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- Expansion by regions for parametric integrals

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Applied Asymptotic Expansions in Momenta and Masses
(STMP 177, Springer 2002)

Analytic Tools for Feynman Integrals (STMP 250, Springer 2013)

Introduction

If a given Feynman integral depends on kinematic invariants and masses which essentially differ in scale, a natural idea is to expand it in ratios of small and large parameters. As a result, the integral is written as a series of simpler quantities than the original integral itself and it can be substituted by a sufficiently large number of terms of such an expansion.

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or the large-mass limit.

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Threshold limit $q^2 \to 4m^2$.

Expansion by subgraphs

Let us consider a Feynman integral F_Γ, corresponding to a graph Γ, in the off-shell large-momentum limit.
$F_\Gamma(Q_1, \ldots, Q_{n_1}, q_1, \ldots, q_{n_2})$
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The sum runs over *asymptotically irreducible (AI)* subgraphs.
Let $\hat{\gamma}$ be the graph that is obtained from a given subgraph γ by identifying all the external vertices associated with the large external momenta.

A subgraph γ is AI if

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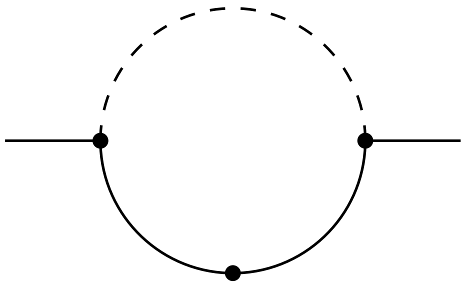
The symbol $\circ$ denotes the insertion of the polynomial which stands to the right of it into the reduced vertex of the graph Γ/γ, i.e. to the vertex to which the subgraph γ was reduced.

Example.



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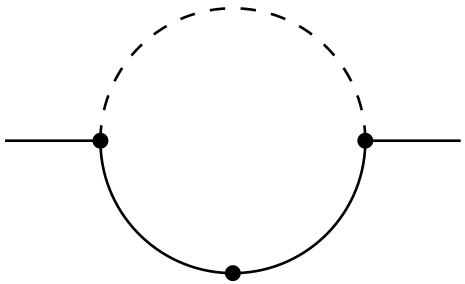

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2} \; .$$

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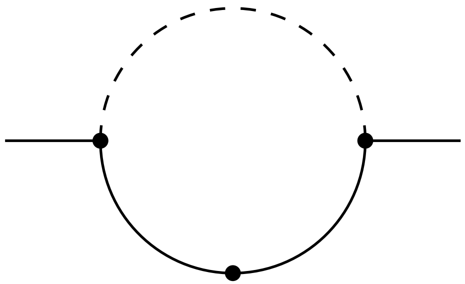

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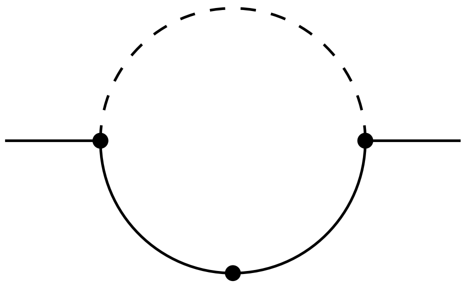

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$\Gamma \to$ expanding the massive propagator in a Taylor series in m.
The contribution from γ is obtained by expanding the propagator $1/(q-k)^2$ in a Taylor series in k:

$$\frac{1}{(q-k)^2} = \frac{1}{q^2} + \frac{2q\cdot k - k^2}{(q^2)^2} + \frac{(2q\cdot k - k^2)^2}{(q^2)^3} + \ldots \; .$$

$$
F_\Gamma(q^2,m^2;d) \sim \int \frac{\mathrm{d}^d k}{(k^2)^2(q-k)^2} - 2m^2 \int \frac{\mathrm{d}^d k}{(k^2)^3(q-k)^2} + \ldots
$$

