin{aligned}
\psi_L^\dagger \sigma_\mu \partial^\mu \cdot \psi_L &\to (e^{i\alpha Y}\psi_L)^\dagger \sigma_\mu \partial^\mu \cdot e^{i\alpha Y}\psi_L & (79)\\
&= \psi_L^\dagger e^{-i\alpha Y}\sigma_\mu e^{i\alpha Y} \cdot (iY\partial^\mu \alpha \psi_L + \partial^\mu \psi_L) & (80)\\
&= \psi_L^\dagger \sigma_\mu \cdot \partial^\mu \psi_L + \psi_L^\dagger \sigma_\mu \cdot (iY\partial^\mu \alpha)\psi_L & (81)
\end{aligned}$$

The kinetic term would be invariant if it weren't for $\partial^\mu \alpha \neq 0$ contribution.

Introduce covariant derivative $D^\mu = \partial^\mu - iYA^\mu$ (introducing gauge field A^μ), and one finds

$$\psi_L^\dagger \sigma_\mu D^\mu \cdot \psi_L, \quad \text{is invariant when} \tag{82}$$

$$A_\mu \to A'_\mu = A_\mu + i\partial_\mu \alpha. \tag{83}$$

Field	$SU(3)$	$SU(2)_L$	T^3	$\frac{Y}{2}$	$Q = T^3 + \frac{Y}{2}$
g^a_μ (gluons)	$\mathbf{8}$	$\mathbf{1}$	0	0	0
$(W^\pm_\mu, W^0_\mu)$	$\mathbf{1}$	$\mathbf{3}$	$(\pm 1, 0)$	0	$(\pm 1, 0)$
B^0_μ	$\mathbf{1}$	$\mathbf{1}$	0	0	0
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
u_R	$\mathbf{3}$	$\mathbf{1}$	0	$\frac{2}{3}$	$\frac{2}{3}$
d_R	$\mathbf{3}$	$\mathbf{1}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
$E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
e_R	$\mathbf{1}$	$\mathbf{1}$	0	-1	-1
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\Phi^c = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Invariants under $SU(3)_c$ and $SU(2)$ are found when tensor products yield a singlet $\mathbf{1}$. Under $SU(3)$ conjugate representations are distinct (i.e., $\bar{Q}_L$ is $\bar{\mathbf{3}}$). Conjugate reps for $SU(2)$ are not distinct.

$$SU(3)_c : \qquad \bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} + \mathbf{8} \tag{84}$$

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \tag{85}$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} + \cdots \tag{86}$$

$$SU(2)_W : \qquad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3} \tag{87}$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \cdots \tag{88}$$