

Method of differential equations

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[A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13]

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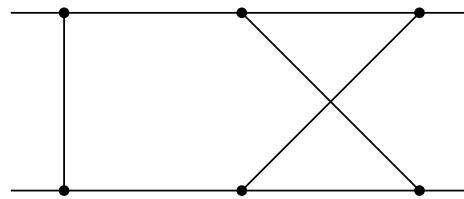
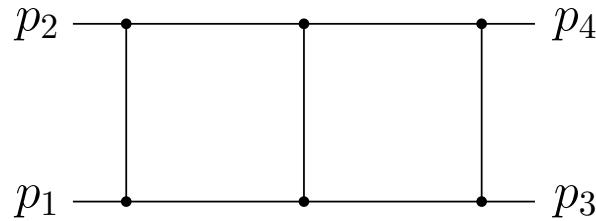
- Take some derivatives of a given master integral in masses or/and kinematic invariants
- Express them in terms of Feynman integrals of the given family with shifted indices
- Apply an IBP reduction to express these integrals in terms of the given master integral and lower master integrals to obtain a differential equation
- Solve DE

The first non-trivial application of the method of differential equations:

massless double boxes with one leg off-shell, $p_1^2 = q^2 \neq 0$,

$$p_i^2 = 0, i = 2, 3, 4$$

[T. Gehrmann & E. Remiddi'01]



Systematic evaluation of master integrals by differential equations.

2dHPL

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Weight for numbers: n for $\zeta(n)$, $\text{Li}_n(1/2)$ etc.

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In other cases, explicit integral representations can be derived, using Feynman parameters or other means.

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DE:

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where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

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Conjecture: one can turn to a new basis by a linear transformation $f \rightarrow Bf$ (resulting in

$A_m \rightarrow B^{-1}A_mB - B^{-1}(\partial_m B)$) such that the DE will take the form

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How to prove it? (A good mathematical problem.)

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A formal solution

$$f(\epsilon, x) = P e^{\epsilon \int_{\mathcal{C}} d\tilde{A}} g(\epsilon),$$

where the integration contour \mathcal{C} connects a base point $x^{(0)}$,
with $g = f(\epsilon, x^{(0)})$, to a given point x .

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where the entries of $d \tilde{A}$ determine the integration kernels.
The problem of evaluating the master integrals f_i in the
 ϵ expansion is essentially solved.

$$P e^{\epsilon \int_{\mathcal{C}} d\tilde{A}} = 1 + \epsilon \int_{0 \leq \tau \leq 1} \mathbf{d}\tilde{A}(\tau) + \epsilon^2 \int_{0 \leq \tau_1 \leq \tau_2 \leq 1} \mathbf{d}\tilde{A}(\tau_2) \cdot \mathbf{d}\tilde{A}(\tau_1) + \dots$$

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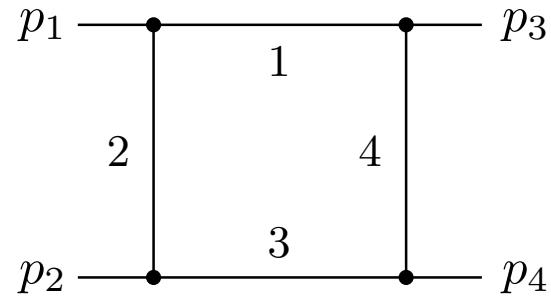
$$g(\epsilon)=g^{(0)}+\epsilon g^{(1)}+\epsilon g^{(2)}+\dots$$

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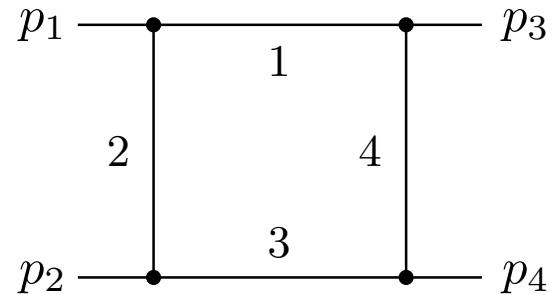
$$g(\epsilon) = g^{(0)} + \epsilon g^{(1)} + \epsilon g^{(2)} + \dots$$

$$\begin{aligned} f(\epsilon) &= g^{(0)} + \epsilon \left(g^{(1)} + \int_{0 \leq \tau \leq 1} \mathbf{d}\tilde{A}(\tau) \cdot g^{(0)} \right) \\ &\quad + \epsilon^2 \left(g^{(2)} + \int_{0 \leq \tau \leq 1} \mathbf{d}\tilde{A}(\tau) \cdot g^{(1)} + \int_{0 \leq \tau_1 \leq \tau_2 \leq 1} \mathbf{d}\tilde{A}(\tau_2) \cdot \mathbf{d}\tilde{A}(\tau_1) \cdot g^{(0)} \right) + \dots \end{aligned}$$

An example: the massless on-shell box diagram, i.e. with
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$$F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)$$

$$= \int \frac{\mathbf{d}^d k}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]^{a_3} [-(k - p_3)^2]^{a_4}} ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

Three master integrals $F(0, 1, 0, 1), F(1, 0, 1, 0), F(1, 1, 1, 1)$.

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$$\int \frac{d^d k}{(-k^2)^{a_1} [-(q-k)^2]^{a_2}} = i\pi^{d/2} \frac{G(a_1, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} ,$$

$$G(a_1, a_2) = \frac{\Gamma(a_1 + a_2 + \epsilon - 2)\Gamma(2 - \epsilon - a_1)\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4 - a_1 - a_2 - 2\epsilon)}$$

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$\Gamma(1 + k\epsilon), \Gamma(k\epsilon)$ are u.t., e.g.

$$\Gamma(1 + \epsilon) = e^{-\gamma_E \epsilon} = \left(1 + \frac{\pi^2 \epsilon^2}{12} - \frac{\epsilon^3 \zeta(3)}{3} + \dots \right)$$

$\Gamma(2 - 2\epsilon) \equiv (1 - 2\epsilon)\Gamma(1 - 2\epsilon)$ is not u.t.

$$G(1, 1) = \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \text{ is u.t.}$$

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Turn to a u.t. basis:

$$f = (-s)^\epsilon \{ \epsilon t F(0, 1, 0, 2), \epsilon s F(1, 0, 2, 0), \epsilon^2 st F(1, 1, 1, 1) \} \equiv \{ f_1, f_2, f_3 \}$$

$$x = t/s, \text{ set } s = -1.$$

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To derive DE apply the projector

$$\mathcal{O}_t = \left(\frac{1}{2t} p_1 + \frac{1}{2(s+t)} p_2 + \frac{s+2t}{2t(s+t)} p_3 \right) \frac{\partial}{\partial p_3}$$

which satisfies $\mathcal{O}_t t = 1$, $\mathcal{O}_t s = \mathcal{O}_t p_i^2 = 0$, $\mathcal{O}_t (p_1 + p_2 + p_3)^2 = 0$

Reduce integrals on the rhs to the master integrals.