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+\frac{1}{q^2}\int \frac{\mathrm{d}^d k}{(k^2-m^2)^2} + \frac{1}{(q^2)^2}\int \frac{(2q\cdot k - k^2)\,\mathrm{d}^d k}{(k^2-m^2)^2} + \ldots \,.
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$$F_\Gamma(q^2,m^2;d) \sim \frac{\mathrm{i}\pi^{d/2}}{(-q^2)^{1+\epsilon}} \frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(1-2\epsilon)} \left(1 + 2\epsilon\frac{m^2}{q^2} + \ldots\right)$$

$$+\frac{\mathrm{i}\pi^{d/2}}{q^2(m^2)^\epsilon}\Gamma(\epsilon)\left(1 + \frac{\epsilon}{1+\epsilon}\frac{m^2}{q^2} + \ldots\right)\ .$$

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$$F_\Gamma(q^2, m^2; 4) \sim \frac{\mathrm{i}\pi^2}{q^2} \left[\ln\left(\frac{-q^2}{m^2}\right) - \frac{m^2}{q^2} + \ldots \right] \,.$$

Expansion by regions

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Expansion by regions. [Beneke & V.S.'98]

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a Taylor series with respect to the parameters that are considered small there.
- Integrate the integrand, expanded in the appropriate way in every region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

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For limits typical of Euclidean space, where we have two scales, for example Q and q, there is a simple equivalence of the two strategies. (Consider the set of

regions labelled by 1PI subgraphs of the given graph.)

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hard, collinear (i.e. almost parallel to two light-like four-vectors), ultrasoft.

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Choose the frame $q = (q_0, \vec{0})$.
Consider the various regions where any loop momentum can be of one of the following four types:

$$
\begin{aligned}
&\text{(hard)}, && k_0 \sim \sqrt{q^2}\,,\ \vec{k} \sim \sqrt{q^2}\,, \\
&\text{(soft)}, && k_0 \sim \sqrt{y}\,,\ \vec{k} \sim \sqrt{y}\,, \\
&\text{(potential)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim \sqrt{y}\,, \\
&\text{(ultrasoft)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim y/\sqrt{q^2}\,.
\end{aligned}
$$

Expansion by regions for parametric integrals

$$F_\Gamma(q_1,\ldots,q_n;d;a_1\ldots,a_L) \quad = \quad \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \delta\left(\sum_{l=1}^L \alpha_l - 1\right) \frac{\prod_l \alpha_l^{a_l-1} \mathcal{U}^{a-(h+1)d/2}}{\left(-\mathcal{V}+\mathcal{U}\sum m_l^2 \alpha_l\right)^{a-hd/2}} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_L$$

where $\mathcal{U}$ and $\mathcal{V}$ are (Kirchhoff, Symanzik) polynomials determined by a given graph.

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Expansion by regions in alpha parameters [VS'99] `asy.m` [Pak & A. Smirnov'11]

a public code to reveal regions.

Input: propagators

Output: regions in the language of alpha parameters

Let
$P(x_1,\ldots,x_n,t) = \sum_{w_1,\ldots,w_n,w_{n+1}} c_{w_1,\ldots,w_n,w_{n+1}} x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}}$
be a polynomial with $c_{\ldots}$>0.

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Conjecture. Let

$$F(t) = \int_0^\infty \ldots \int_0^\infty (P(x_1,\ldots,x_n,t))^\lambda \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \;,$$

where $t > 0$, and $\lambda \in \mathbb{C}$ is such that the integral is absolutely convergent.

Then the asymptotic expansion of the function $F(t)$ in the limit $t \to +0$ is given by

$$F(t) \sim \sum_i \int_0^\infty \ldots \int_0^\infty \left[M_i \left(P(x_1, \ldots, x_n, t) \right)^\lambda \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

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where the sum runs over facets of the Newton polytope of P, for which the normal vectors $r_i = (r_{i,1}, \ldots, r_{i,n}, r_{i,n+1}), i = 1, \ldots, N$ oriented inside the polytope have $r_{i,n+1} > 0$.

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Let us normalize these vectors by $r_{i,n+1} = 1$.

