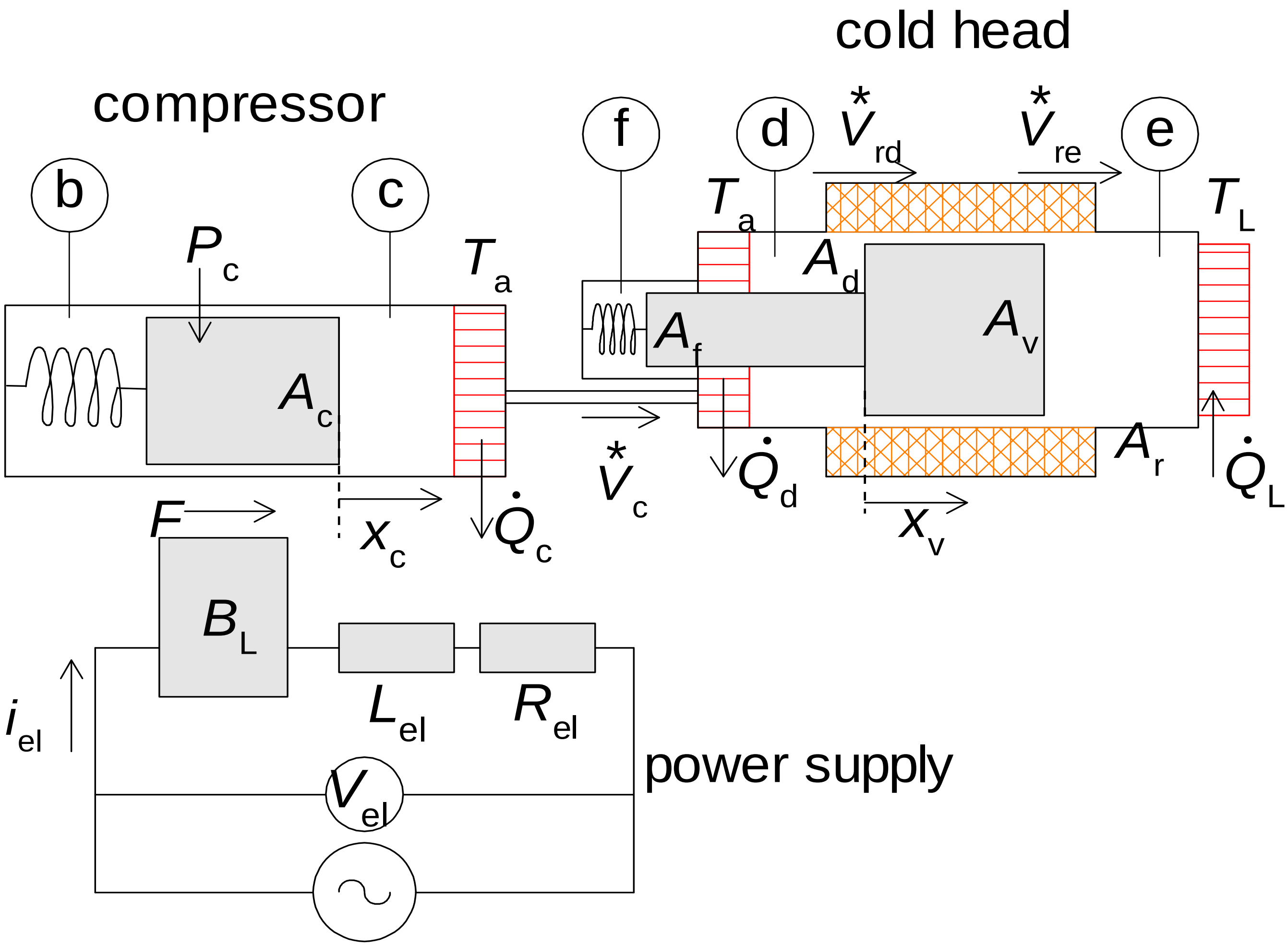
modeling/thermoacoustics

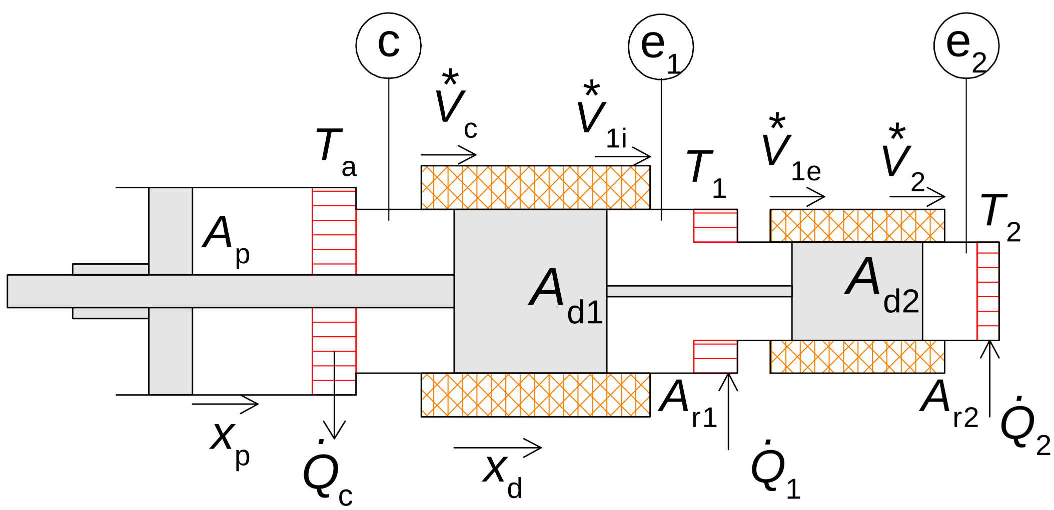
in the field of cryocoolers there are many cases of small periodic variations such as