DE in the new basis in differential form,

$$\mathbf{d} f(\epsilon, x) = \epsilon \mathbf{d} \tilde{A}(x) f(\epsilon, x)$$

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$$\int_{0 \leq \tau_1 \leq \dots \leq \tau_k \leq x} d\tilde{A}(\tau_k) \dots \cdot \dots d\tilde{A}(\tau_1)$$

→ a linear combination of integrals

$$\int_{0 \leq \tau_1 \leq \dots \leq \tau_k \leq x} \frac{d\tau_k}{\tau_k + a_k} \dots \frac{d\tau_1}{\tau_1 + a_1}$$

where $a_i = 0$ or 1 .

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HPLs

$$H(a_1, a_2, \dots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \dots, a_n; t) dt,$$

where $f(\pm 1; t) = 1/(1 \mp t)$, $f(0; t) = 1/t$,

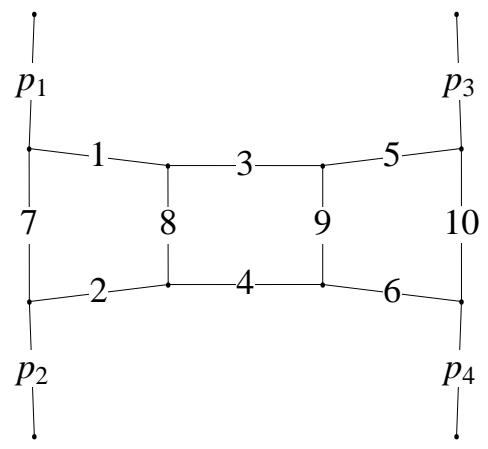
The result is $f_3 = \sum_{j=0} c_j(x, L) \epsilon^j$, with

$$c_0 = 4 \quad c_1 = 2L, \quad c_2 = -\frac{4}{3}\pi^2,$$

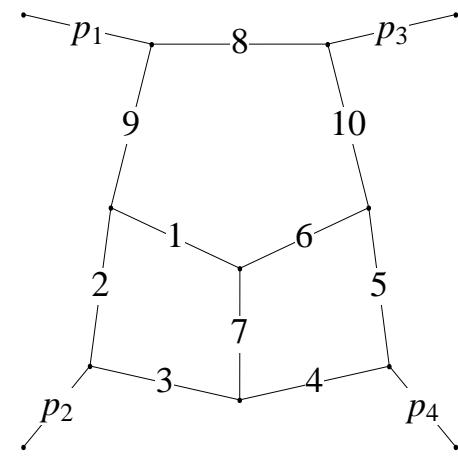
$$c_3 = \pi^2 H_1(x) + 2H_{0,0,1}(x) - \frac{7}{6}\pi^2 L + 2H_{0,1}(x)L + H_1(x)L^2 - \frac{1}{3}L^3 - \frac{34}{3}\zeta_3,$$

$$\begin{aligned} c_4 = & -2H_{1,0,0,1}(x) - 2H_{0,0,1,1}(x) - 2H_{0,1,0,1}(x) - 2H_{0,0,0,1}(x) - 2H_{0,1,1}(x)L \\ & - 2H_{1,0,1}(x)L + H_{0,1}(x)L^2 - H_{1,1}(x)L^2 + \frac{2}{3}H_1(x)L^3 - \frac{1}{6}L^4 \\ & - \pi^2 H_{1,1}(x) + \pi^2 H_1(x)L - \frac{1}{2}\pi^2 L^2 + 2H_1(x)\zeta_3 - \frac{20}{3}L\zeta_3 - \frac{41}{360}\pi^4 + \dots \end{aligned}$$

with $L = \log x$.



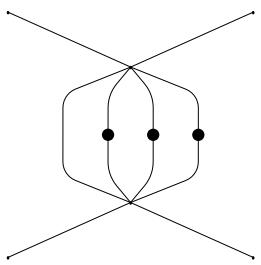
(A)



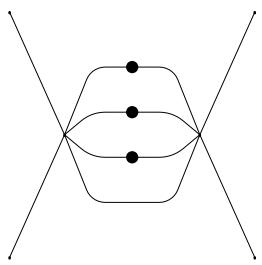
(E)

$$\begin{aligned}
F_{a_1, \dots, a_{15}}^A(s, t; D) &= \int \int \int \frac{d^D k_1 d^D k_2 d^D k_3}{(-k_1^2)^{a_1} [-(p_1 + p_2 + k_1)^2]^{a_2} (-k_2^2)^{a_3}} \\
&\times \frac{[-(k_1 - p_3)^2]^{-a_{11}} [-(p_1 + k_2)^2]^{-a_{12}} [-(k_2 - p_3)^2]^{-a_{13}}}{[-(p_1 + p_2 + k_2)^2]^{a_4} (-k_3^2)^{a_5} [-(p_1 + p_2 + k_3)^2]^{a_6} [-(p_1 + k_1)^2]^{a_7}} \\
&\times \frac{[-(p_1 + k_3)^2]^{-a_{14}} [-(k_1 - k_3)^2]^{-a_{15}}}{[-(k_1 - k_2)^2]^{a_8} [-(k_2 - k_3)^2]^{a_9} [-(k_3 - p_3)^2]^{a_{10}}} ,
\end{aligned}$$

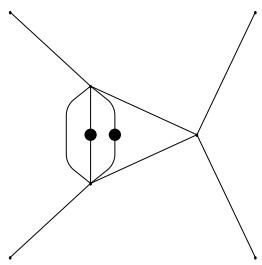
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&\times \frac{[-(p_1 + p_2 + k_3)^2]^{-a_{11}} [-(p_1 + k_2)^2]^{-a_{12}} [-(k_1 - p_3)^2]^{-a_{13}}}{[-(p_1 + p_2 + k_1)^2]^{a_3} [-(p_1 + p_2 + k_2)^2]^{a_4} [-(k_2 - p_3)^2]^{a_5} [-(k_2 - k_3)^2]^{a_6}} \\
&\times \frac{(-k_1^2)^{-a_{14}} (-k_2^2)^{-a_{15}}}{[-(k_1 - k_2)^2]^{a_7} (-k_3^2)^{a_8} [-(p_1 + k_3)^2]^{a_9} [-(k_3 - p_3)^2]^{a_{10}}} .
\end{aligned}$$



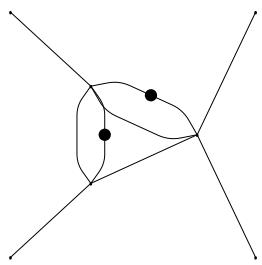
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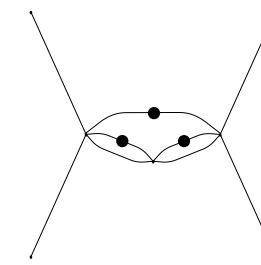
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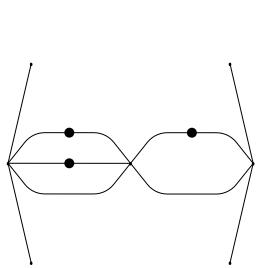
(3)