Let

$$P_i(x_1, \ldots, x_n, t, \rho) = P(\rho^{r_{i,1}} x_1, \ldots, \rho^{r_{i,n}} x_n, \rho t)$$

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so that

$$P_i(x_1,\ldots,x_n,t,\rho) = \sum_{m=m_{i,1}}^{m_{i,2}} \rho^m Q_{i,m}(x_1,\ldots,x_n,t)\ ,$$

where the sum runs over rational numbers from a certain set determined by P and its i-th facet.

$$M_i\left(P(x_1,\ldots,x_n,t)\right)^\lambda = \rho^{m_{i,1}\lambda}\mathcal{T}_\rho \left(\sum_{m=m_{i,1}}^{m_{i,2}} \rho^{m-m_{i,1}} Q_{i,m}(x_1,\ldots,x_n,t)\right)^\lambda\Bigg|_{\rho=1}$$

$$= \mathcal{T}_\rho \left(Q_{i,m_{i,1}}(x_1,\ldots,x_n,t) + \sum_{m=m_{i,1}+1}^{m_{i,2}} \rho^{m-m_{i,1}} Q_{i,m}(x_1,\ldots,x_n,t)\right)^\lambda\Bigg|_{\rho=1}$$

$$= \left(Q_{i,m_{i,1}}(x_1,\ldots,x_n,t)\right)^\lambda + \ldots$$

where the operator $\mathcal{T}_\rho$ performs an asymptotic expansion in powers of ρ at $\rho = 0$.

Example. Let us expand

$$F(t) = \int_0^\infty \frac{\mathrm{d}x}{P(x,t)} \ ,$$

with $P(x,t) = t^3 + tx + x^2 + x^3$, in the limit $t \to +0$.

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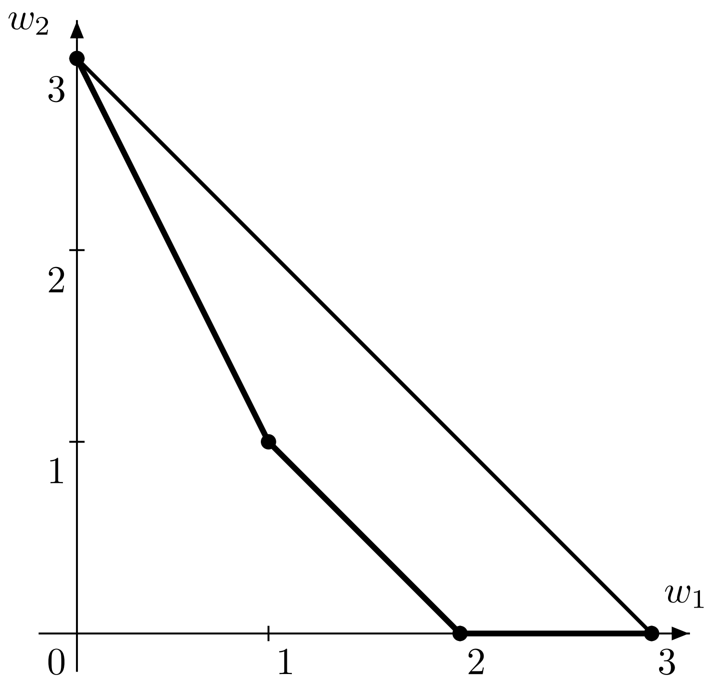
Introduce a regularization:

$$F(t,\lambda) = \int_0^\infty \frac{\mathrm{d}x}{P(x,t)^{1+\lambda}} \ .$$

The Newton polytope of P



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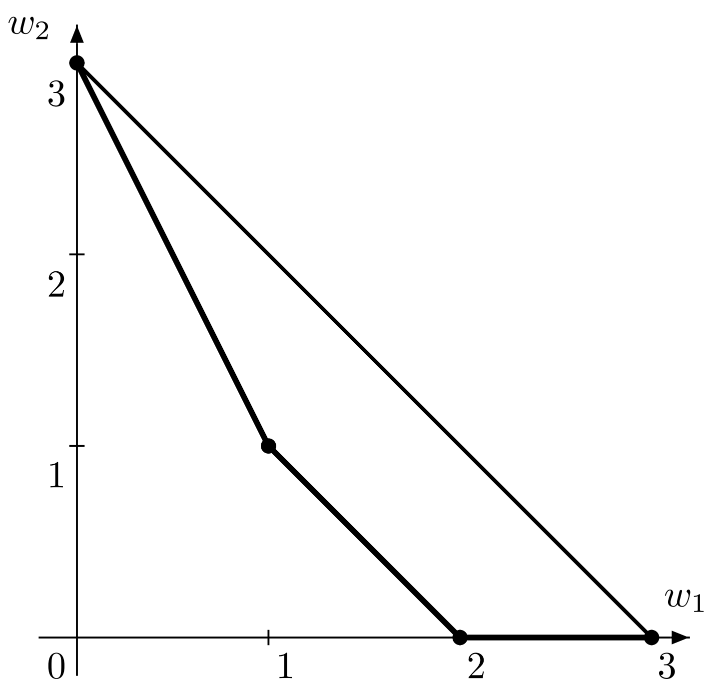

Normal vectors for the three facets contributing to the expansion:

$r_1 = (2, 1)$, $r_2 = (1, 1)$, $r_3 = (0, 1)$

Facet 1. $x \to \rho^2 x, t \to \rho t$.

$P(x,t) \to \rho^3 t^3 + \rho^3 tx + \rho^4 x^2 + \rho^6 x^3 = \rho^3(t^3 + tx + \rho x^2 + \rho^3 x^3)$

LO:

$$\int_0^\infty \frac{\mathrm{d}x}{(t^3+tx)^{1+\lambda}} = \frac{1}{\lambda}\frac{1}{t^{1+3\lambda}}$$

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Facet 2. $x \to \rho x, t \to \rho t$.
$P(x,t) \to \rho^3 t^3 + \rho^2 tx + \rho^2 x^2 + \rho^3 x^3 = \rho^2 (tx + x^2 + \ldots)$
LO:

$$\int_0^\infty \frac{\mathrm{d}x}{(tx+x^2)^{1+\lambda}} = \frac{\Gamma(-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)} \frac{1}{t^{1+2\lambda}}$$

Facet 3. $t \to \rho t$. Just an expansion in t.

LO:

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The result in LO

$$F(t) = \int_0^\infty \frac{\mathrm{d}x}{P(x,t)} \sim -\frac{\ln t}{t} + O(\ln t)$$

Normal vectors of the faces $\leftrightarrow$ regions

Let $r = (r_1, \ldots, r_n)$.
Perform the scaling $x_j \to \rho^{r_j} x_j, i = 1, \ldots, n$ and $t \to \rho t$
Then

$$c x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}} \to c(\rho^{r_1} x_1)^{w_1} \ldots (\rho^{r_n} x_n)^{w_n} (\rho t)^{w_{n+1}}$$
$$= c\rho^{r_1 w_1 + \ldots + r_n w_n + w_{n+1}} x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}}$$
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$= c\rho^{r_1 w_1 + \ldots + r_n w_n + w_{n+1}} x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}}$
$= c\rho^{\vec{r}\vec{w}} x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}}$
Keeping the minimal power of ρ in the polynomial reduced to taking terms corresponding to a face of the Newton polytope (to which r is normal).

If for a given r we obtain a face of dimension $< n$ then the corresponding contribution is a zero scaleless integral.

Example.

$$F(t) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{P(x_1, x_2, t)} \;,$$

with $P(x_1, x_2, t) = (x_1 + t)(x_2 + t)(x_1 + x_2 + 1)$, in the limit $t \to +0$.

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Example.

$$F(t) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{P(x_1,x_2,t)} \;,$$

with $P(x_1,x_2,t) = (x_1+t)(x_2+t)(x_1+x_2+1)$, in the limit $t \to +0$.

Introduce a regularization:

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{P(x_1,x_2,t)^{1+\lambda}}$$

`asy.m` $\to$ four regions $(0,0,1),(0,1,1),(1,0,1),(1,1,1)$.