free-piston Stirling refrigerator

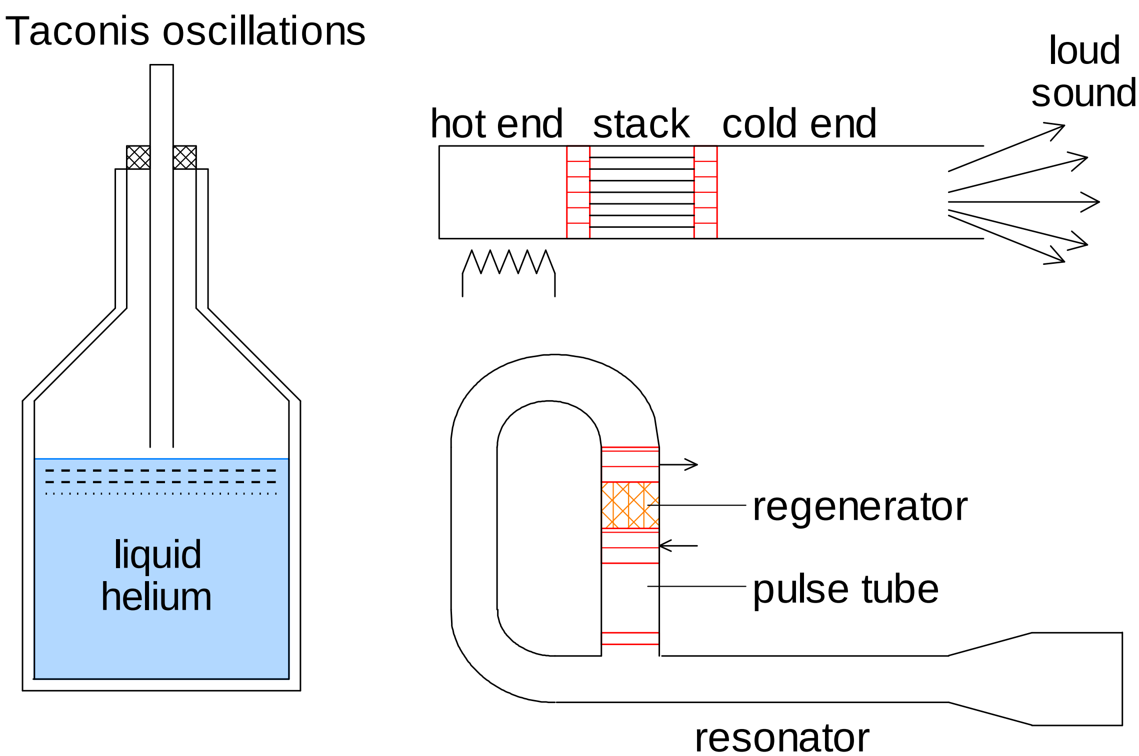


StCryoSph01.TC

two-stage crank-driven Stirling refrigerator



thermoacoustic systems



harmonic approximation

$$p = p_0 + \delta p$$

small pressure variations ($p_1 \ll p_0$) and steady state

$$\delta p = p_1 \cos(\omega t + \varphi) = \mathrm{Re}\left(\hat{p}\mathrm{e}^{\mathrm{i}\omega t}\right)$$

with the complex amplitude

$$\hat{p} = p_\mathrm{r} + \mathrm{i}p_\mathrm{i}$$

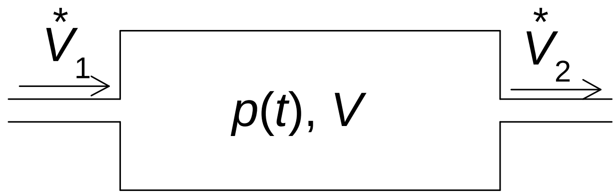
and

$$\mathrm{e}^{\mathrm{i}\omega t} = \cos\omega t + \mathrm{i}\sin\omega t$$

replaces e.g. $\mathrm{d}p/\mathrm{d}t$ by $\mathrm{i}\omega\hat{p}$: a set of differential equations becomes a set of linear equations

a complicated system can be analyzed in a few microseconds!

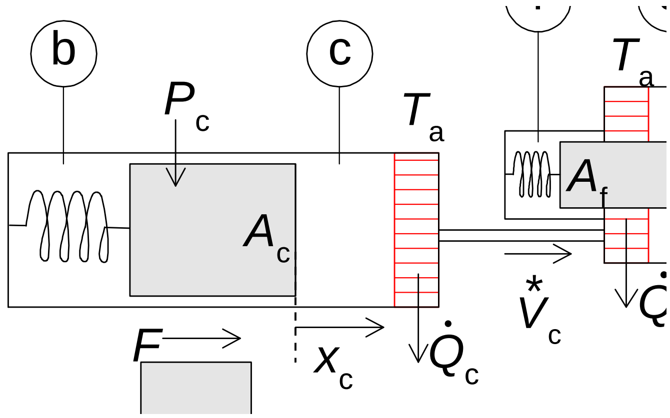
for higher accuracy but (much) longer calculation times: Sage/Regen/DeltaE/CFD volume-flow equation