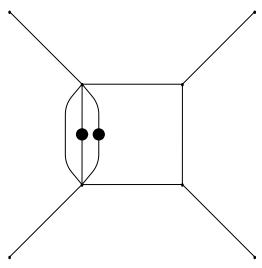
(4)



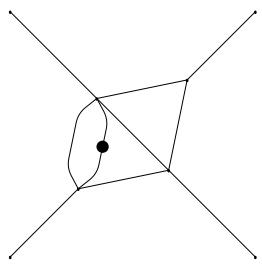
(5)*



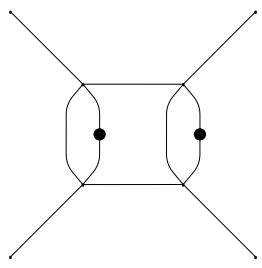
(6)



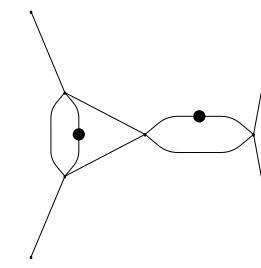
(7)



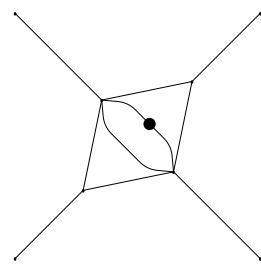
(8)



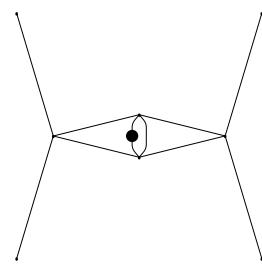
(9), (14)*



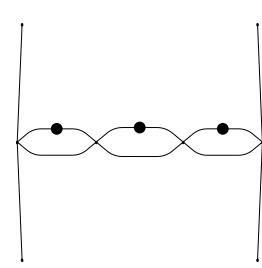
(10)



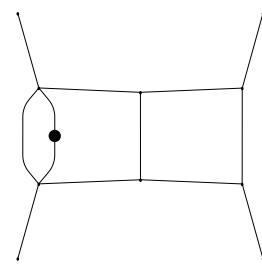
(11)



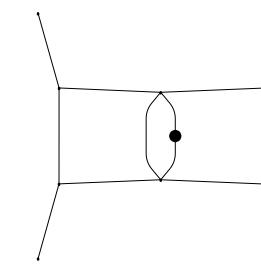
(12)



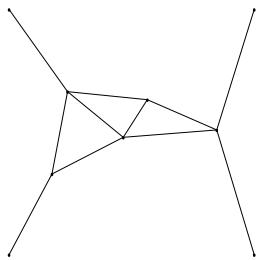
(13)



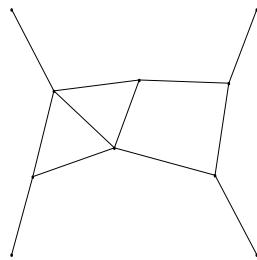
(18)*, (19)



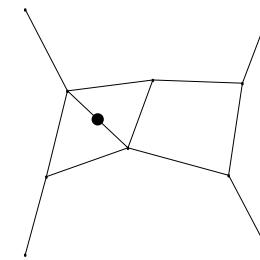
(22), (23)*



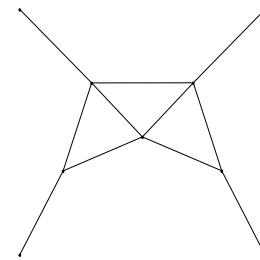
(17)



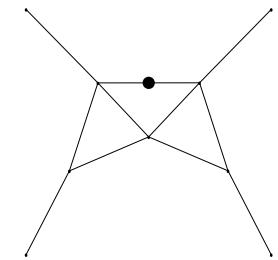
(20)



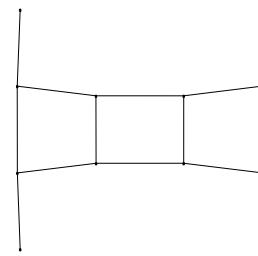
(21)



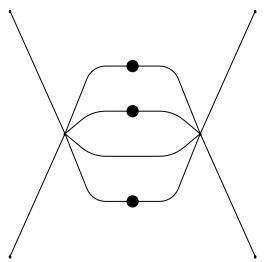
(15)



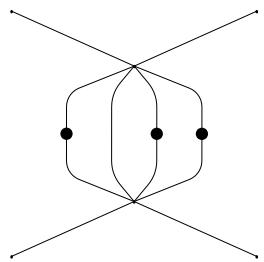
(16)



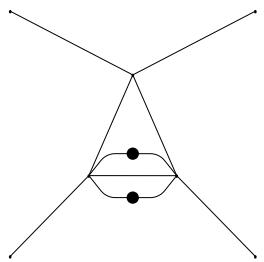
(24), (25)*,
(26)*



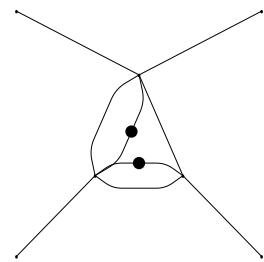
(1)



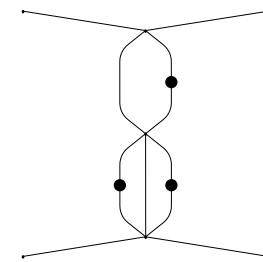
(2)



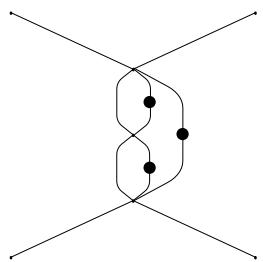
(3)



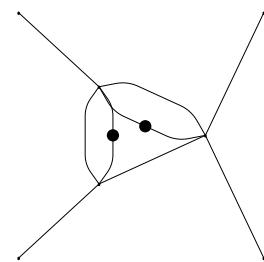
(4)



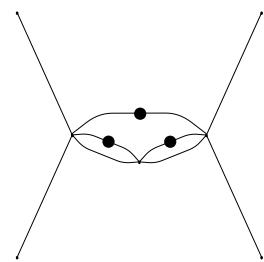
(5)



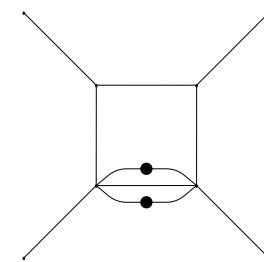
(6)*



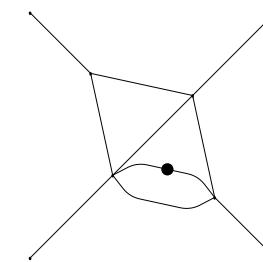
(7)



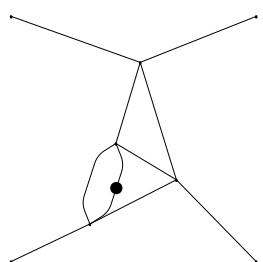
(8)*



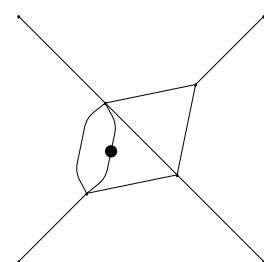
(9)



