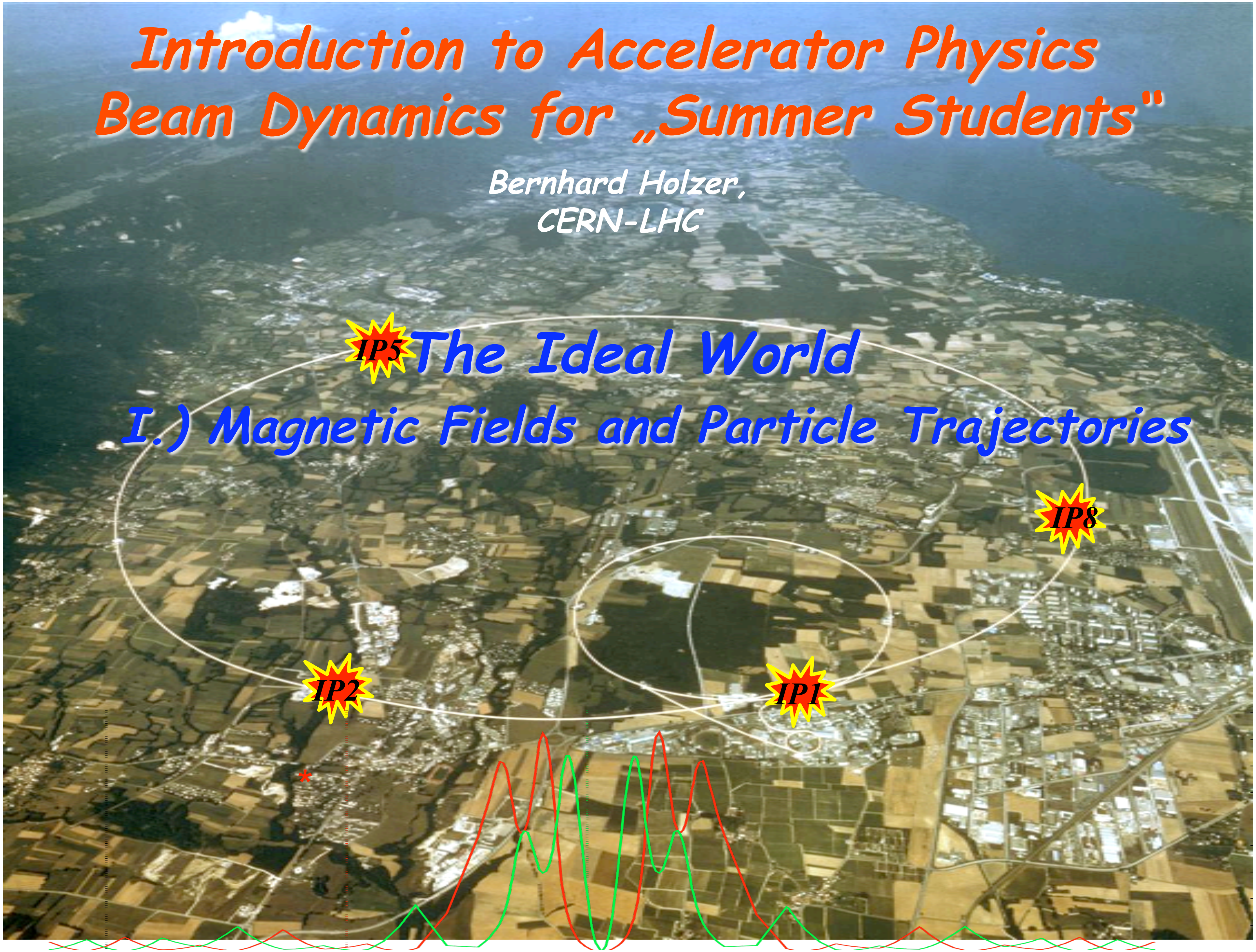
II A Bit of Theory

Introduction to Accelerator Physics
Beam Dynamics for „Summer Students"

Bernhard Holzer,
CERN-LHC

The Ideal World

I.) Magnetic Fields and Particle Trajectories



Luminosity Run of a typical storage ring:

LHC Storage Ring: Protons ***accelerated and*** *stored for 12 hours*

distance of particles travelling at about $v \approx c$

$L = 10^{10}\text{-}10^{11}$ *km*

... several times Sun - Pluto and back ♪

→ *guide the particles* ***on a well defined orbit*** *(„design orbit")*

→ *focus the particles* ***to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.***

1.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine"

→ need transverse deflecting force

Lorentz force $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3*10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3*10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

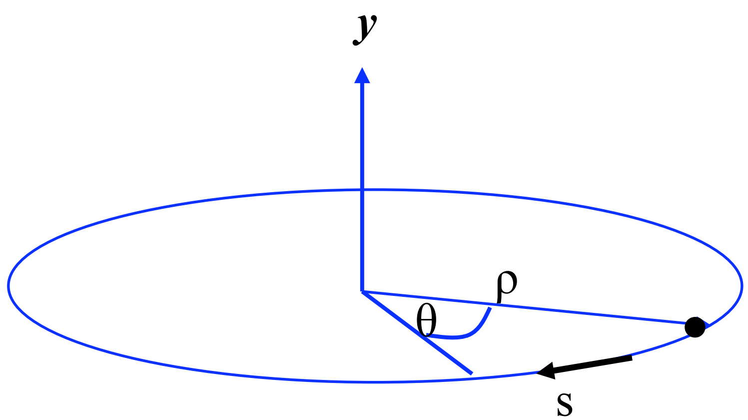
equivalent el. field ... E

technical limit for el. field:

$$E \le 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

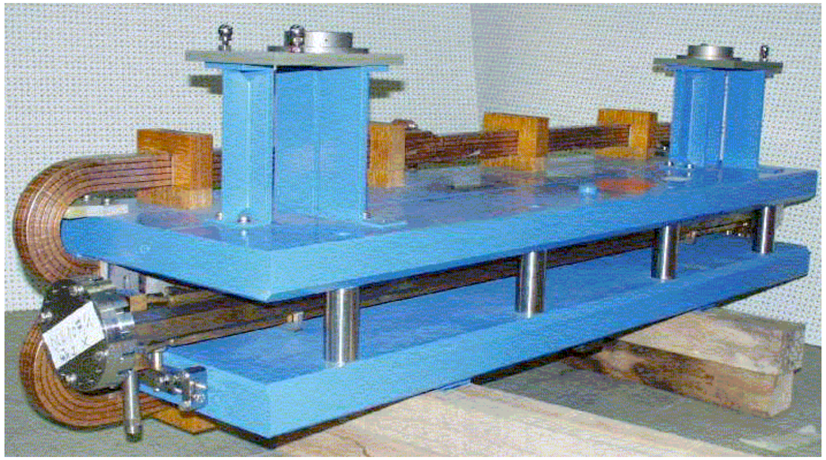
centrifugal force $F_{centr} = \dfrac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma\, m_0\, \cancel{v^2}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

2.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \longrightarrow \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

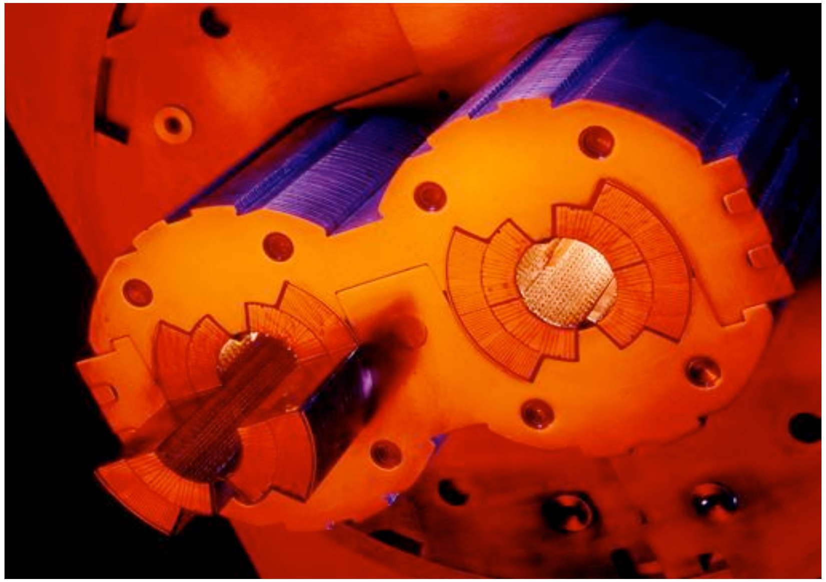
Example LHC:

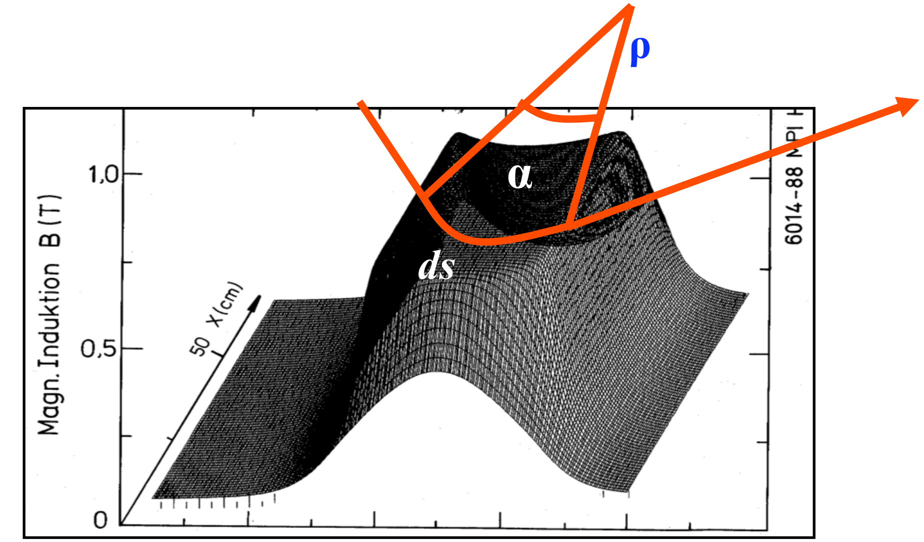
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{Vs}/{m^2}}{7000*10^9\,{eV}/{c}} = \frac{8.3\,s*3*10^8\,{m}/{s}}{7000*10^9\,m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,{1}/{m}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$\rho = 2.53\ \boldsymbol{km} \longrightarrow$ $\boldsymbol{2\pi\rho = 17.6\ km}$

$\boldsymbol{\approx 66\%}$

$\boldsymbol{B} \approx 1 \ldots 8\ \boldsymbol{T}$

rule of thumb: $\boxed{\dfrac{1}{\rho} \approx 0.3\,\dfrac{B[T]}{p[GeV/c]}}$

„normalised bending strength"

The Problem:

LHC Design Magnet current: I=11850 A

and the machine is 27 km long !!!

