Parity of Dirac spinors

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Let us start with the Dirac equation, which we write as

$$i\frac{\partial}{\partial t}\psi = -i\boldsymbol{\alpha}.\boldsymbol{\nabla}\psi + \beta m\psi\,. \tag{1}$$

The parity operation transforms

$$\boldsymbol{x} \mapsto \boldsymbol{x}' = -\boldsymbol{x} \tag{2}$$

$$t \mapsto t' = t \tag{3}$$

$$\psi \mapsto \psi_P\,. \tag{4}$$

We wish to find the form of ψ_P such that the transformed system satisfies the *same* equation, *ie.*

$$i\frac{\partial}{\partial t'}\psi_P(\boldsymbol{x}', t') = -i\boldsymbol{\alpha}.\boldsymbol{\nabla}'\psi_P(\boldsymbol{x}', t') + \beta m\psi_P(\boldsymbol{x}', t')\,. \tag{5}$$

It is obvious that $\boldsymbol{\nabla}' = -\boldsymbol{\nabla}$, and therefore if we left-multiply by β and take advantage of the anticommutation relations between β and $\boldsymbol{\alpha}$, we find

$$i\frac{\partial}{\partial t}\left(\beta\psi_P(\boldsymbol{x}', t)\right) = -i\boldsymbol{\alpha}.\boldsymbol{\nabla}\left(\beta\psi_P(\boldsymbol{x}', t)\right) + \beta m\left(\beta\psi_P(\boldsymbol{x}', t)\right)\,, \tag{6}$$

from which it is straightforward to identify that

$$\beta\psi_P(\boldsymbol{x}', t) = \psi(\boldsymbol{x}, t)\,, \tag{7}$$

or

$$\psi_P(\boldsymbol{x}, t) = \beta\psi(-\boldsymbol{x}, t)\,. \tag{8}$$

[In fact, most generally we can allow the parity transformation to introduce an arbitrary phase. However, we will neglect this minor complication.]

Let us now consider the effect of the parity operation on a general (positive energy) solution of the Dirac equation:

$$\psi(\boldsymbol{x}, t) = (E+m)^{1/2}\begin{pmatrix}\phi\\ \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E+m)}\phi\end{pmatrix}e^{-iEt+i\boldsymbol{p}.\boldsymbol{x}}\,, \tag{9}$$

so that

$$\psi_P(\boldsymbol{x}, t) = \beta\psi(-\boldsymbol{x}, t) = (E+m)^{1/2}\begin{pmatrix}\phi\\ -\frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E+m)}\phi\end{pmatrix}e^{-iEt-i\boldsymbol{p}.\boldsymbol{x}}\,, \tag{10}$$

which is equivalent to $\boldsymbol{p} \mapsto -\boldsymbol{p}$. Note that the top two components of the spinor (*ie.* the "particle" part) transform oppositely to the bottom two components (*ie.* the "antiparticle" part). This is the origin of the statement that particle and antiparticle have opposite parity.

We could also consider the effect of parity on solutions of the Klein-Gordon equation. Here the solution is straightforward since

$$(\partial_\mu\partial^\mu + m^2)\phi = 0\,, \tag{11}$$

is invariant under parity, we simply have

$$\phi_P(\boldsymbol{x}, t) = \phi(-\boldsymbol{x}, t) \tag{12}$$

(up to an arbitrary phase).