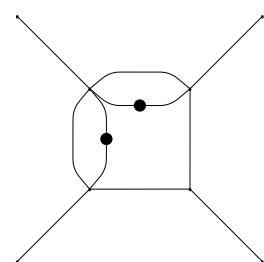
(10)



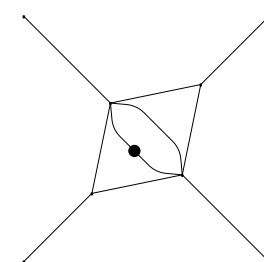
(11)



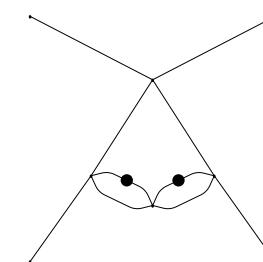
(12)



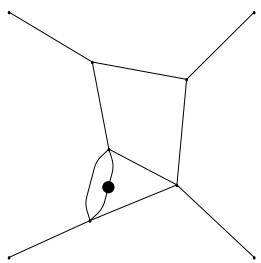
(13)



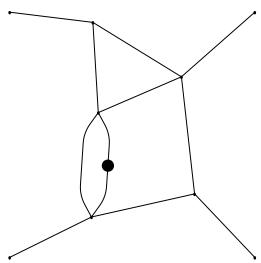
(14)



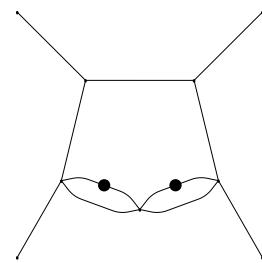
(17)*



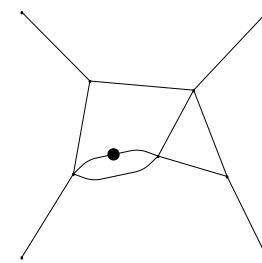
(18)



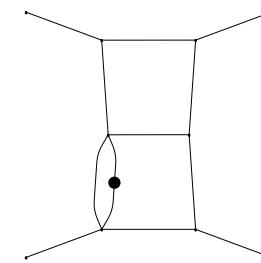
(19)



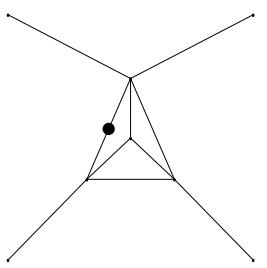
(25)*



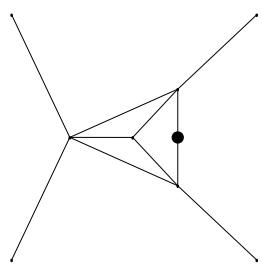
(26)



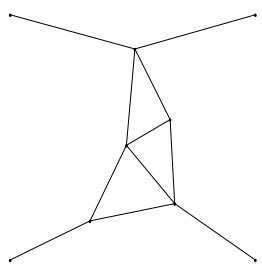
(29), (30)*



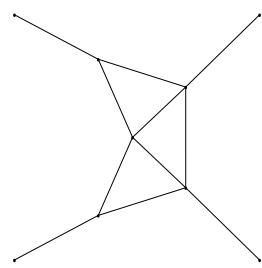
(15)



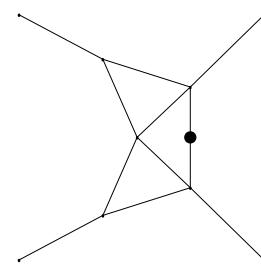
(16)



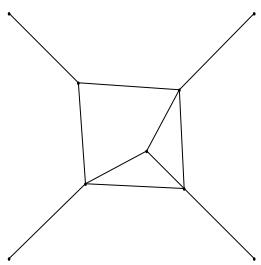
(20)



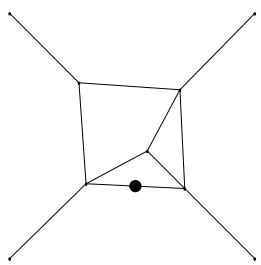
(21)



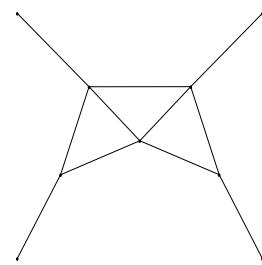
(22)



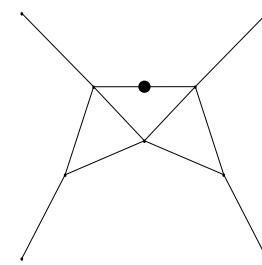
(23)



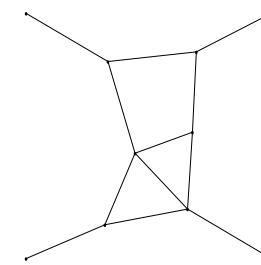
(24)



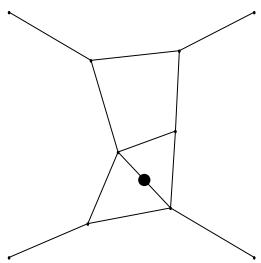
(27)



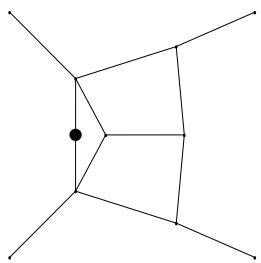
(28)



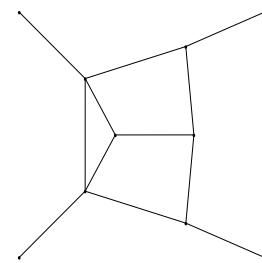
(31)



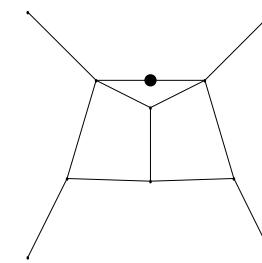
(32)



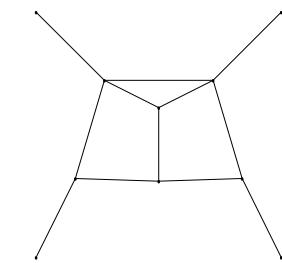
(33)*



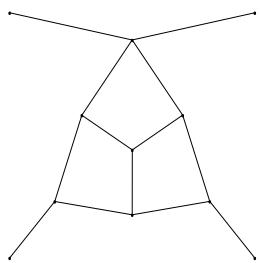
(34)*



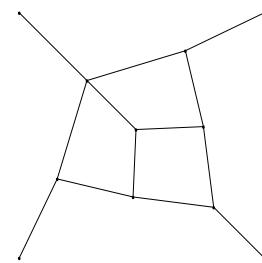
(35)*



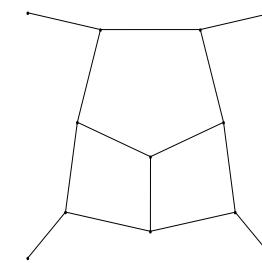
(36)*



(37)*



(38)*



(39)*,
(40)*, (41)*

$$f_i^A = \epsilon^3 (-s)^{3\epsilon} \frac{e^{3\epsilon\gamma_E}}{(i\pi^{D/2})^3} g_i^A.$$

The factor $(-s)^{3\epsilon}$ is to make the basis functions f_i^A dimensionless.