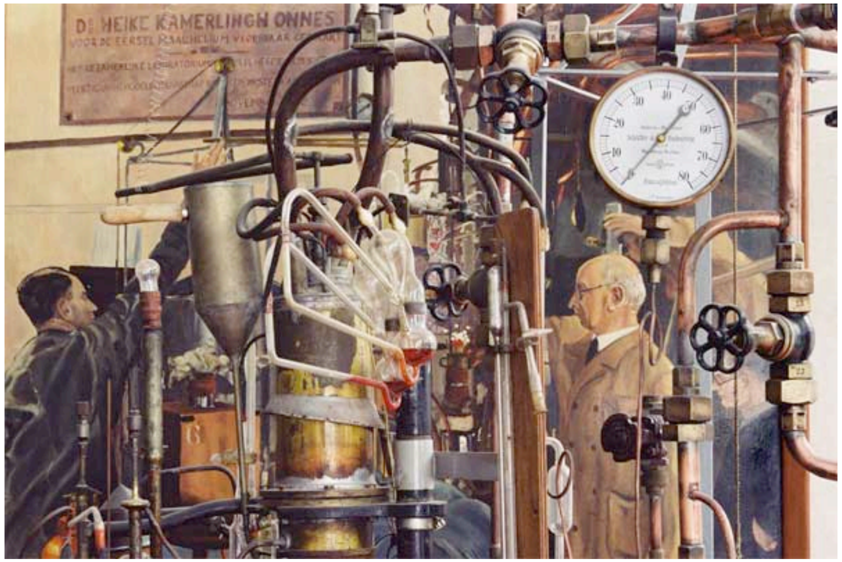
Ohm's law: $U = R * I, \qquad P = R * I^2$

Problem:

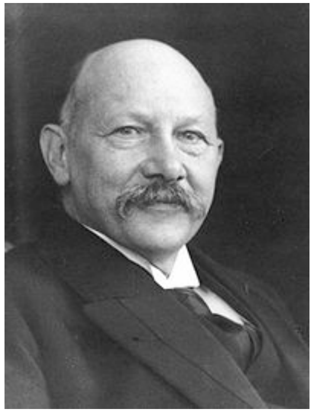
reduce ohmic losses to the absolute minimum

The Solution:

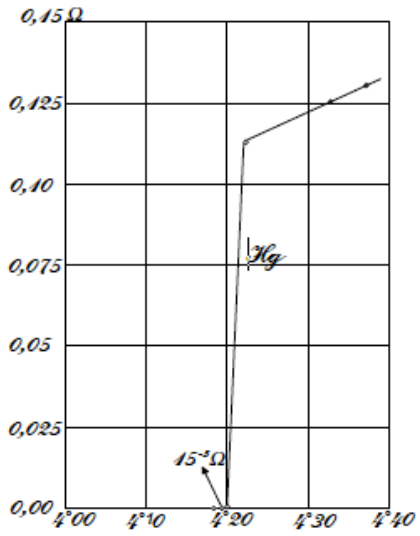
super conductivity

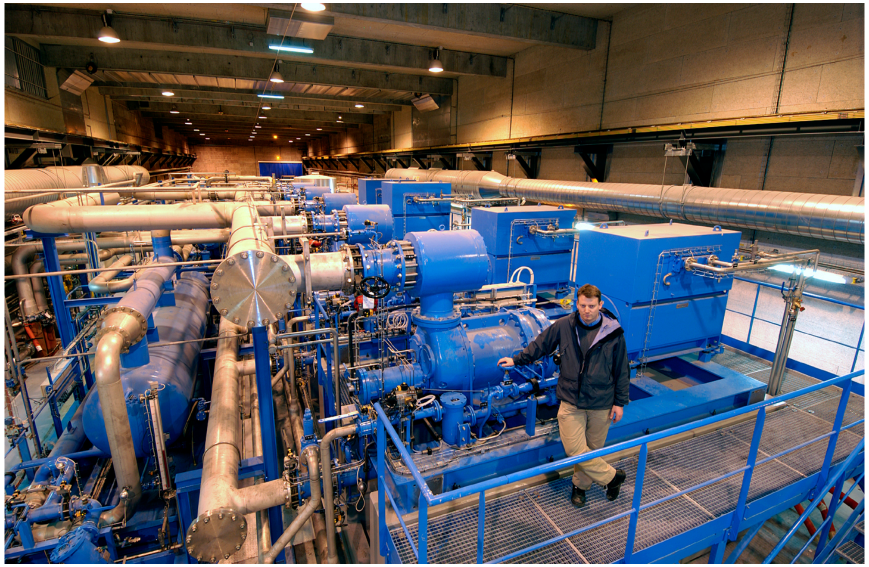


Super Conductivity

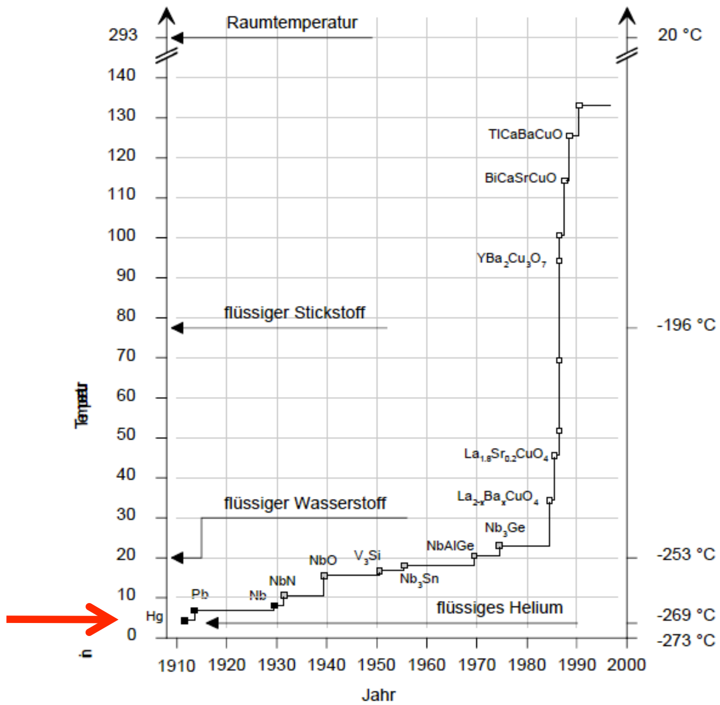


discovery of sc. by
H. Kammerling Onnes,
Leiden 1911



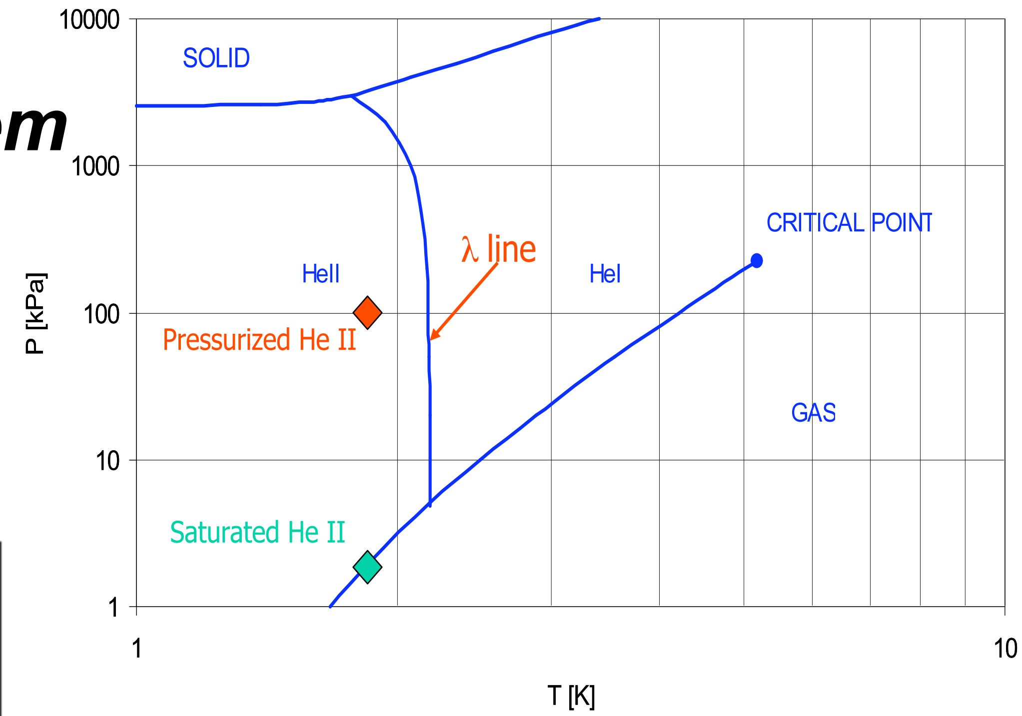


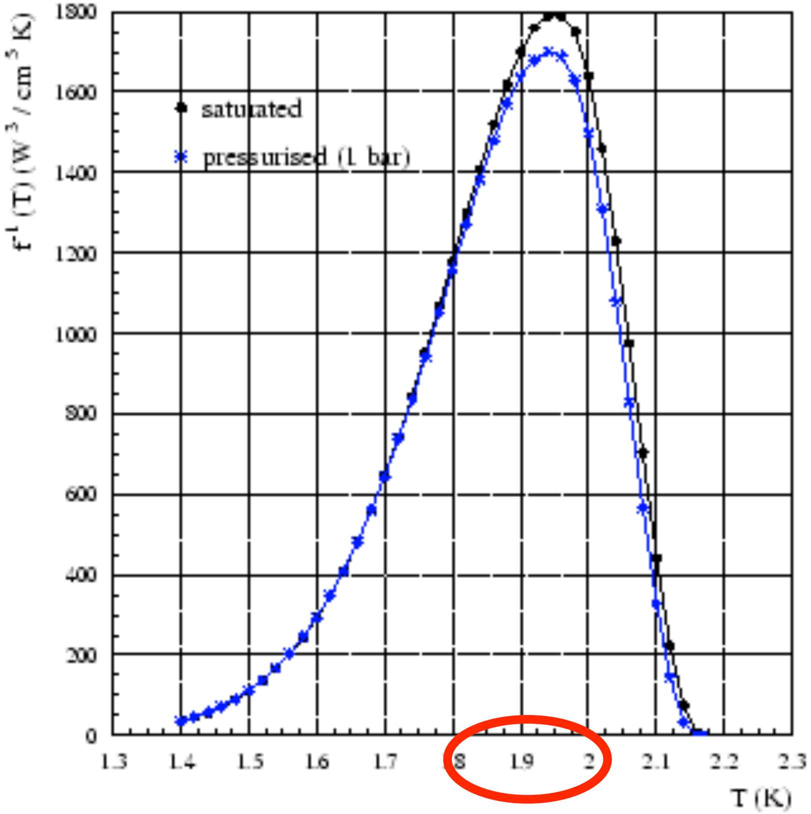
LHC 1.9 K cryo plant



Superfluid helium: 1.9 K cryo system

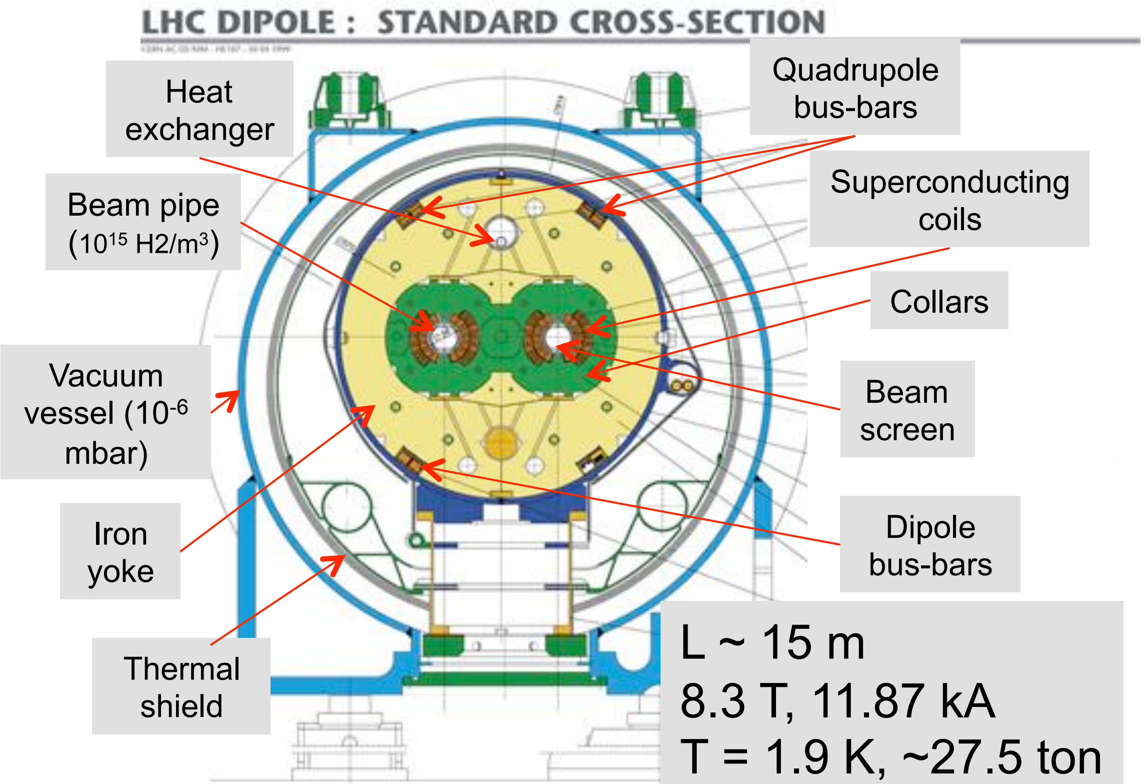
Phase diagramm of Helium



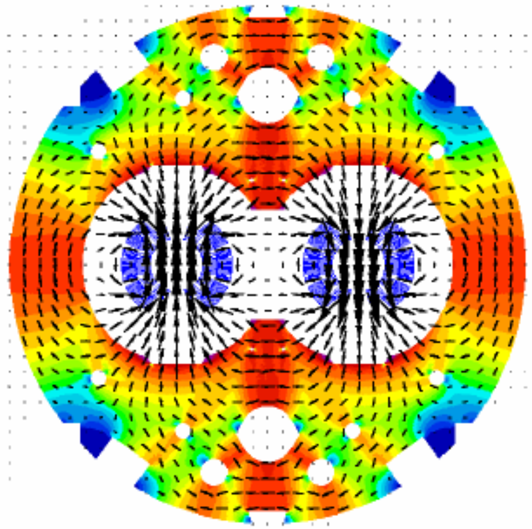


thermal conductivity of fl. Helium in supra fluid state

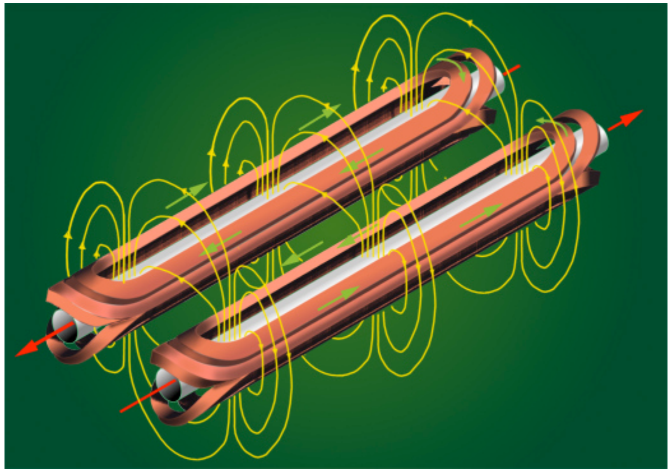
LHC: The -1232- Main Dipole Magnets

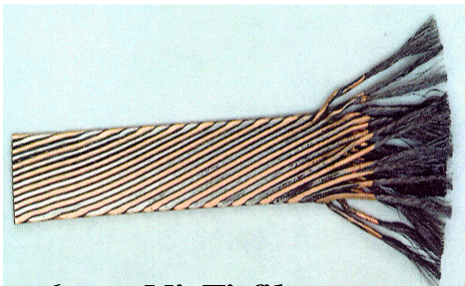


L ~ 15 m
8.3 T, 11.87 kA
T = 1.9 K, ~27.5 ton

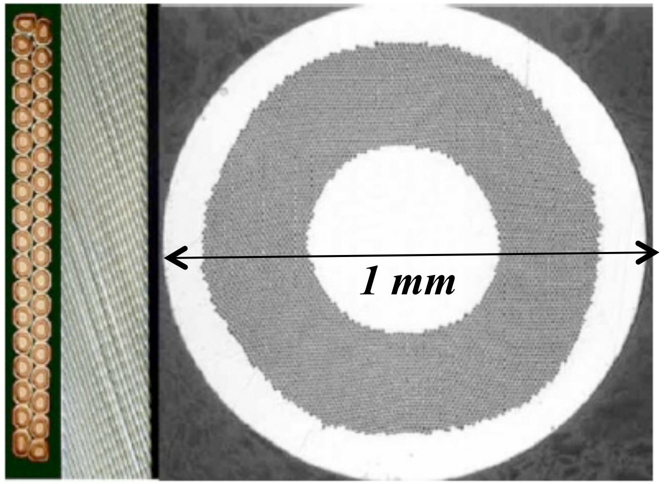


required field quality:
$\Delta B/B=10^{-4}$





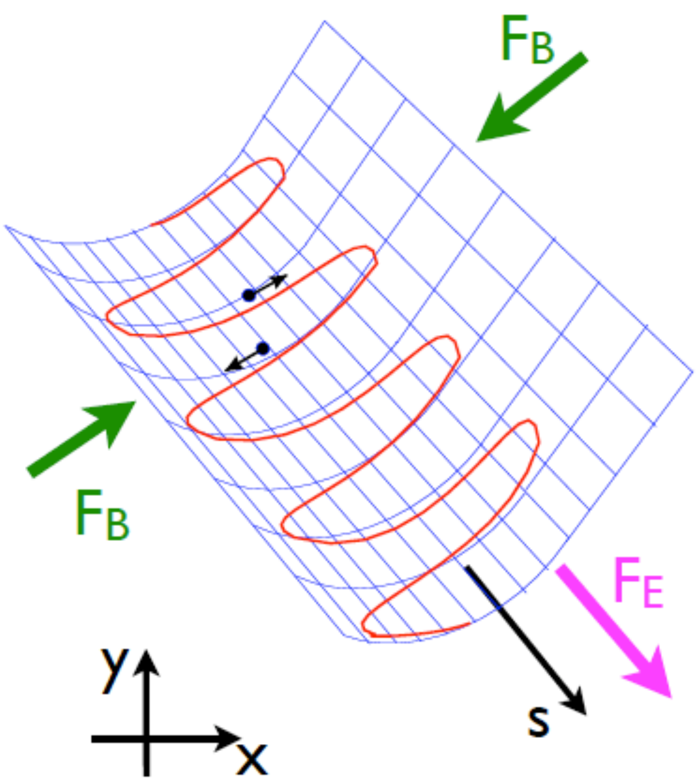
6 μm Ni-Ti filament



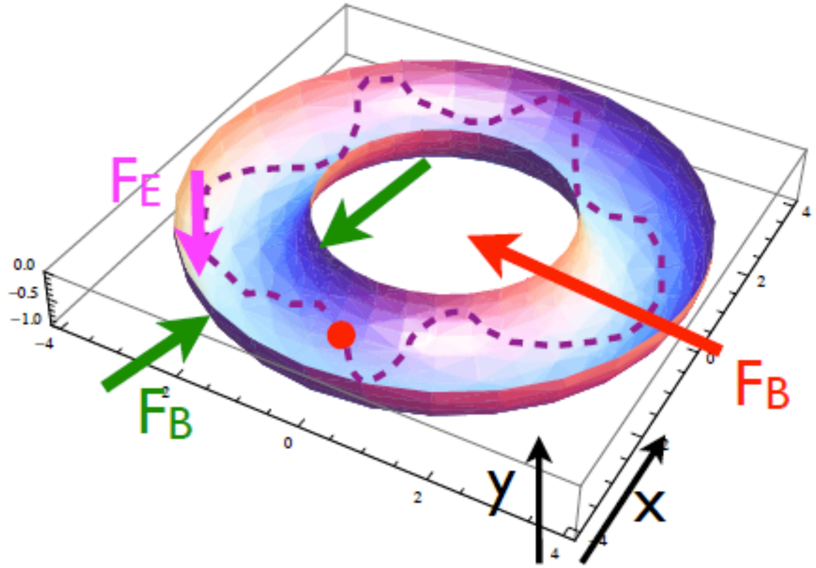
2.) Focusing Properties – Transverse Beam Optics

$$\overline{F(t)} = \underbrace{q\left(\overline{E(t)}\right.}_{F_E} + \underbrace{\overline{v(t)} \otimes}_{F_B} \left.\overline{B(t)}\right)$$

Linear Accelerator

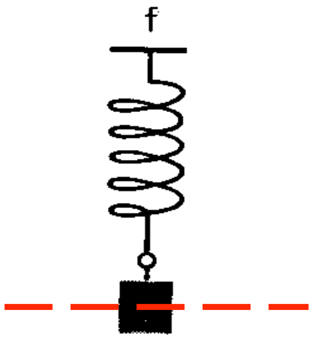


Circular Accelerator



2.) Focusing Properties – Transverse Beam Optics

classical mechanics: pendulum



there is a restoring force, proportional to the elongation x:

$$m * \frac{d^2 x}{dt^2} = -c * x$$

general solution: free harmonic oszillation

$$x(t) = A * \cos(\omega t + \varphi)$$

Storage Ring:* we need a *Lorentz force* that rises as a function of the *distance to .... ?