$(0,0,1)$

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\left(x_1 x_2 (x_1 + x_2 + 1)\right)^{1+\lambda}} = \frac{\Gamma(-\lambda)^2 \Gamma(1+3\lambda)}{\Gamma(1+\lambda)}$$

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$(0,1,1)$; $x_2 \to \rho x_2, t \to \rho t$

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 (x_2 + t)(x_1 + 1))^{1+\lambda}} = \frac{\Gamma(-\lambda)\Gamma(1+2\lambda)}{\lambda \Gamma(1+\lambda)} t^{-\lambda}$$

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$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\left((x_1 + t)(x_2 + t)\right)^{1+\lambda}} = \frac{1}{\lambda^2} t^{-2\lambda}$$

Summing up the four LO contributions $\to$

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$$F(t) = -\frac{1}{1-2t}\left(2\mathrm{Li}_2(t) + 2\ln t \ln(1-t) - \ln^2 t - \frac{\pi^2}{6}\right)$$

Back to Feynman integrals:

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)[(q-k)^2 - m^2]}$$

in the large-momentum limit so that the corresponding scaling is $m^2 \to \rho m^2$.

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We have $\mathcal{U} = x_1 + x_2$,

$\mathcal{W} = -\mathcal{V} + m^2(x_1 + x_2)\mathcal{U} \to \rho m^2 x_1^2 + \rho m^2 x_2^2 + (2\rho m^2 - q^2)x_1 x_2$

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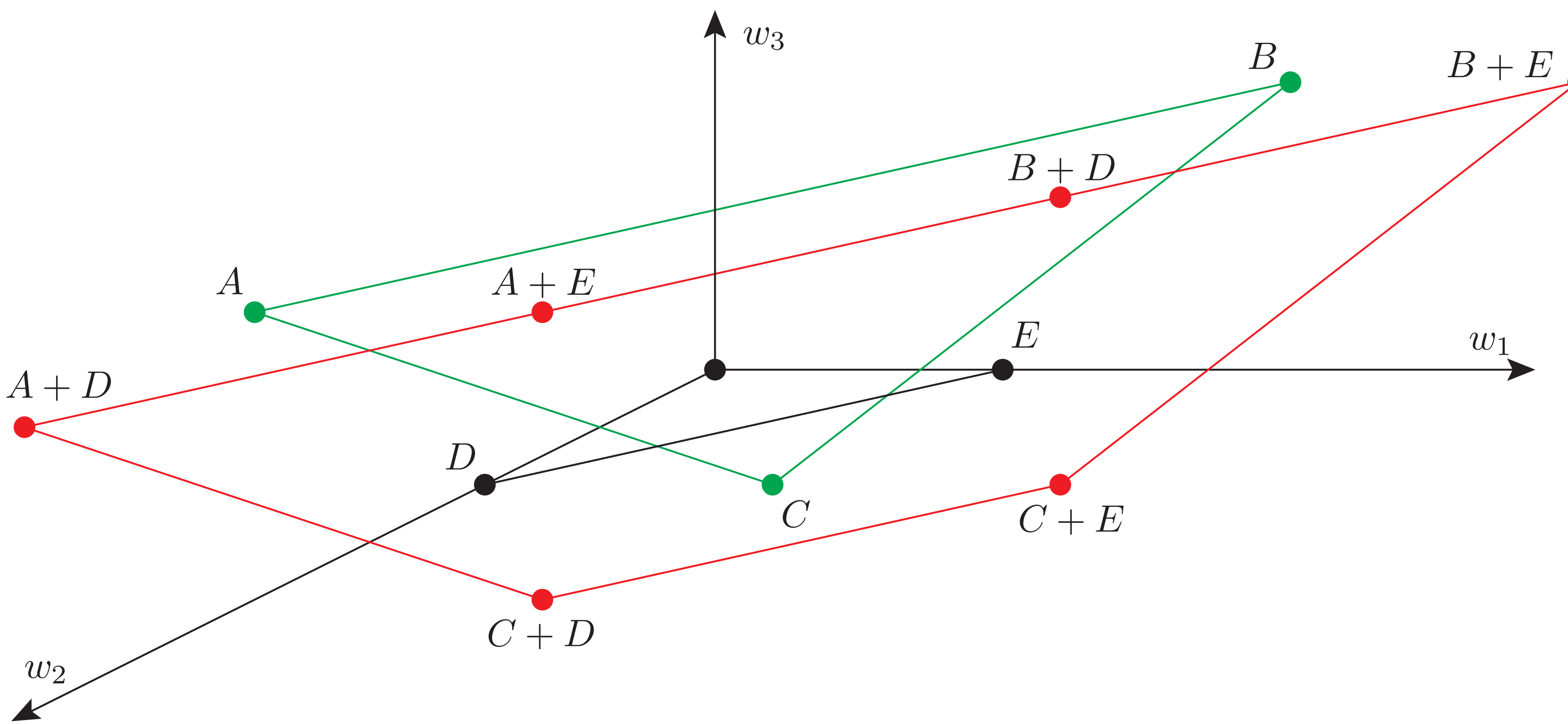
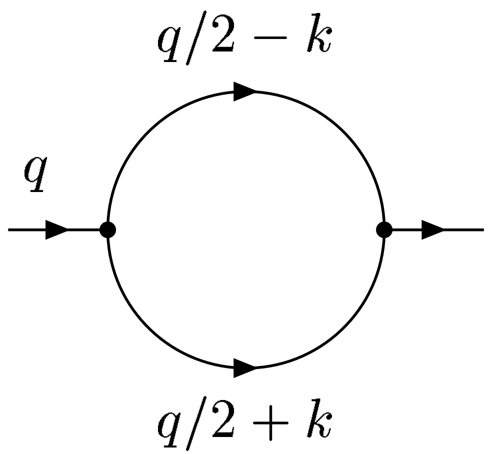
The weights of the polynomials:



Revealing potential and Glauber regions



$$F(q^2, y; d) = \int \frac{\mathrm{d}^d k}{(k^2 + q\cdot k - y)(k^2 - q\cdot k - y)} \;,$$

in the threshold limit $y = m^2 - q^2/4 \to 0$

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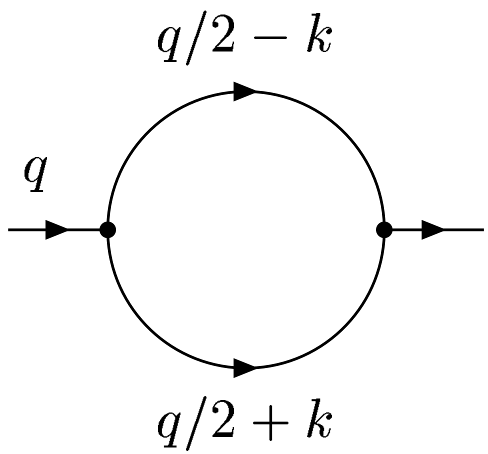

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in the threshold limit $y = m^2 - q^2/4 \to 0$

`asy.m` reported only about the hard region $k \sim q$.

$$F(q^2,y) = \mathrm{i}\pi^{d/2}\,\Gamma(\epsilon)$$
$$\times \int_0^\infty \int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\epsilon-2}\;\delta\left(\alpha_1+\alpha_2-1\right)\;\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}(\alpha_1-\alpha_2)^2+y(\alpha_1+\alpha_2)^2-\mathrm{i}0\right]^\epsilon}\;,$$

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The domain where $\alpha_1 \approx \alpha_2$ causes problems. Decompose the integration domain into $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, with equal contributions. Turn to new variables by $\alpha_1 = \alpha_1'/2,\ \alpha_2 = \alpha_2' + \alpha_1'/2$,

$$\mathrm{i}\pi^{d/2}\,\frac{\Gamma(\epsilon)}{2} \int_0^\infty \int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\epsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\; \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2 + y(\alpha_1+\alpha_2)^2 - \mathrm{i}0\right]^\epsilon}\;.$$

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`asy2.m`

[Jantzen, A. Smirnov & VS'2012]

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It checks all pairs of variables (say, x and y) which are part of monomials with opposite sign. For all those pairs the code tries to build a linear combination z of x and y such that in the variables x and z or y and z this monomial disappears.