The factor ϵ^3 ensures that all basis functions admit a Taylor expansion around $\epsilon = 0$.

$$\begin{aligned}
g_1^A &= t F_{0,0,0,0,0,0,2,2,2,1,0,0,0,0,0}^A, & g_2^A &= s F_{0,2,0,0,1,0,0,2,2,0,0,0,0,0,0}^A, \\
g_3^A &= \epsilon s F_{0,0,0,0,1,1,2,2,1,0,0,0,0,0,0}^A, & g_4^A &= \epsilon s F_{0,0,0,1,2,0,2,1,1,0,0,0,0,0,0}^A, \\
g_5^A &= s F_{0,1,2,-1,0,1,0,2,2,0,0,0,0,0,0,0}^A, & g_6^A &= s^2 F_{0,2,2,0,2,1,0,1,0,0,0,0,0,0,0}^A, \\
g_7^A &= \epsilon s t F_{0,0,0,0,1,1,2,2,1,1,0,0,0,0,0,0}^A, & g_8^A &= \epsilon^2 (s+t) F_{0,0,0,1,1,0,2,1,1,1,0,0,0,0,0,0}^A, \\
g_9^A &= \epsilon s t F_{0,0,1,1,0,0,2,1,1,2,0,0,0,0,0,0}^A, & g_{10}^A &= \epsilon s^2 F_{0,0,1,1,2,1,2,1,0,0,0,0,0,0,0,0}^A, \\
g_{11}^A &= \epsilon^2 (s+t) F_{0,1,0,0,1,0,1,1,2,1,0,0,0,0,0,0}^A, & g_{12}^A &= -\epsilon(2\epsilon-1) s F_{1,1,0,0,1,1,0,2,1,0,0,0,0,0,0,0}^A, \\
g_{13}^A &= s^3 F_{2,1,2,1,2,1,0,0,0,0,0,0,0,0,0,0}^A, & g_{14}^A &= \epsilon s F_{0,0,1,1,0,0,2,1,1,2,0,0,-1,0,0}^A, \\
g_{15}^A &= \epsilon^3 t F_{0,1,1,0,0,1,1,1,1,1,0,0,0,0,0,0}^A, & g_{16}^A &= \epsilon^2 s^2 F_{0,1,2,0,0,1,1,1,1,1,0,0,0,0,0,0}^A, \\
g_{17}^A &= \epsilon^3 s F_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0,0}^A, & g_{18}^A &= \epsilon^2 s^2 F_{0,0,1,1,1,1,2,1,1,1,0,0,-1,0,0}^A, \\
g_{19}^A &= \epsilon^2 s^2 t F_{0,0,1,1,1,1,2,1,1,1,0,0,0,0,0,0}^A, & g_{20}^A &= \epsilon^3 s(s+t) F_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0,0}^A, \\
g_{21}^A &= \epsilon^2 s^2 t F_{0,1,1,0,1,1,1,2,1,1,0,0,0,0,0,0}^A, & g_{22}^A &= \epsilon^2 s^2 t F_{1,1,0,0,1,1,1,2,1,1,0,0,0,0,0,0}^A, \\
g_{23}^A &= \epsilon^2 s^2 F_{1,1,0,0,1,1,1,2,1,1,-1,0,0,0,0,0}^A, & g_{24}^A &= \epsilon^3 s^3 t F_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0}^A, \\
g_{25}^A &= \epsilon^3 s^3 F_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0,0}^A, & g_{26}^A &= \epsilon^3 s^3 F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,0}^A
\end{aligned}$$

With the variable $x = t/s$, the differential equations take the following form,

$$\partial_x f(x, \epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f(x, \epsilon).$$

where a and b are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E.

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The matrices a and b for case A are on the next slide.

-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$\frac{4}{3}$	0	0	0	0	0	0	0	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$-\frac{1}{6}$	0	0	0	-1	0	0	0	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$-\frac{1}{3}$	0	$\frac{1}{3}$	0	0	0	0	0	0	0	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$\frac{1}{3}$	0	0	-8	$-\frac{8}{3}$	0	0	0	0	0	0	0	0	0	0	0	12	-3	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	0	$-\frac{2}{3}$	$-\frac{40}{9}$	0	0	-1	-24	-2	$\frac{4}{3}$	0	0	0	$-\frac{8}{3}$	0	0	0	1	$\frac{2}{3}$	0	0	0	0	0	0	0	0	0	
-2	0	0	8	0	0	0	$\frac{3}{2}$	24	$-\frac{2}{3}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$\frac{23}{27}$	$\frac{17}{54}$	$-\frac{1}{6}$	$-\frac{56}{9}$	$-\frac{14}{9}$	$\frac{1}{6}$	-1	$-\frac{20}{3}$	0	0	-2	0	0	0	$\frac{8}{3}$	-2	$-\frac{2}{3}$	0	0	1	$\frac{1}{3}$	0	0	0	0	0	0	0	0
$-\frac{4}{3}$	$-\frac{54}{3}$	0	0	0	0	3	0	0	0	12	0	0	0	0	0	0	0	0	0	$-\frac{3}{2}$	0	0	0	0	0	0	0	
$-\frac{4}{3}$	$-\frac{4}{3}$	0	0	0	0	3	0	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$\frac{20}{9}$	$\frac{19}{9}$	-2	0	0	0	-3	0	0	0	-20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	-4	2	0	-3	0	0	0	
$-\frac{8}{3}$	$\frac{41}{18}$	$-\frac{7}{2}$	$\frac{68}{9}$	$\frac{14}{9}$	$\frac{7}{2}$	0	48	4	3	-12	3	1	0	0	0	-2	0	-6	0	6	-2	0	2	1	0	0	0	
$-\frac{28}{9}$	$-\frac{7}{6}$	$\frac{9}{2}$	$\frac{20}{3}$	$\frac{22}{9}$	$\frac{13}{2}$	3	16	0	3	12	0	1	0	-16	4	6	0	-2	-12	2	-3	-3	1	1	0	0	0	