.... the design orbit

$$F(x) = q * v * B(x)$$

Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

linear increasing Lorentz force

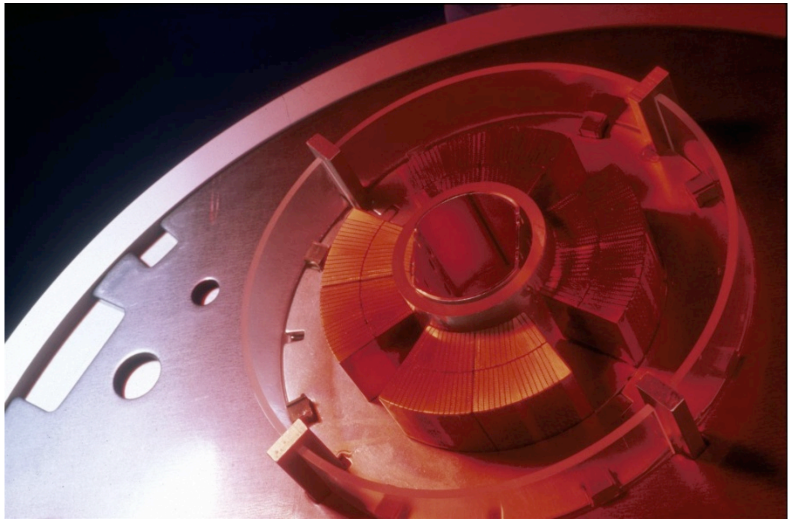
linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

$$\longrightarrow \qquad k = \frac{g}{p/e}$$

simple rule:

$$\boxed{k = 0.3\,\frac{g(T/m)}{p(GeV/c)}}$$



LHC main quadrupole magnet

$$g \approx 25 \ldots 220 \;\; T/m$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = g$$

Focusing forces and particle trajectories:

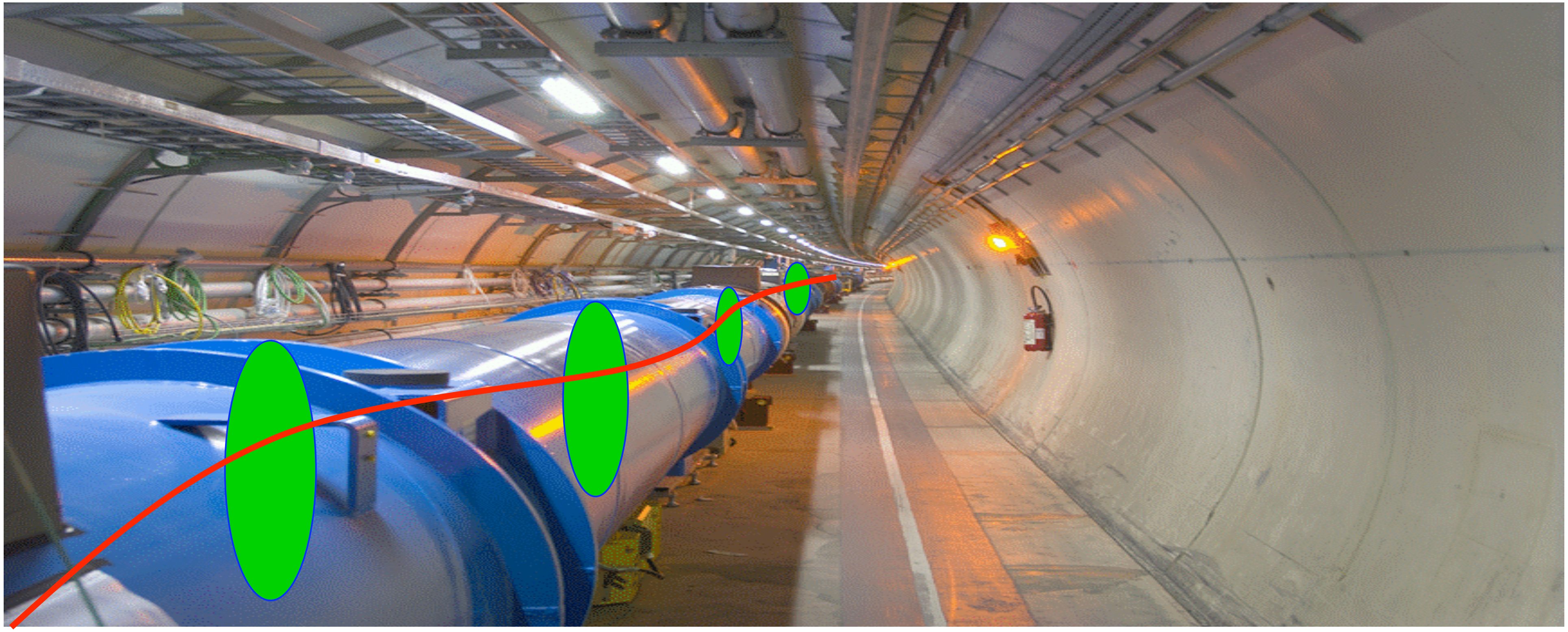
normalise magnet fields to momentum
(remember: ***B*ρ = p / q*** *)*

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

Quadrupole Magnet

$$k := \frac{g}{p/q}$$

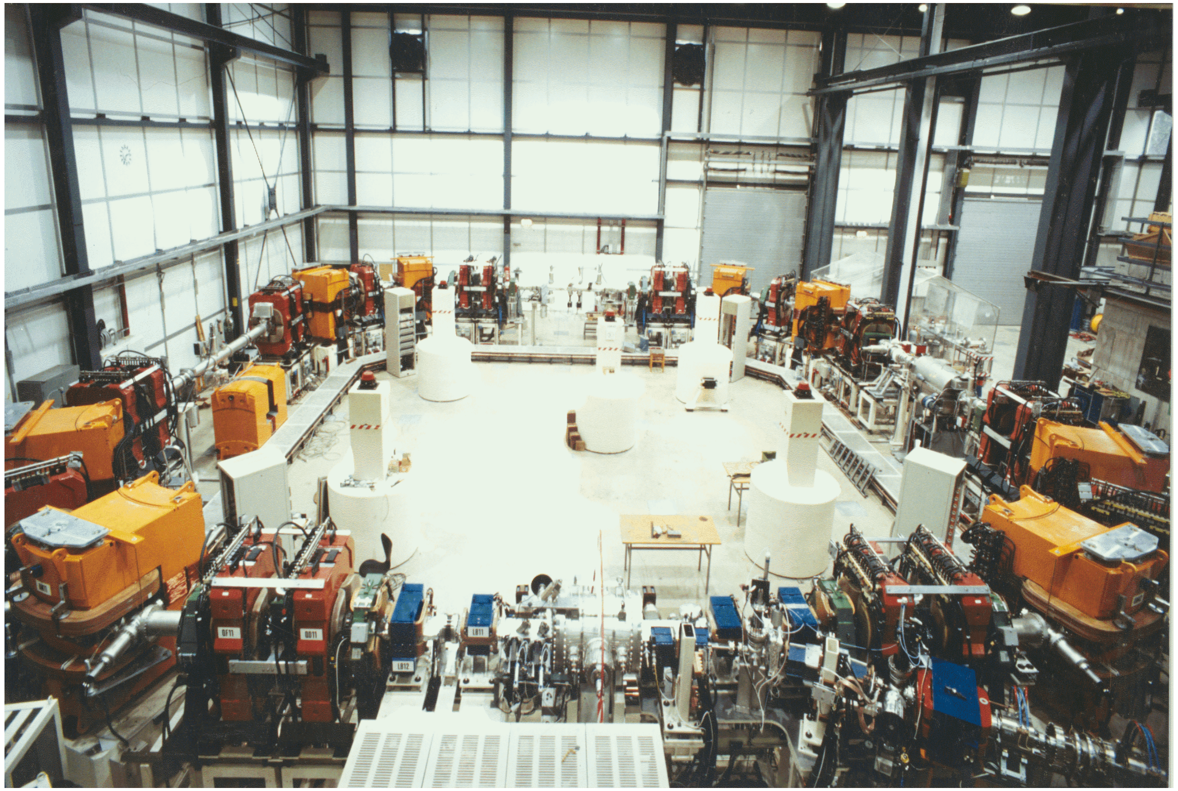


3.) The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}m\,x^2} + \cancel{\frac{1}{3!}n\,x^3} + \ldots$$

only terms linear in x, y taken into account *dipole fields*

quadrupole fields



Separate Function Machines:

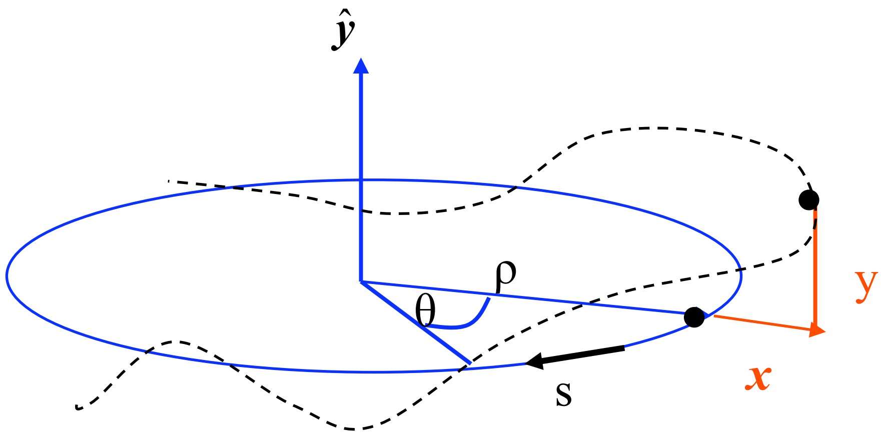
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur dipole und quads → linear*

The Equation of Motion:

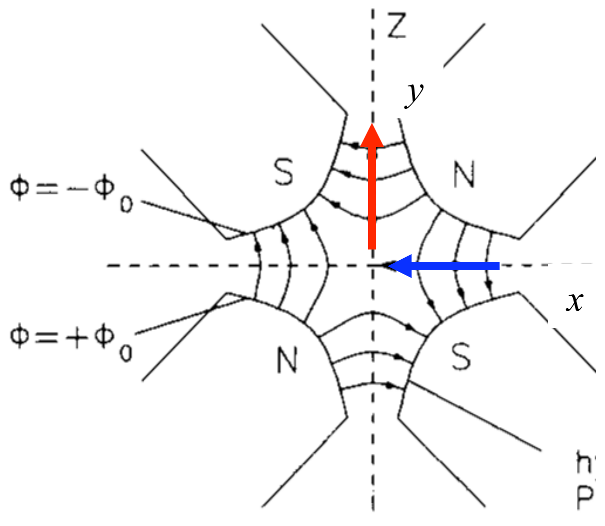


* *Equation for the horizontal motion:*

$$\boxed{x'' + x\,(\frac{1}{\rho^2} + k) = 0}$$

x =   particle amplitude
x' = angle of particle trajectory (wrt ideal path line)

* *Equation for the vertical motion:*



$$\frac{1}{\rho^2} = 0$$ ***no dipoles … in general …***

$\boldsymbol{k \;\leftrightarrow\; -k}$ ***quadrupole field changes sign***

$$\boxed{y'' - k\,y = 0}$$

4.) Solution of Trajectory Equations

Define … hor. plane: K= 1/ρ² + k

… vert. Plane: K = - k

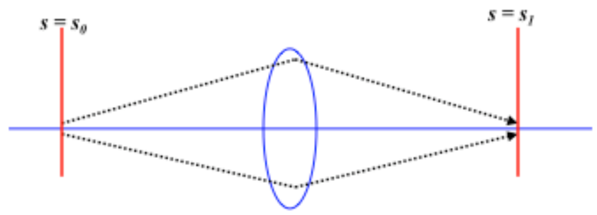
$$x'' + K\,x = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: ***Hor. Focusing Quadrupole K > 0:***

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x_0' \cdot \cos(\sqrt{|K|}s)$$