$$\overset{*}{V}_{\mathbf{1}} = \overset{*}{V}_{\mathbf{2}} + \frac{V}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}$$

with $\hat{U}$ the complex amplitude of $\overset{*}{V}$ becomes

$$\hat{U}_{\mathbf{1}} = \hat{U}_{\mathbf{2}} + \frac{V}{\gamma p_0}\mathrm{i}\omega\hat{p}$$



equation of motion of the piston

$$m_{\mathsf{p}}\frac{\mathsf{d}^2 x_{\mathsf{c}}}{\mathsf{d}t^2} = B_{\mathsf{L}} i_{\mathsf{el}} - A_{\mathsf{c}}\left(p_{\mathsf{b}} - p_{\mathsf{c}}\right) - f_{\mathsf{r}}\frac{\mathsf{d}x_{\mathsf{c}}}{\mathsf{d}t} - k_{\mathsf{c}} x_{\mathsf{c}}$$

with $\mathsf{d}/\mathsf{d}t \rightarrow \mathsf{i}\omega$ this becomes

$$-\omega^2 m_{\mathsf{p}} \hat{x}_{\mathsf{c}} = B_{\mathsf{L}} \hat{\imath}_{\mathsf{el}} - A_{\mathsf{c}}\left(\hat{p}_{\mathsf{b}} - \hat{p}_{\mathsf{c}}\right) - \mathsf{i}\omega f_{\mathsf{r}} \hat{x}_{\mathsf{c}} - k_{\mathsf{c}} \hat{x}_{\mathsf{c}}$$