3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
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$-\frac{1}{3}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$\frac{1}{12}$	0	0	-2	$-\frac{2}{3}$	0	0	-2	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	-4	$\frac{16}{9}$	0	0	0	0	2	0	0	0	0	0	0	0	0	-3	1	0	0	0	0	0	0	0	0	0	
$-\frac{23}{27}$	$-\frac{17}{54}$	$\frac{1}{6}$	$\frac{56}{9}$	$\frac{14}{9}$	$-\frac{1}{6}$	1	$\frac{20}{3}$	0	0	2	0	0	0	$-\frac{8}{3}$	2	$\frac{2}{3}$	0	0	-2	$-\frac{1}{3}$	0	0	0	0	0	0	0	
$-\frac{9}{9}$	$-\frac{7}{9}$	$-\frac{40}{9}$	$-\frac{3}{3}$	-4	-2	-3	-16	0	0	-12	0	0	0	16	-4	-4	0	0	12	$\frac{2}{3}$	0	0	0	0	0	0	0	
0	$-\frac{1}{3}$	-6	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	-3	0	0	0	0	
0	-1	2	0	0	0	0	0	0	0	0	-3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	
0	$\frac{17}{9}$	-7	$\frac{40}{9}$	$\frac{28}{9}$	7	0	0	6	0	6	2	0	0	0	-4	0	0	0	0	0	0	0	0	0	-2	-2	0	
$-\frac{28}{9}$	$-\frac{16}{9}$	6	$\frac{32}{9}$	$\frac{28}{9}$	-5	3	16	0	-2	12	-6	-2	0	-16	4	8	-6	0	-12	-2	0	0	0	0	0	2	0	
0	0	0	0	0	-3	0	0	0	-6	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

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A solution near $D = 4$ dimensions, so we parametrize, e.g. for family A ,

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The iterative solution in ϵ for all functions f_i can be expressed in terms of harmonic polylogarithms of argument x and with indices drawn from $0, -1$, up to boundary constants.

For planar graphs we expect the limit $u \rightarrow 0$, i.e. $x \rightarrow -1$ to be finite.

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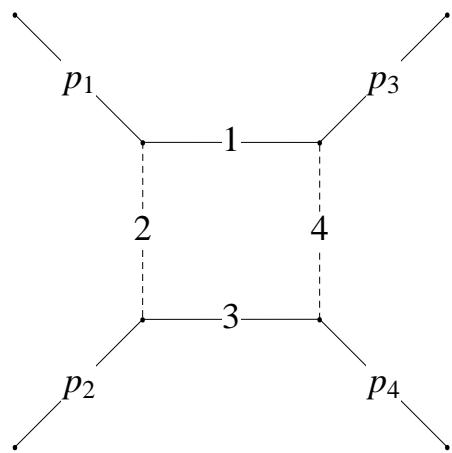
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These conditions fix almost everything:

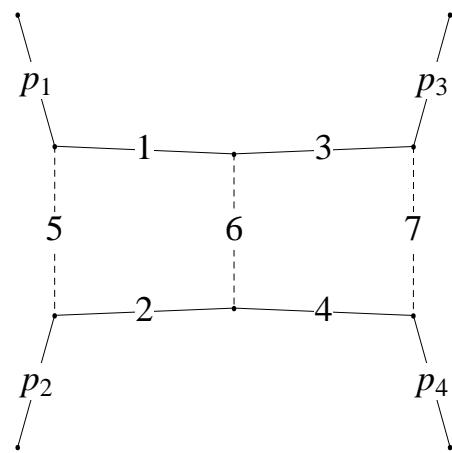
the only additional information needed can easily be obtained from f_1 :

$$\begin{aligned}
f_1^A &= e^{3\epsilon\gamma_E} \Gamma^4(1-\epsilon) \Gamma(1+3\epsilon) / \Gamma(1-4\epsilon) \\
&= 1 - \epsilon^2 \frac{\pi^2}{4} - 29\epsilon^3 \zeta_3 - \epsilon^4 \frac{71}{160} \pi^4 + \epsilon^5 \left(\frac{29}{4} \pi^2 \zeta_3 - \frac{1263}{5} \zeta_5 \right) \\
&\quad + \epsilon^6 \left(-\frac{11539}{24192} \pi^6 + \frac{841}{2} \zeta_3^2 \right) + \mathcal{O}(\epsilon^7).
\end{aligned}$$

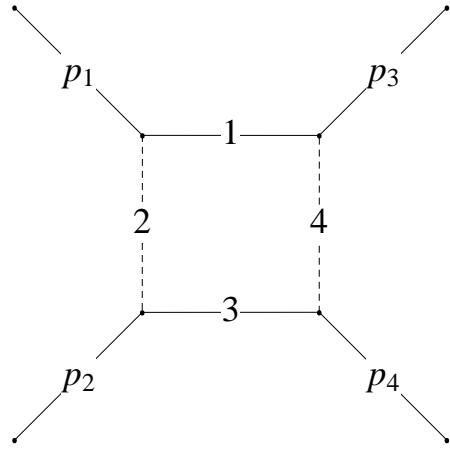
$$\begin{aligned}
f_{26}^A(x, \epsilon) = & -\frac{4}{9} + \frac{13\pi^2\epsilon^2}{36} + \frac{1}{2}\epsilon H_{\{0\}}(x) \\
& + \epsilon^3 \left(\frac{9}{4}\pi^2 H_{\{-1\}}(x) - \frac{15}{8}\pi^2 H_{\{0\}}(x) + \frac{9}{2}H_{\{-1,0,0\}}(x) \right. \\
& \quad \left. - \frac{9}{2}H_{\{0,0,0\}}(x) - \frac{71\zeta_3}{18} \right) \\
& + \epsilon^4 \left(\frac{61\pi^4}{720} + \frac{21}{4}\pi^2 H_{\{-1,-1\}}(x) - \frac{25}{4}\pi^2 H_{\{-1,0\}}(x) \right. \\
& \quad \left. - \frac{21}{4}\pi^2 H_{\{0,-1\}}(x) + \frac{25}{4}\pi^2 H_{\{0,0\}}(x) \right. \\
& \quad \left. + \frac{21}{2}H_{\{-1,-1,0,0\}}(x) - 27H_{\{-1,0,0,0\}}(x) \right. \\
& \quad \left. - \frac{21}{2}H_{\{0,-1,0,0\}}(x) + 27H_{\{0,0,0,0\}}(x) + \frac{21}{2}H_{\{-1\}}(x)\zeta_3 \right. \\
& \quad \left. - 2H_{\{0\}}(x)\zeta_3 \right) + \dots
\end{aligned}$$



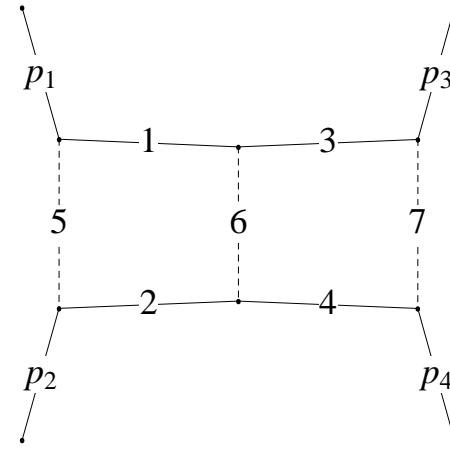
(1)



(2a)



(1)



(2a)

$$G_{a_1, \dots, a_4}(s, t, m^2; D)$$

$$= \int \frac{d^D k}{[-k^2 + m^2]^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2 + m^2]^{a_3} [-(k - p_3)^2]^{a_4}} ,$$

$$\begin{aligned}
G_{a_1, a_2, \dots, a_9}(s, t, m^2; D) &= \int \int \frac{d^D k_1 d^D k_2}{(-k_1^2 + m^2)^{a_1} [-(k_1 + p_1 + p_2)^2 + m^2]^{a_2}} \\
&\times \frac{[-(k_2 + p_1)^2]^{-a_8} [-(k_1 - p_3)^2]^{-a_9}}{[-k_2^2 + m^2]^{a_3} [-(k_2 + p_1 + p_2)^2 + m^2]^{a_4} [-(k_1 + p_1)^2]^{a_5} [-(k_1 - k_2)^2]^{a_6} [-(k_2 - p_3)^2]^{a_7}}
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\end{aligned}$$

Results for some of the master integrals for 2a

[VS'02, G. Heinrich & VS'04, M. Czakon, J. Gluza & T. Riemann'04–06]

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Due to invariance under inversions of x and y , it is sufficient to consider $|x| < 1, |y| < 1$.