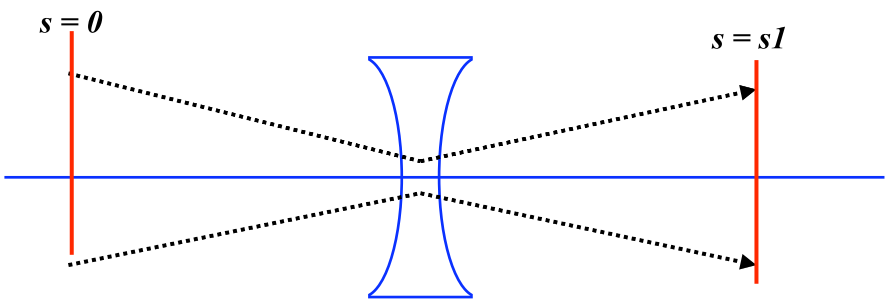
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos\left(\sqrt{|K|}\,l\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}\,l\right) \\ -\sqrt{|K|} \sin\left(\sqrt{|K|}\,l\right) & \cos\left(\sqrt{|K|}\,l\right) \end{pmatrix}$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



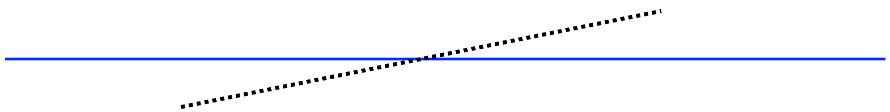
Ansatz: ***Remember from school***

$$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$$K = 0$$



$$x(s) = x_0' * s$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

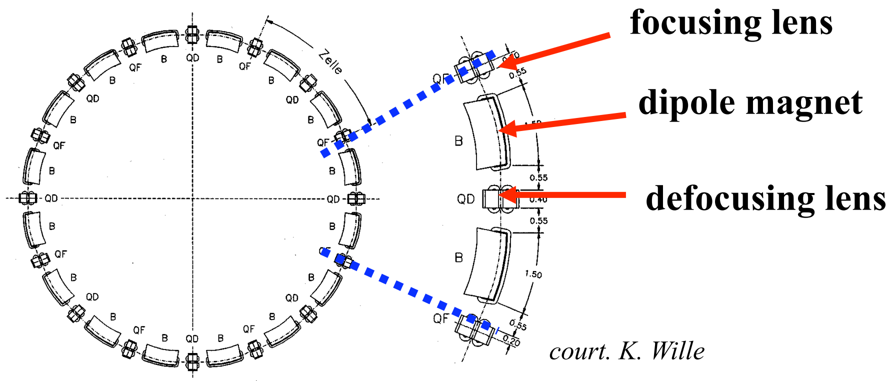
! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

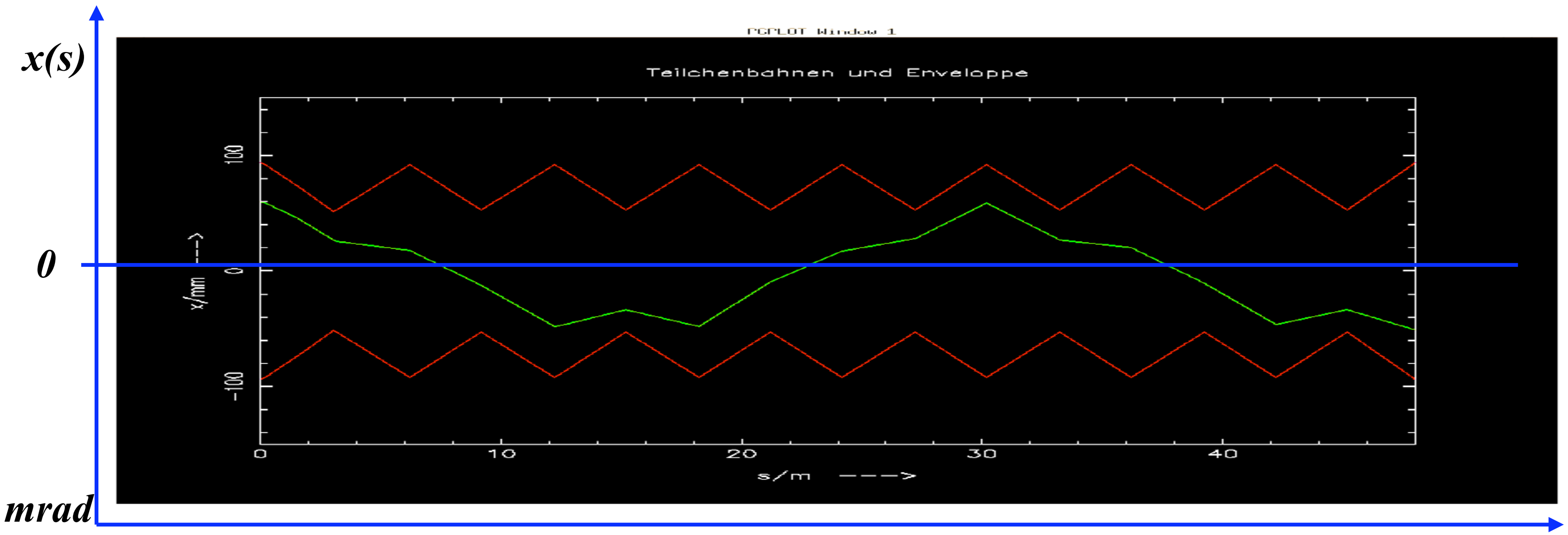
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

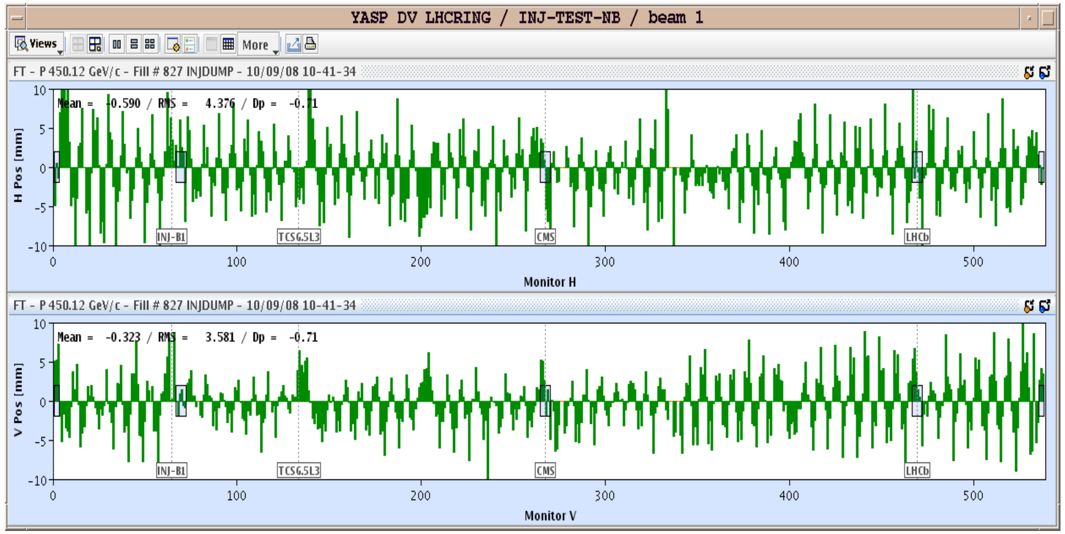
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values
in a strong
foc. machine:

x ≈ mm, x´ ≤ mrad

5.) Orbit & Tune:



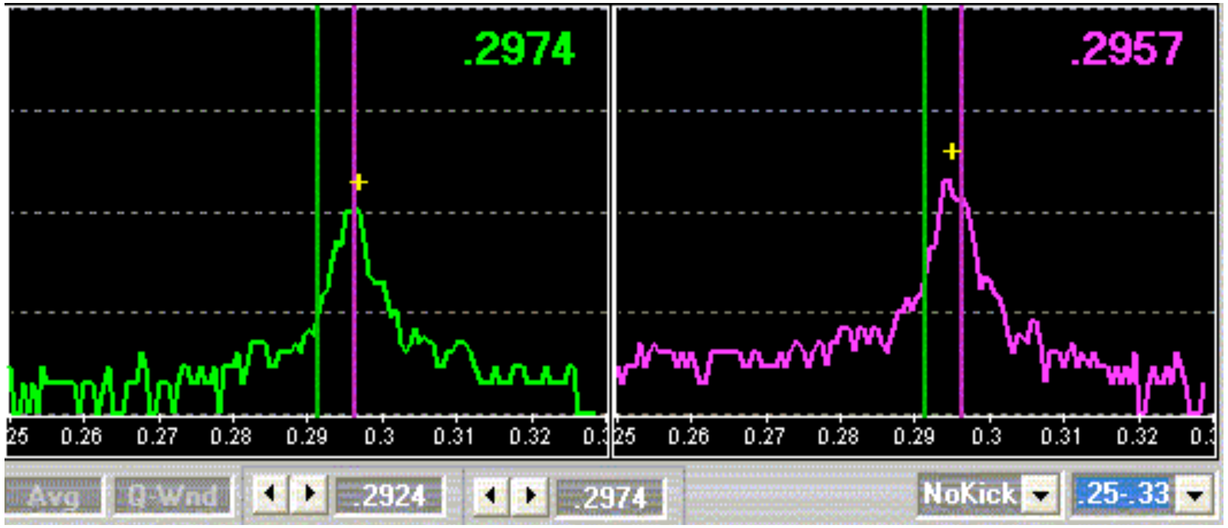
Tune: number of oscillations per turn

64.31
59.32

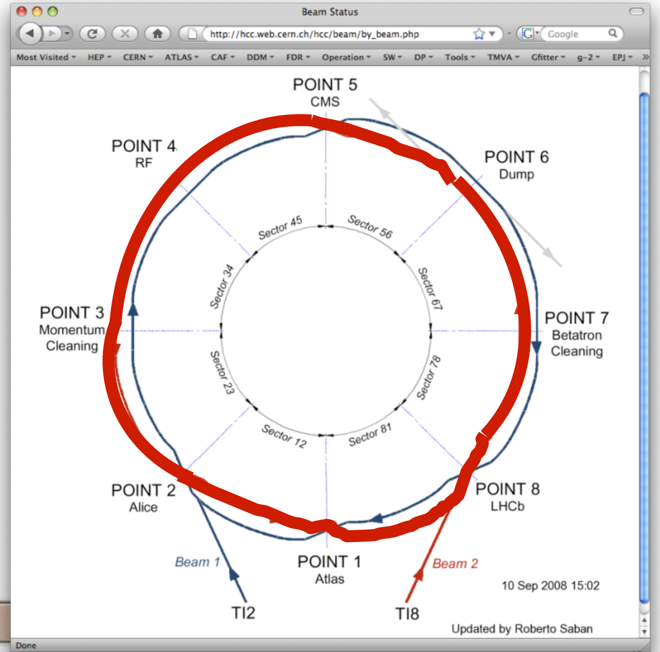
Relevant for beam stability:
non integer part