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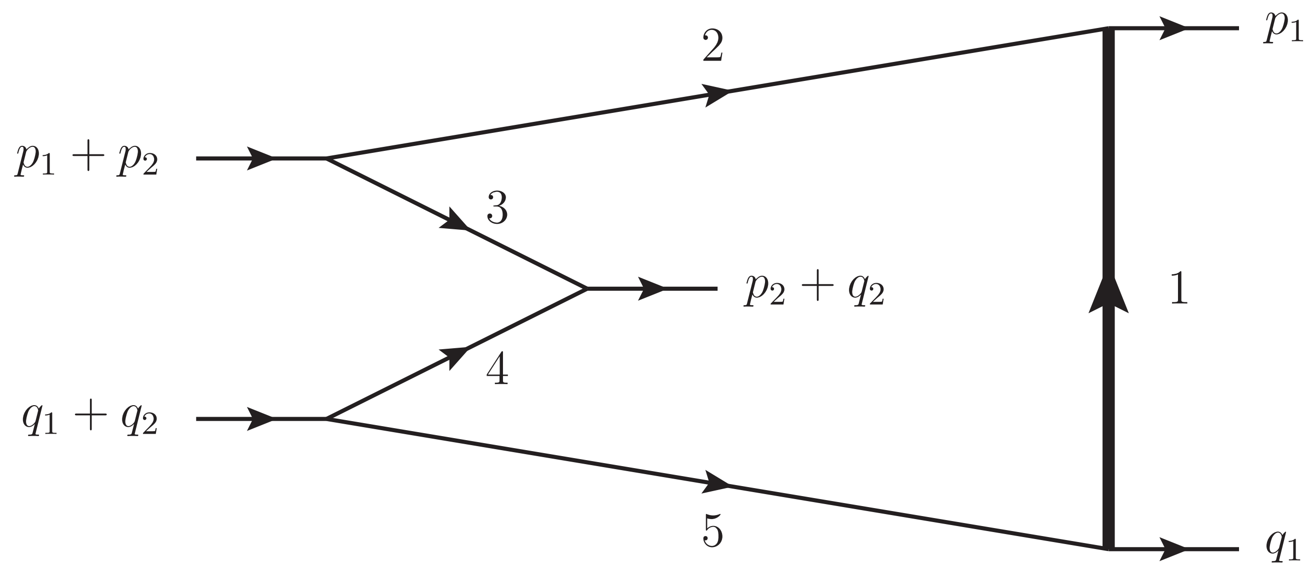
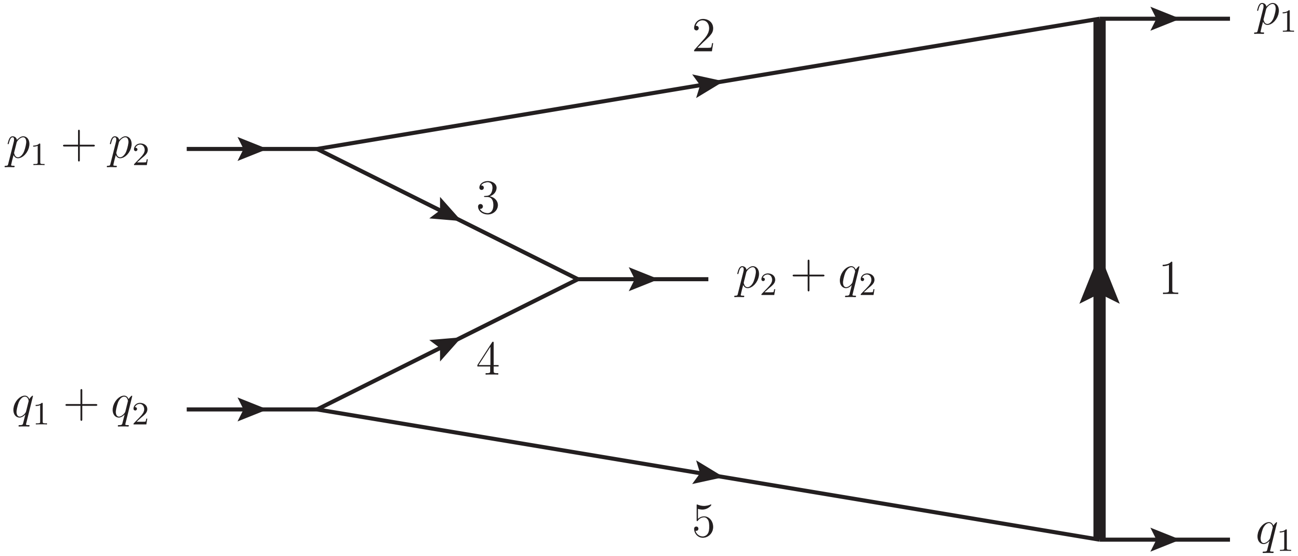
The code checks whether in the new variables the number of monomials with opposite sign decreases.