Singular points

$$x = 0 \leftrightarrow s = \infty, \quad x = 1 \leftrightarrow s = 0 \quad x = -1 \leftrightarrow s = 4m^2$$

A branch cut in the s -channel starting at $s = 4m^2$ and
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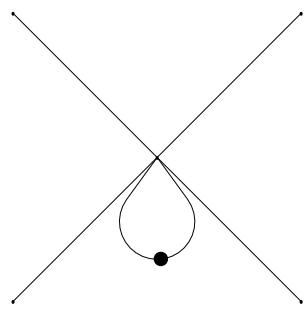
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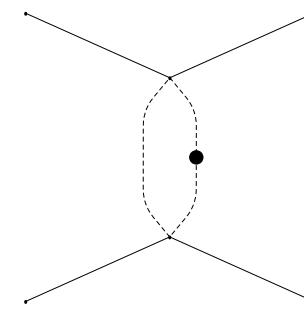
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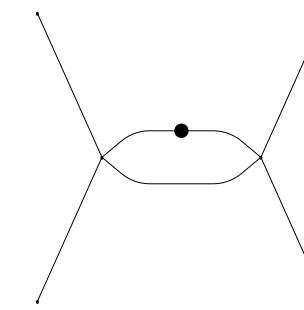
The analytic result should be real-valued in the $s < 0, t < 0$,
i.e. $0 < x < 1, 0 < y < 1$.



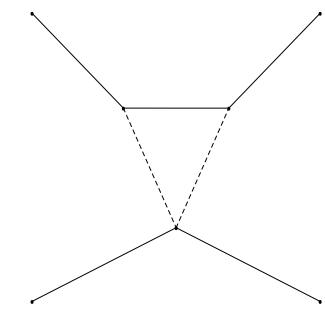
(b) (1)



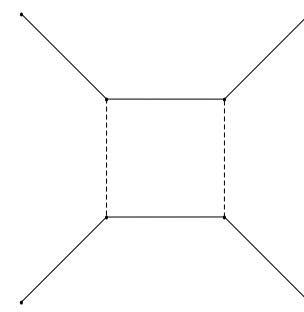
(b) (2)



(b) (3)



(b) (4)



(b) (5)

$$f_i = (m^2)^\epsilon e^{2\epsilon\gamma_E} g_i$$

with

$$g_1 = \epsilon G_{2,0,0,0},$$

$$g_2 = \epsilon t G_{0,2,0,1},$$

$$g_3 = \epsilon \sqrt{(-s)(4m^2 - s)} G_{2,0,1,0},$$

$$g_4 = -2\epsilon^2 (4m^2 - t)(-t) G_{1,1,0,1},$$

$$g_5 = -2\epsilon^2 \sqrt{(-s)(4m^2 - s)} t G_{1,1,1,1}.$$

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We obtain

$$\mathbf{d} f = \epsilon \mathbf{d} \tilde{A} f$$

with

$$\begin{aligned}
\tilde{A} = & \left[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -2 \end{pmatrix} \log(1+x) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{pmatrix} \log y \right. \\
& + \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \log(1+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \log(1-y) + \right. \\
& \left. + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{pmatrix} \log(x+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \log(1+xy) \right].
\end{aligned}$$

A solution in terms of Chen iterated integrals

$$f(x, y, \epsilon) = \mathbb{P} e^{\epsilon \int_C d\tilde{A}} g(\epsilon),$$

which can be evaluated in terms of multiple polylogarithms.
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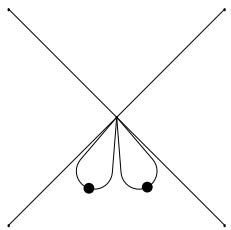
$$\begin{aligned} f_5 = & \epsilon \left[4H_0(x) \right] + \epsilon^2 \left[4G_0(y)H_0(x) - 8G_1(y)H_0(x) \right] \\ & + \epsilon^3 \left[-8G_0(y)H_{-1,0}(x) + 4G_0(y)H_{0,0}(x) - 8H_0(x)G_{1,0}(y) + 16H_0(x)G_{1,1}(y) \right. \\ & + 4H_0(x)G_{-\frac{1}{x},0}(y) - 8H_0(x)G_{-\frac{1}{x},1}(y) + 4H_0(x)G_{-x,0}(y) - 8H_0(x)G_{-x,1}(y) \\ & + 8H_{-1,0}(x)G_{-\frac{1}{x}}(y) + 8H_{-1,0}(x)G_{-x}(y) - 4H_{0,0}(x)G_{-\frac{1}{x}}(y) - 4H_{0,0}(x)G_{-x}(y) \\ & + 4G_{-\frac{1}{x},0,0}(y) - 8G_{-\frac{1}{x},0,1}(y) - 4G_{-x,0,0}(y) + 8G_{-x,0,1}(y) + 8H_{-2,0}(x) \\ & - 16H_{-1,-1,0}(x) + 8H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{10}{3}\pi^2 G_{-\frac{1}{x}}(y) - 2\pi^2 G_{-x}(y) \\ & \left. - \frac{2}{3}\pi^2 G_0(y) - \frac{4}{3}\pi^2 H_{-1}(x) - \frac{7}{3}\pi^2 H_0(x) + 8\zeta_3 \right] + \mathcal{O}(\epsilon^4). \end{aligned}$$

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t),$$

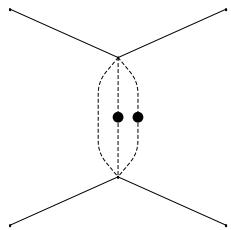
with

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}, \quad a_1 \neq 0.$$

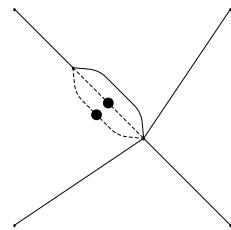
For $a_1 = 0$, we have $G(\vec{0}_n; x) = 1/n! \log^n(x)$.