LHC revolution frequency: 11.3 kHz

$$0.31 * 11.3 = 3.5 kHz$$

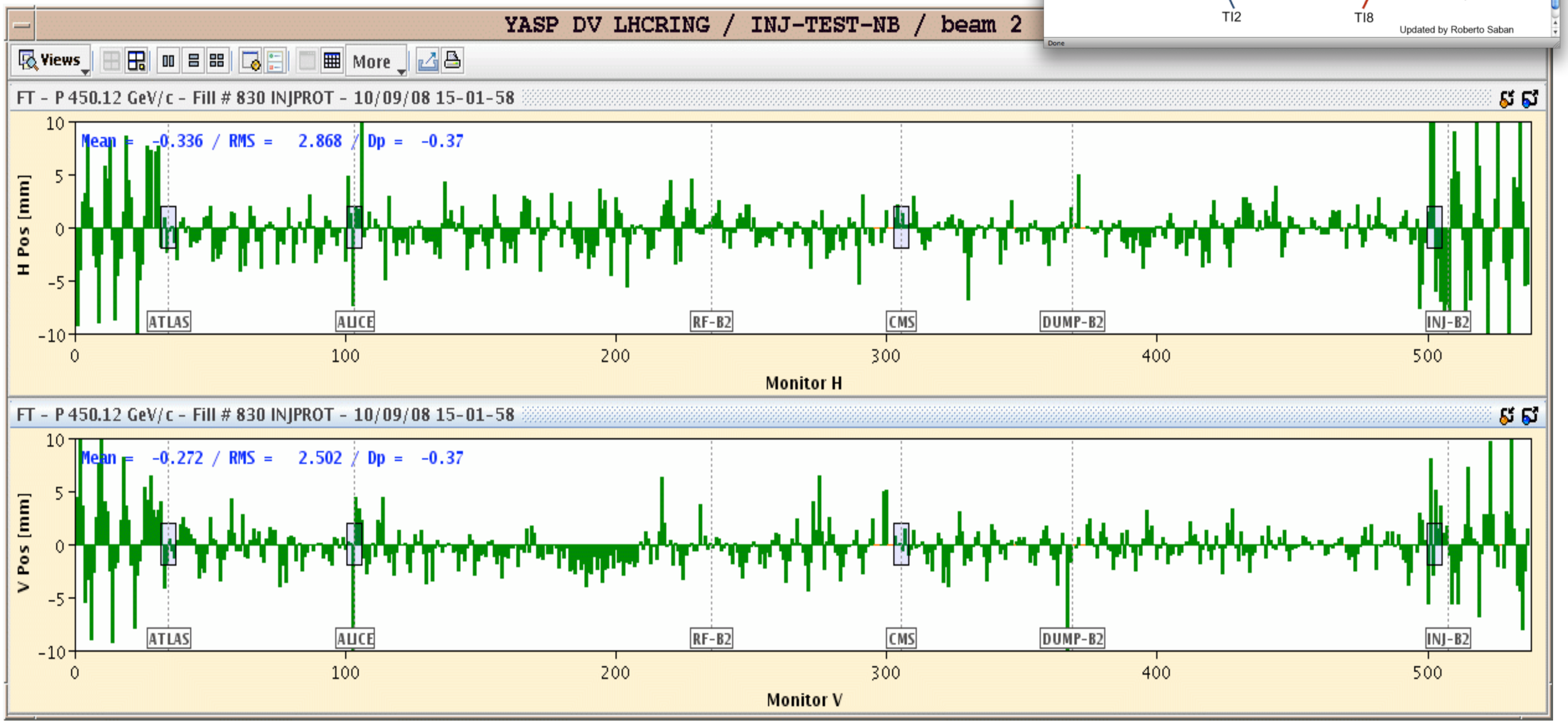


LHC Operation: Beam Commissioning



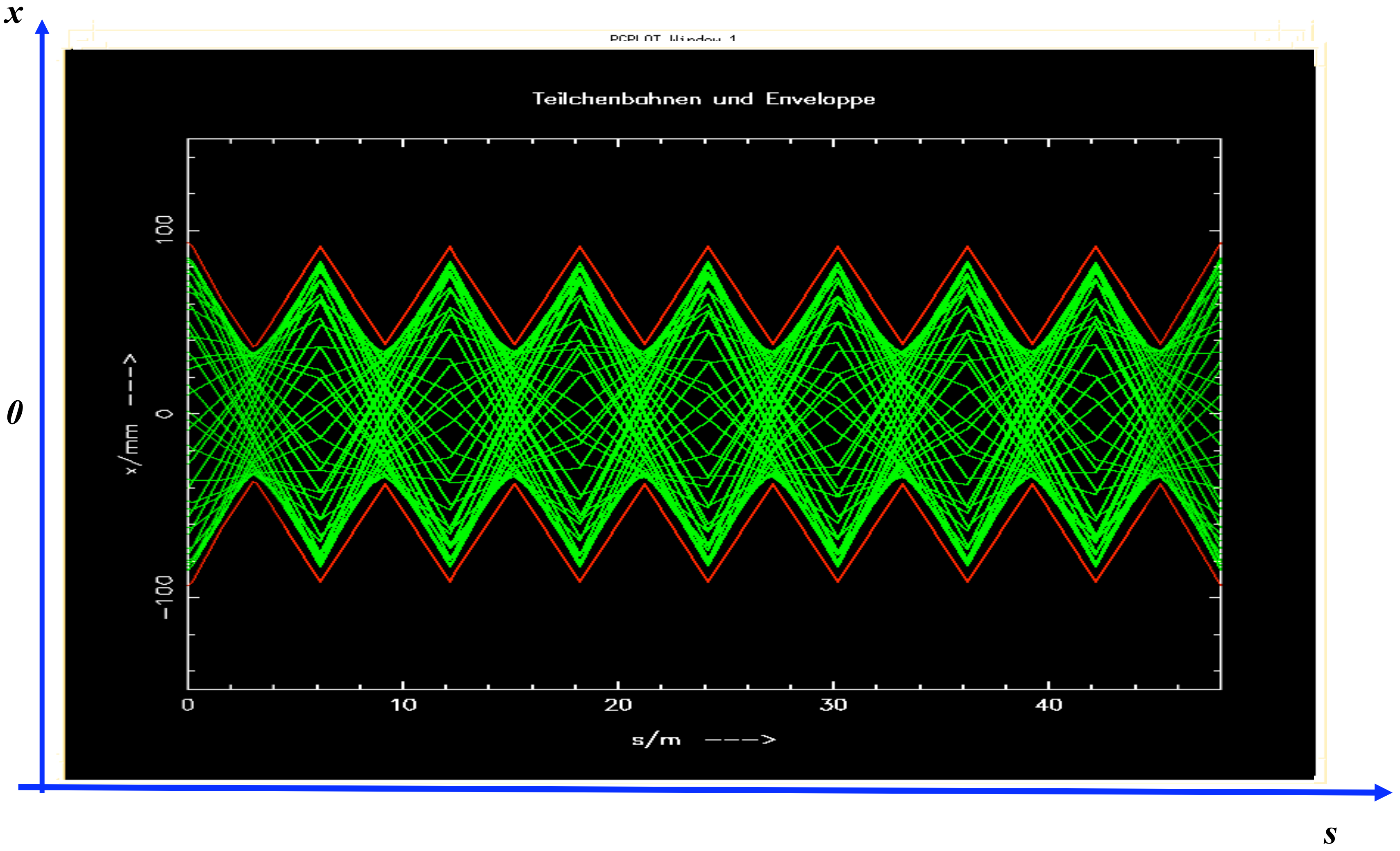
First turn steering "by sector:"

- One beam at the time
- Beam through 1 sector (1/8 ring),
correct trajectory, open collimator and move on.



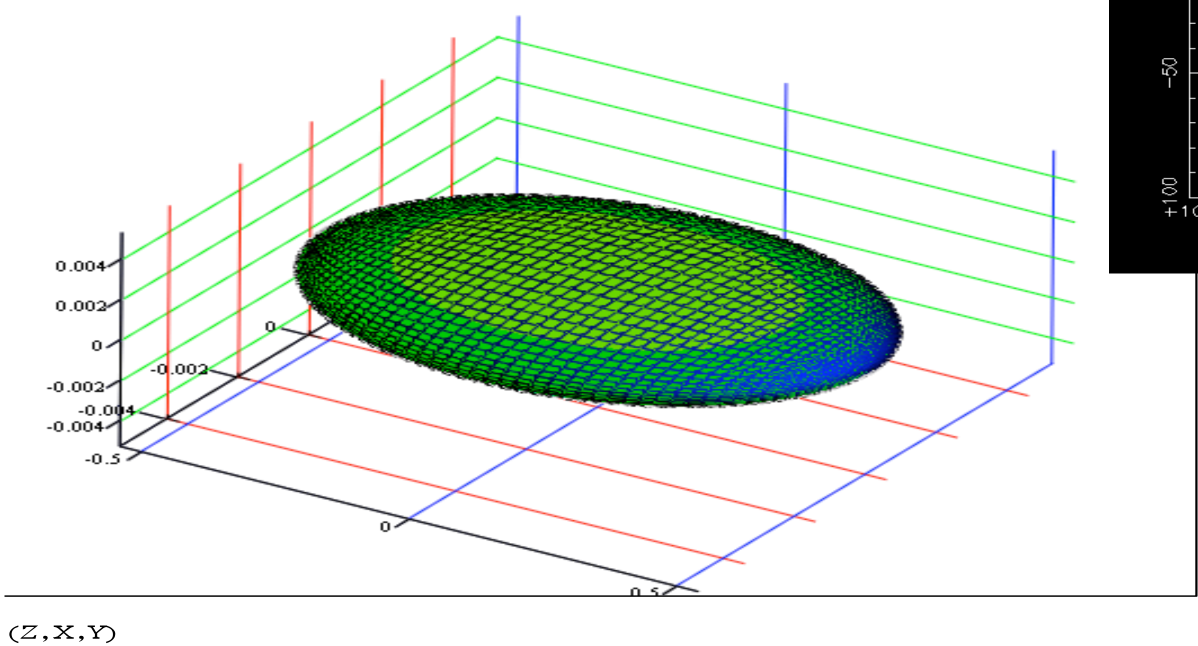
Question: what will happen, if the particle performs a second turn ?