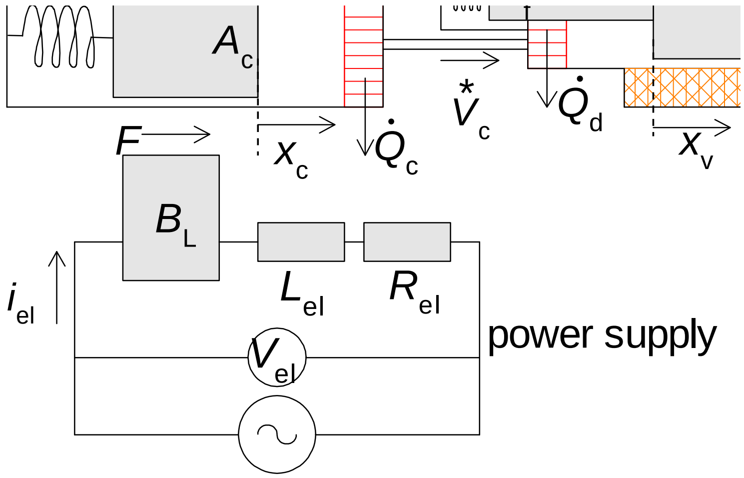
or

$$0 = B_{\mathsf{L}} \hat{\imath}_{\mathsf{el}} - A_{\mathsf{c}}\left(\hat{p}_{\mathsf{b}} - \hat{p}_{\mathsf{c}}\right) + \left(\omega^2 m_{\mathsf{p}} - \mathsf{i}\omega f_{\mathsf{r}} - k_{\mathsf{c}}\right)\hat{x}_{\mathsf{c}}$$

or

$$0 = B_{\mathsf{L}} \hat{\imath}_{\mathsf{el}} - A_{\mathsf{c}}\left(\hat{p}_{\mathsf{b}} - \hat{p}_{\mathsf{c}}\right) - k_{\mathsf{ef}} \hat{x}_{\mathsf{c}}$$

power supply



$$V_{\mathsf{el}} = B_{\mathsf{L}} \frac{\mathrm{d}x_{\mathsf{c}}}{\mathrm{d}t} + L_{\mathsf{el}} \frac{\mathrm{d}i_{\mathsf{el}}}{\mathrm{d}t} + R_{\mathsf{el}} i_{\mathsf{el}}$$

becomes

$$\hat{V}_{\mathsf{el}} = \left(R_{\mathsf{el}} + \mathrm{i}\omega L_{\mathsf{el}} \right) \hat{\imath}_{\mathsf{el}} + \mathrm{i}\omega B_{\mathsf{L}} \hat{x}_{\mathsf{c}}$$

or

$$\hat{V}_{\mathsf{el}} = Z_{\mathsf{el}} \hat{\imath}_{\mathsf{el}} + \mathrm{i}\omega B_{\mathsf{L}} \hat{x}_{\mathsf{c}}$$

average powers

applied power

$$P = \overline{Fv} = A\overline{v\delta p}$$

with

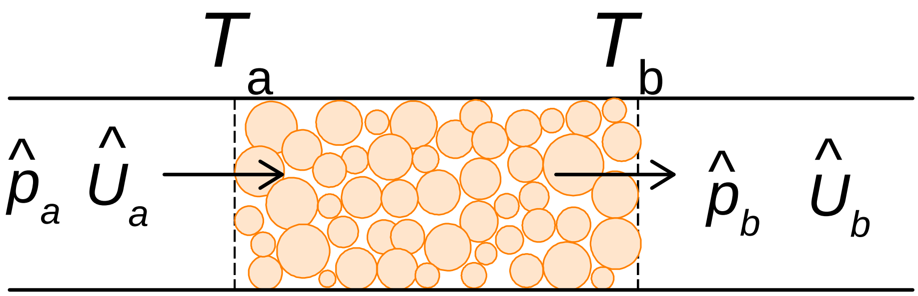
$$\overline{v\delta p} = \overline{\mathsf{Re}\left(\hat{v}\mathsf{e}^{\mathsf{i}\omega t}\right)\mathsf{Re}\left(\hat{p}\mathsf{e}^{\mathsf{i}\omega t}\right)}$$

gives

$$\overline{v\delta p} = \frac{1}{2}\left(v_{\mathsf{r}}p_{\mathsf{r}} + v_{\mathsf{i}}p_{\mathsf{i}}\right)$$

choose your phase here?