For all such pairs the code recursively repeats the

initial procedure in the new variables.



in the simplified kinematics $p_1 = p_2 = p$ and $q_1 = q_2 = q$ with $p^2 = q^2 = 0$ and $(p+q)^2 = 2p \cdot q = Q^2$ in the limit $m^2/Q^2 \to 0$:



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$$F(Q^2, m^2) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)(k^2 - 2p \cdot k)(k^2 + 2p \cdot k)} \times \frac{1}{(k^2 - 2q \cdot k)(k^2 + 2q \cdot k)} \, .$$

[Jantzen'2011]

- a hard region where $k \sim Q$,
- a 1-collinear region where $k^2 \sim p \cdot k \sim m^2$ and $q \cdot k \sim Q^2$,
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The Glauber region: LO $(m^2)^{-2-\epsilon}$
The collinear contributions: $(m^2)^{-1-\epsilon}$

The hard contribution: $(m^2)^0$

$$F(Q^2,m^2) = -\mathrm{i}\pi^{d/2}\,\Gamma(3+\epsilon)\int_0^\infty \ldots \int_0^\infty \mathsf{d}\alpha_1 \cdots \mathsf{d}\alpha_5$$

$$\times\ \frac{\delta\left(\sum_i \alpha_i - 1\right)\ (\alpha_1+\ldots+\alpha_5)^{1+2\epsilon}}{\left[\alpha_1(\alpha_1+\ldots+\alpha_5)m^2 + (\alpha_2-\alpha_3)(\alpha_4-\alpha_5)Q^2 - \mathrm{i}0\right]^{3+\epsilon}}\ .$$

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the domain $(\alpha_2-\alpha_3)\sim(m^2)^1$ or $(\alpha_4-\alpha_5)\sim(m^2)^1$ is responsible for the Glauber contribution

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decompose the integral into four parts corresponding to the domains where $(\alpha_2-\alpha_3)$ and $(\alpha_4-\alpha_5)$ are either positive or negative and then introduce new variables in such a way that this product takes the form $\pm\alpha_2'\alpha_4'$.

For example, in the domain $\alpha_2 \leq \alpha_3$, $\alpha_5 \leq \alpha_4$ change the variables by $\alpha_2 = \alpha_3'/2,\ \alpha_3 = \alpha_2' + \alpha_3'/2$ and by $\alpha_4 = \alpha_4' + \alpha_5'/2,\ \alpha_5 = \alpha_5'/2$

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$$F(Q^2, m^2) = 2(I_+ + I_-),$$

$$I_\pm = -\mathrm{i}\pi^{d/2} \frac{\Gamma(3+\epsilon)}{4} \int_0^\infty \ldots \int_0^\infty \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_5$$
$$\times \frac{\delta(\alpha_1 - 1)\ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)^{1+2\epsilon}}{[\alpha_1(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)m^2 \pm \alpha_2\alpha_4 Q^2 - \mathrm{i}0]^{3+\epsilon}},$$

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This agrees with the leading contribution of the Glauber region in the momentum-space expansion.
When revealing Glauber regions the preresolution algorithm of `asy2.m` tries to eliminate monomials with opposite sign by automatically separating the integration into domains

and performing changes of variables.

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  For example, to parametrical integrals for Wilson loops
  [Del Duca, Duhr & VS'11]
- The mathematical status is unclear.
  To prove expansion by regions is an interesting mathematical problem.