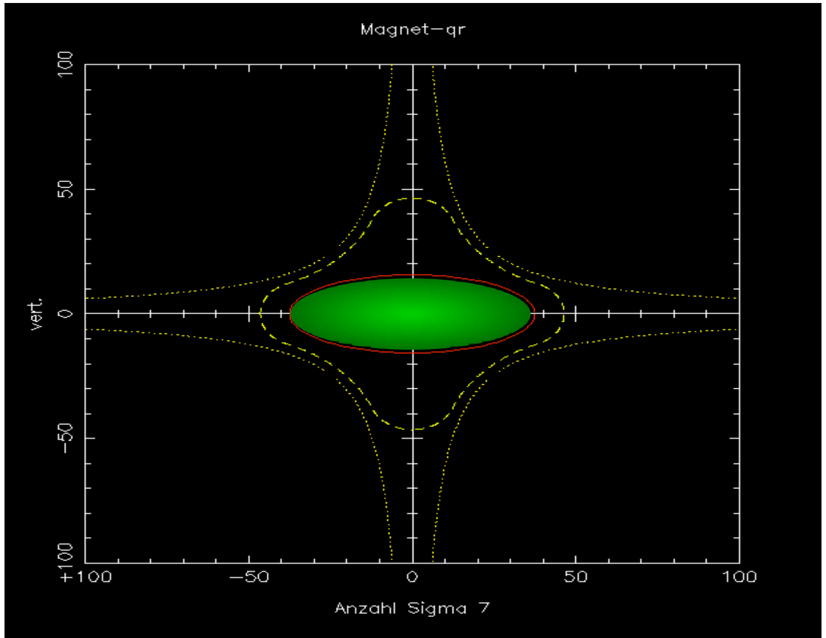
... or a third one or ... 10^{10} turns



II.) The Ideal World:

Particle Trajectories, Beams & Bunches





Bunch in a Storage Ring

Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

6.) The Beta Function

„it is convenient to see“

... after some beer ... general solution of Mr Hill can be written in the form:

Ansatz:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties. scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice. For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi} \cdot \int \frac{ds}{\beta(s)}$$

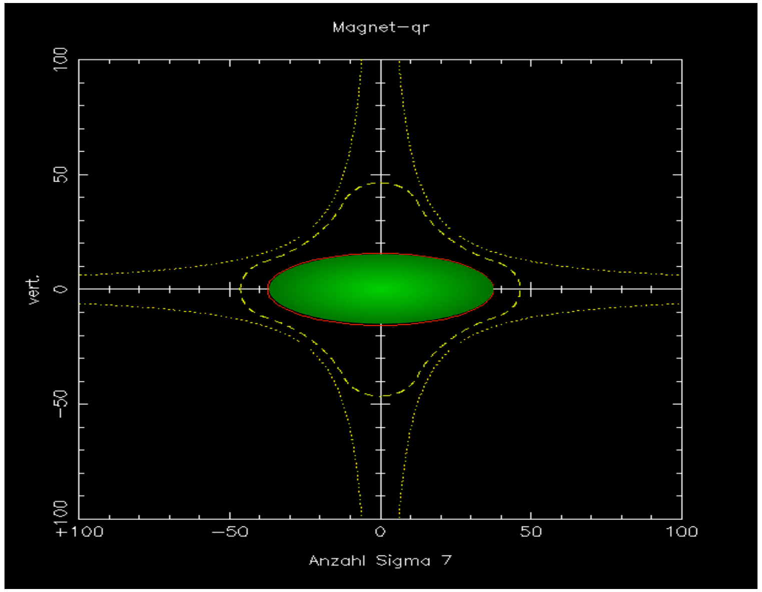
6.) The Beta Function

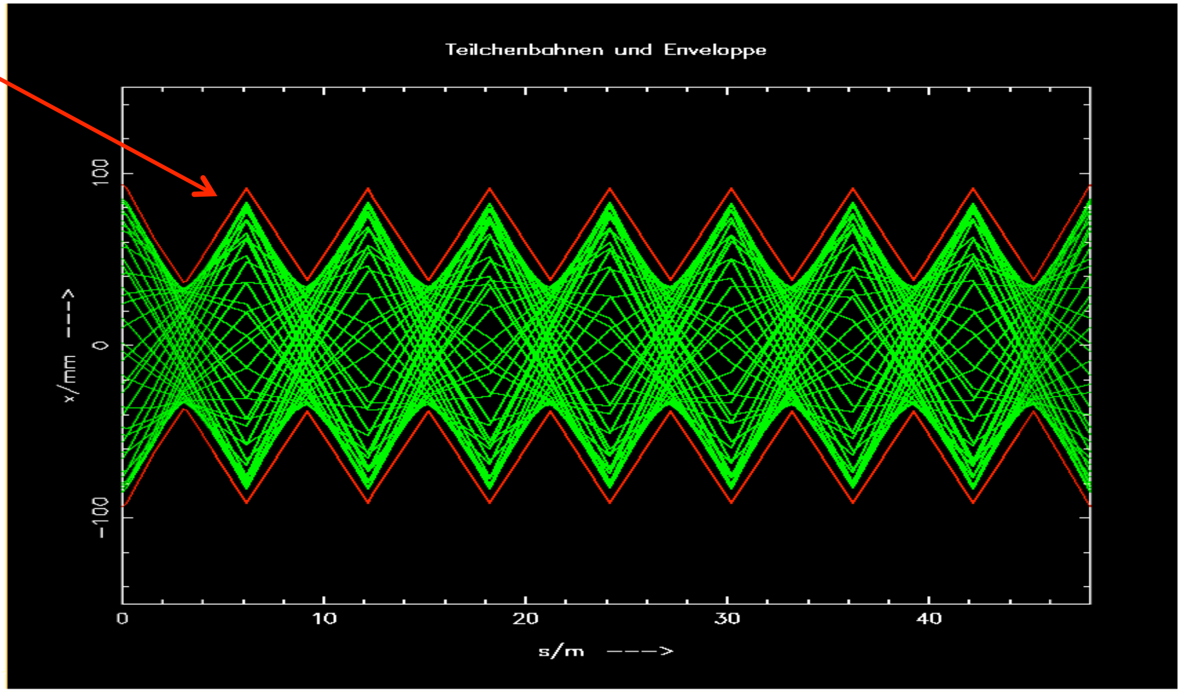
Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



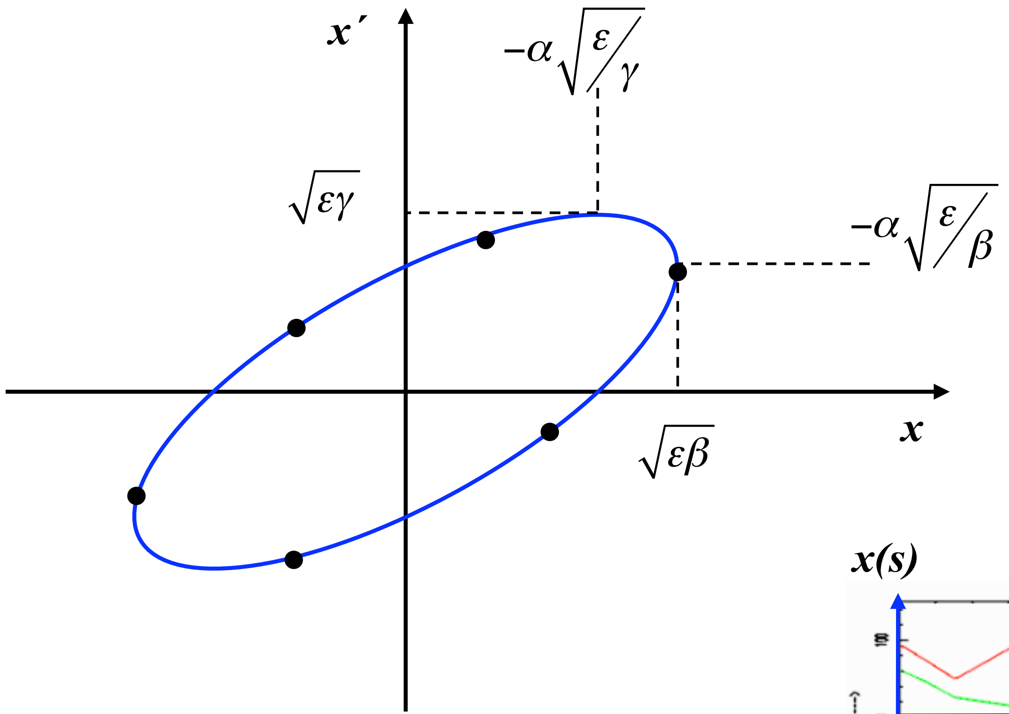


β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

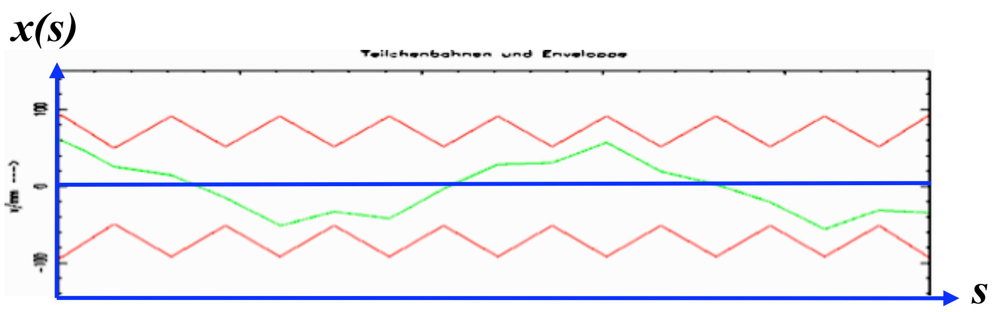
7.) Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

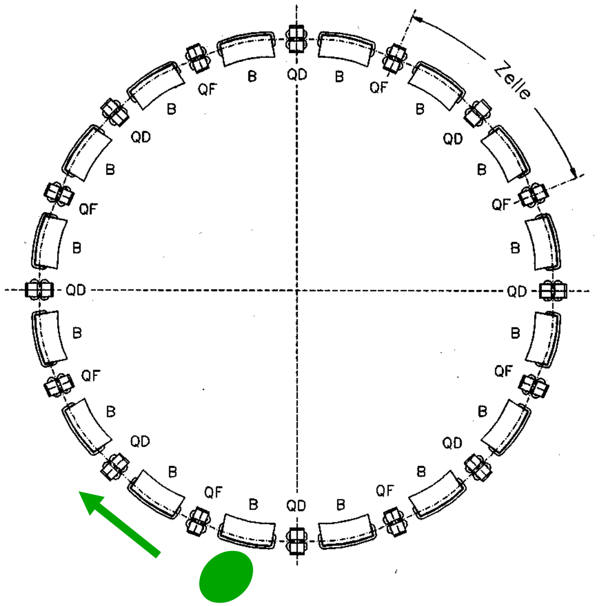


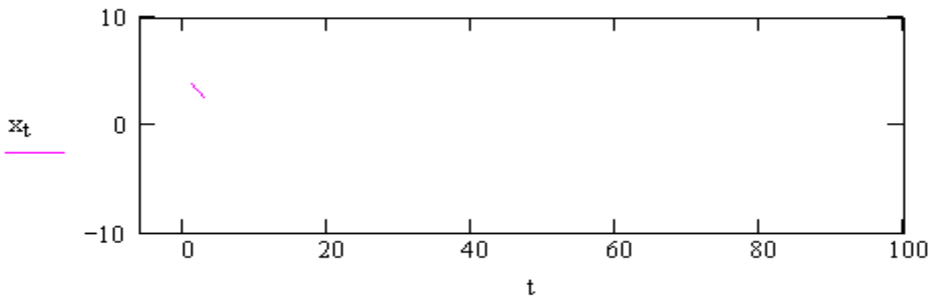
ε ***beam emittance* = *woozilycity* of the particle ensemble, *intrinsic beam parameter*,**
cannot be changed by the foc. properties.
***Scientifiquely spoken:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

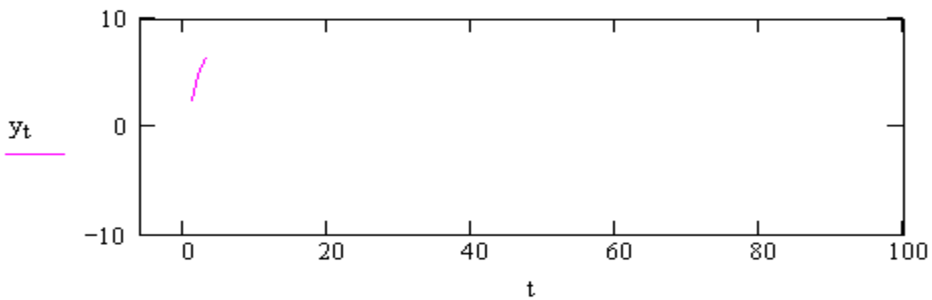
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