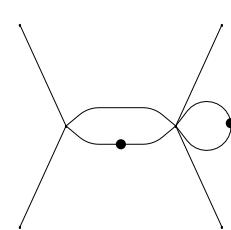
(1)



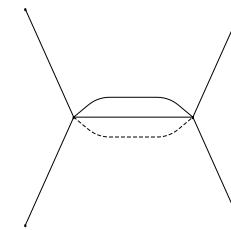
(2)



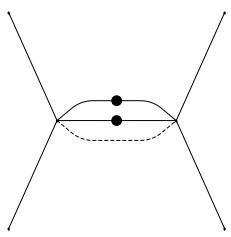
(3)



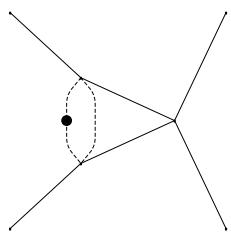
(4)



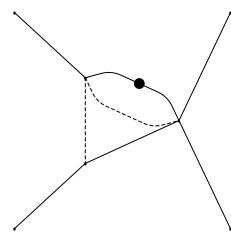
(5)[†]



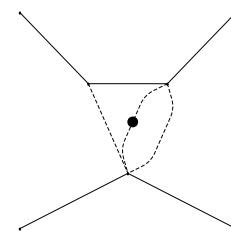
(6)



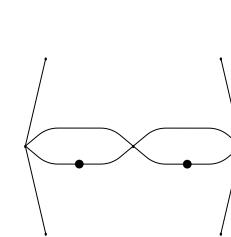
(7)



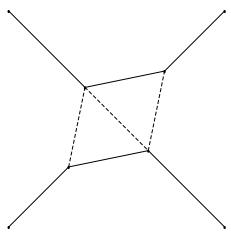
(8)



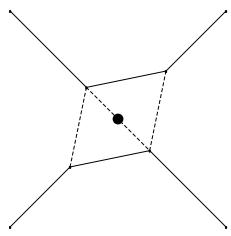
(9)



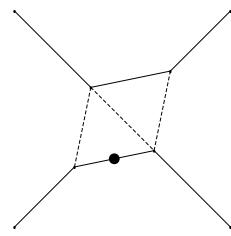
(10)



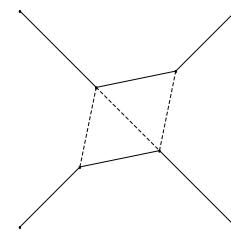
(11)



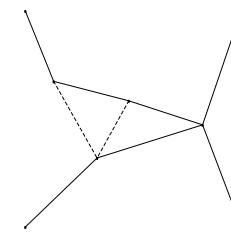
(12)



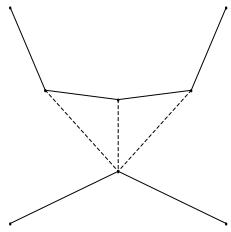
(13)



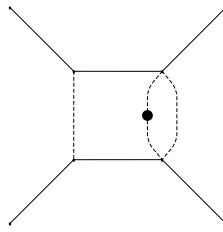
(14)[†]



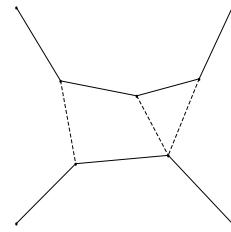
(15), (16)[†]



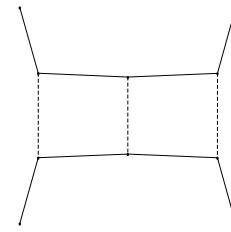
(17),(18)[†]



(19)



(20),(21)[†]



(22),(23)^{*}

$$\mathrm{d} f = \epsilon \mathrm{d} \tilde{A} f$$

with

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with

$$\begin{aligned}
\tilde{A} = & B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\
& + B_6 \log(1-y) + B_7 \log(x+y) + B_8 \log(1+xy) \\
& + B_9 \log(x+y - 4xy + x^2y + xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\
& + B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)
\end{aligned}$$

$$\mathbf{d} f = \epsilon \mathbf{d} \tilde{A} f$$

with

$$\begin{aligned}\tilde{A} = & B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\ & + B_6 \log(1-y) + B_7 \log(x+y) + B_8 \log(1+xy) \\ & + B_9 \log(x+y - 4xy + x^2y + xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\ & + B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)\end{aligned}$$

$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y - 4xy + x^2y + xy^2}},$$

- At order ϵ, ϵ^2 and ϵ^3 , the symbol alphabet remains the same as at one loop.
- At order ϵ^4 all functions except f_{11} have the same symbol alphabet as at one loop.

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- At order ϵ^4 all functions except f_{11} have the same symbol alphabet as at one loop.

For example,

$$\begin{aligned}
f_{23} = & \epsilon^2 \left[-12H_{0,0}(x) \right] + \epsilon^3 \left[-16G_0(y)H_{0,0}(x) + 32G_1(y)H_{0,0}(x) + 8H_{2,0}(x) \right. \\
& + 16H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{4}{3}\pi^2 H_0(x) + 4\zeta_3 \left. \right] + \epsilon^4 \left[32G_0(y)H_{-2,0}(x) \right. \\
& - 32H_{-2,0}(x)G_{-\frac{1}{x}}(y) - 32H_{-2,0}(x)G_{-x}(y) + 64G_{1,0}(y)H_{0,0}(x) - 128G_{1,1}(y)H_{0,0}(x) \\
& - 32H_{0,0}(x)G_{-\frac{1}{x},0}(y) + 64H_{0,0}(x)G_{-\frac{1}{x},1}(y) - 32H_{0,0}(x)G_{-x,0}(y) \\
& + 64H_{0,0}(x)G_{-x,1}(y) - 16H_0(x)G_{-\frac{1}{x},0,0}(y) + 32H_0(x)G_{-\frac{1}{x},0,1}(y) \\
& + 16H_0(x)G_{-x,0,0}(y) - 32H_0(x)G_{-x,0,1}(y) + 64G_0(y)H_{-1,0,0}(x) \\
& - 64H_{-1,0,0}(x)G_{-\frac{1}{x}}(y) - 64H_{-1,0,0}(x)G_{-x}(y) \\
& - 48G_0(y)H_{0,0,0}(x) + 48H_{0,0,0}(x)G_{-\frac{1}{x}}(y) + 48H_{0,0,0}(x)G_{-x}(y) - 120H_{-3,0}(x) \\
& + \frac{52}{3}\pi^2 H_{0,0}(x) + 48H_{3,0}(x) + 128H_{-2,-1,0}(x) - 120H_{-2,0,0}(x) - 48H_{-2,1,0}(x) \\
& + 64H_{-1,-2,0}(x) - 32H_{-1,2,0}(x) - 48H_{2,-1,0}(x) + 32H_{2,0,0}(x) + 16H_{2,1,0}(x) \\
& + 64H_{-1,-1,0,0}(x) - 80H_{-1,0,0,0}(x) + 76H_{0,0,0,0}(x) + \frac{8}{3}\pi^2 G_0(y)H_0(x) \\
& - \frac{40}{3}\pi^2 H_0(x)G_{-\frac{1}{x}}(y) + 8\pi^2 H_0(x)G_{-x}(y) - 16\zeta_3 H_{-1}(x) - 28\zeta_3 H_0(x) \\
& \left. + \frac{8}{3}\pi^2 H_{-2}(x) - \frac{4}{3}\pi^2 H_2(x) - \frac{4\pi^4}{15} \right] + \mathcal{O}(\epsilon^5)
\end{aligned}$$

to be continued