transfer matrix/functions



connects $\hat{p}_\mathsf{a}$ and $\hat{U}_\mathsf{a}$ to $\hat{p}_\mathsf{b}$ and $\hat{U}_\mathsf{b}$ via

$$\begin{pmatrix} \hat{p} \\ \hat{U} \end{pmatrix}_\mathsf{a} = \overrightarrow{\overrightarrow{T}} \begin{pmatrix} \hat{p} \\ \hat{U} \end{pmatrix}_\mathsf{b} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{U} \end{pmatrix}_\mathsf{b}$$

means

$$\begin{aligned} \hat{p}_\mathsf{a} &= T_{11}\hat{p}_\mathsf{b} + T_{12}\hat{U}_\mathsf{b} \\ \hat{U}_\mathsf{a} &= T_{21}\hat{p}_\mathsf{b} + T_{22}\hat{U}_\mathsf{b} \end{aligned}$$

components is series give products of transfer matrices orifice with flow resistance R_{O}

$$p_{\mathsf{a}} - p_{\mathsf{b}} = R_{\mathsf{O}} \overset{*}{V}$$

so

$$\begin{aligned} \hat{p}_{\mathsf{a}} &= 1 \times \hat{p}_{\mathsf{b}} + R_{\mathsf{O}} \hat{U}_{\mathsf{b}} \\ \hat{U}_{\mathsf{a}} &= 0 + 1 \times \hat{U}_{\mathsf{b}} \end{aligned}$$

so

$$\overset{\rightarrow}{\vec{T}} = \begin{pmatrix} 1 & R_{\mathsf{O}} \\ 0 & 1 \end{pmatrix}$$

volume-flow equation

$$\overset{*}{V}_{\mathsf{a}} = \overset{*}{V}_{\mathsf{b}} + \frac{V}{\gamma p} \frac{\mathsf{d}p}{\mathsf{d}t}$$

so

$$\begin{aligned} \hat{p}_{\mathsf{a}} &= 1 \times \hat{p}_{\mathsf{b}} + 0 \\ \hat{U}_{\mathsf{a}} &= \mathsf{i}\omega \frac{V}{\gamma p_0} \hat{p}_{\mathsf{b}} + 1 \times \hat{U}_{\mathsf{b}} \end{aligned}$$

so

$$\overset{\rightarrow}{\vec{T}} = \begin{pmatrix} 1 & 0 \\ \mathsf{i}\omega \frac{V}{\gamma p_0} & 1 \end{pmatrix}$$

ideal regenerator

$$\begin{aligned}\hat{p}_{\mathsf{a}} &= \hat{p}_{\mathsf{b}} + 0 \\ \hat{U}_{\mathsf{a}} &= 0 + \frac{T_{\mathsf{a}}}{T_{\mathsf{b}}}\hat{U}_{\mathsf{b}}\end{aligned}$$

so

$$\overrightarrow{\overrightarrow{T}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{T_{\mathsf{a}}}{T_{\mathsf{b}}} \end{pmatrix}$$

regenerator with flow resistance and a linear temperature profile $\overrightarrow{\overrightarrow{T}}$ contains Bessel functions of complex parameters makes everybody nervous!