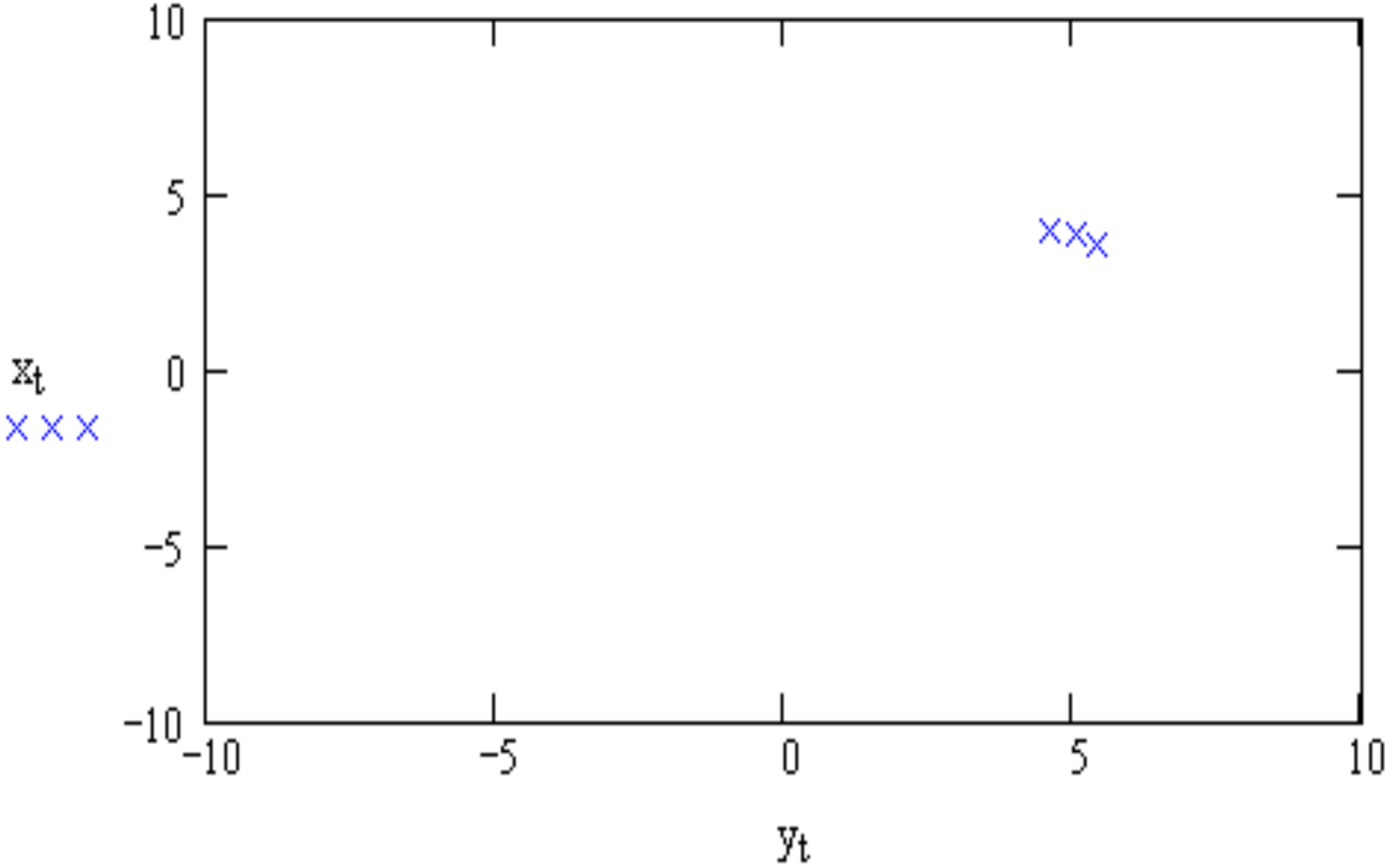




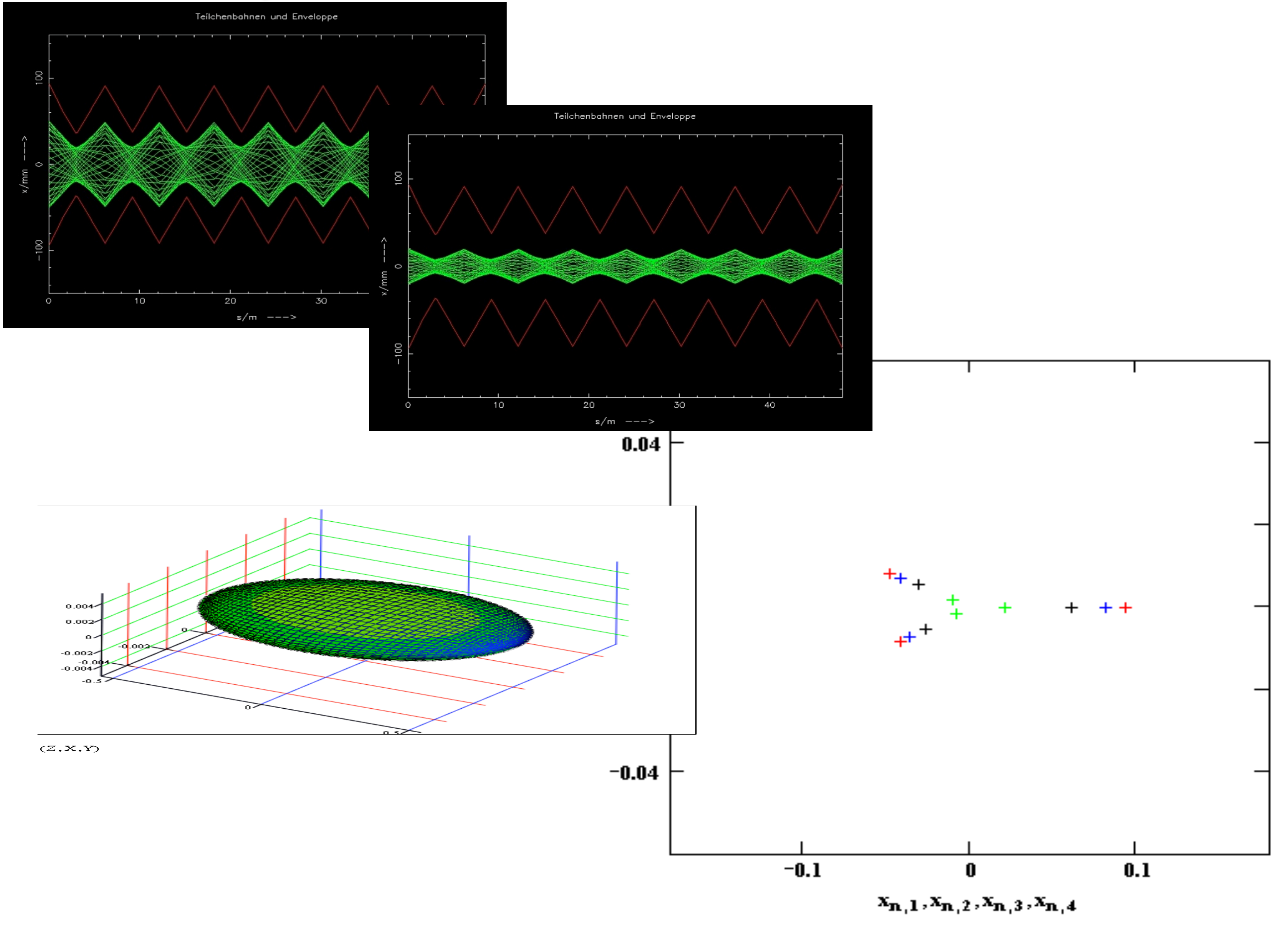


… and now the ellipse:

note for each turn x, x´at a given position „s_1“ and plot in the phase space diagram

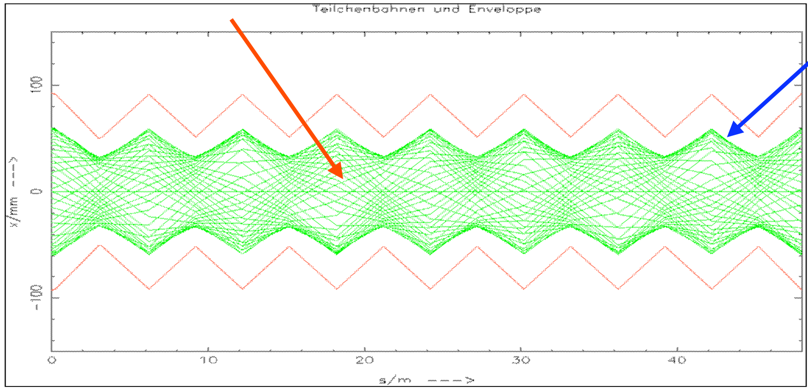


Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad\qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

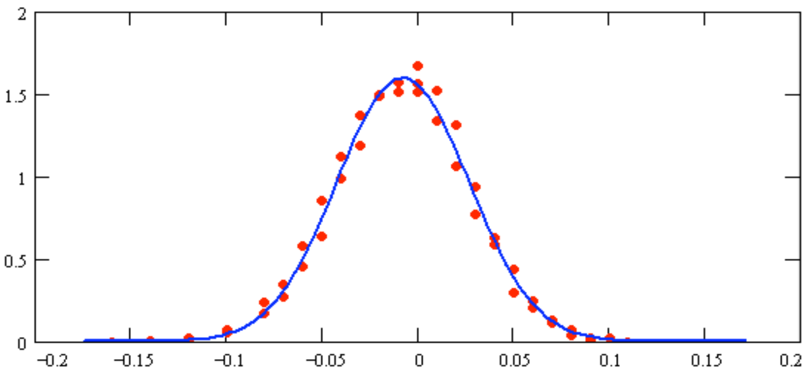
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

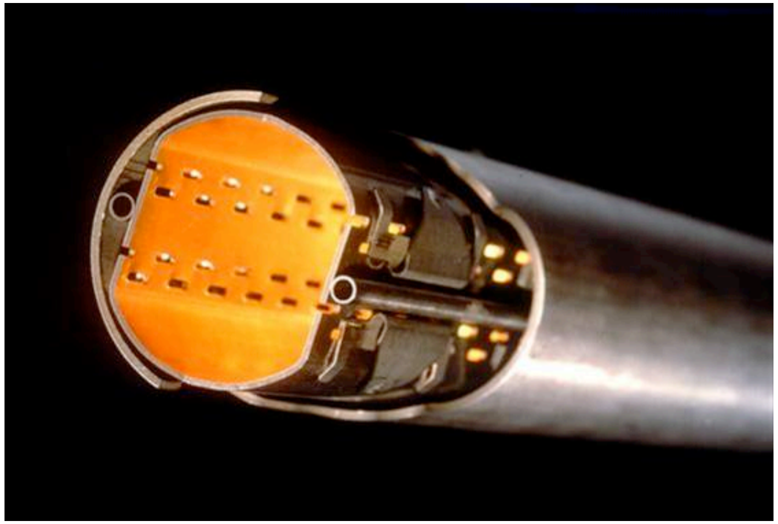
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$