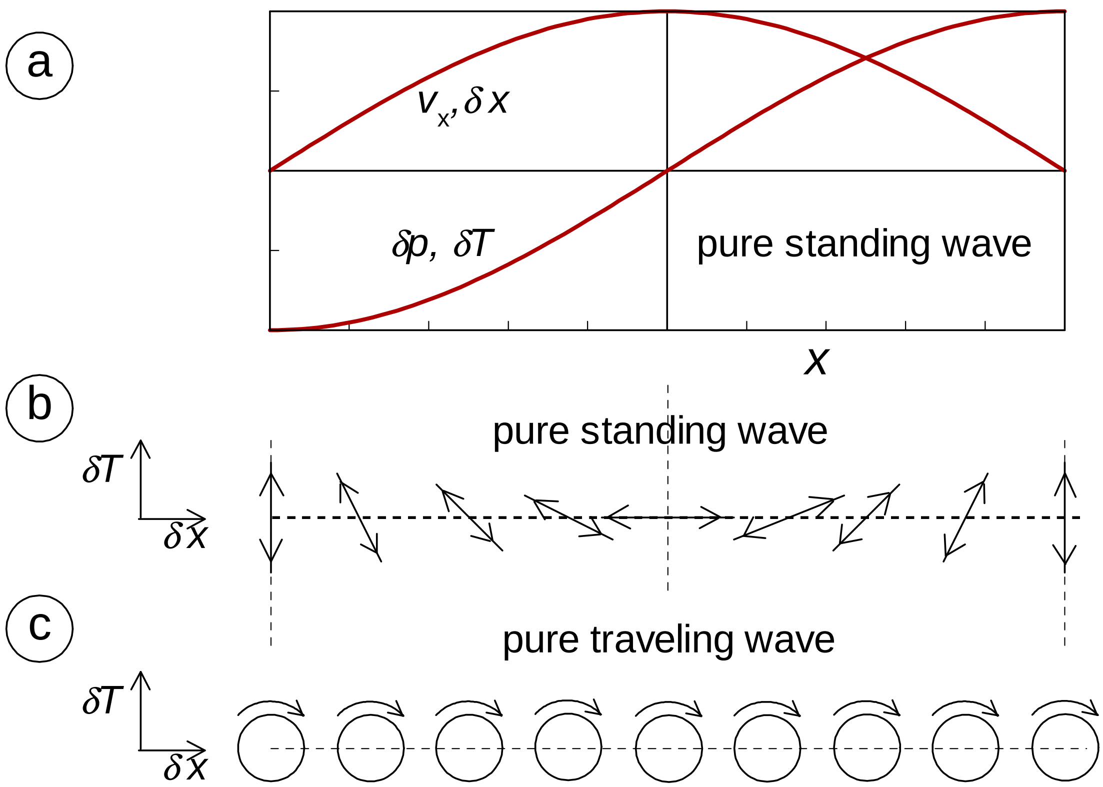
Let commercial software solve your problem!

sound duct of length L (no proof here!)

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \cos\frac{\omega L}{c} & \mathrm{i}\frac{c\rho_0}{A}\sin\frac{\omega L}{c} \\ \mathrm{i}\frac{A}{c\rho_0}\sin\frac{\omega L}{c} & \cos\frac{\omega L}{c} \end{pmatrix}$$

plate stack: Rott functions

SOUND



TaStandWave.TC

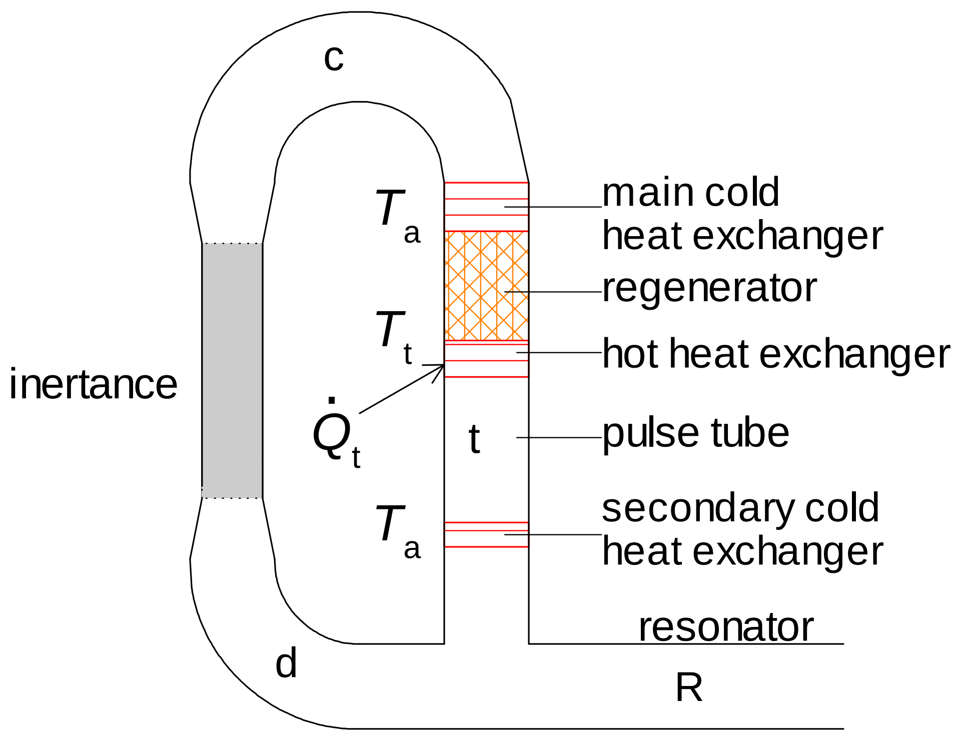
average enthalpy flow (acoustic power) is

$$\overline{\overset{*}{H}} = \overline{\overset{*}{n} H_{\mathsf{m}}} = \overline{\frac{p_0 \overset{*}{V}}{R T_0} C_{\mathsf{p}} \delta T} = \overline{\overset{*}{V} \delta p}$$

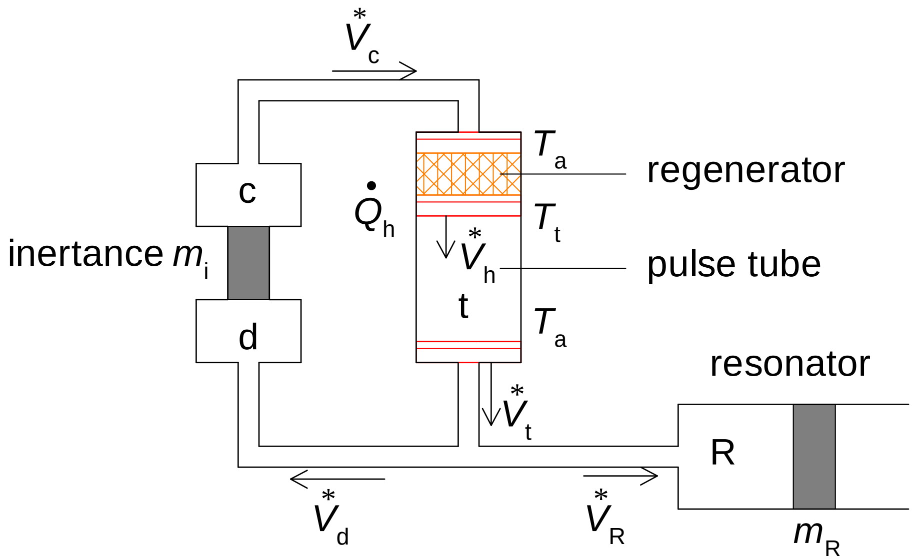
with

$$\delta p = \frac{C_{\mathsf{p}} p_0}{R T_0} \delta T$$

calculation of onset of oscillations



model



$$\frac{\mathrm{d}^4\delta p_\mathrm{t}}{\mathrm{d}t^4} + a_3\frac{\mathrm{d}^3\delta p_\mathrm{t}}{\mathrm{d}t^3} + a_2\frac{\mathrm{d}^2\delta p_\mathrm{t}}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}\delta p_\mathrm{t}}{\mathrm{d}t} + a_0\delta p_\mathrm{t} = 0$$

try

$$\delta p_\mathrm{t} = \sum_{n=1}^{4} c_\mathrm{n}\mathrm{e}^{z_\mathrm{n}t} = \sum_{n=1}^{4} c_\mathrm{n}\mathrm{e}^{-\alpha_\mathrm{n}t}\mathrm{e}^{\mathrm{i}\omega_\mathrm{n}t}$$

$\alpha_\mathrm{n} > 0$ dying oscillations

$\alpha_\mathrm{n} < 0$ growing amplitudes

$\alpha_\mathrm{n} = 0$ steady oscillations at ω_n