

Computing methods for multiloop Feynman integrals

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lists of misprints

<http://theory.sinp.msu.ru/~smirnov>

- Feynman integrals: basic notation, definitions and properties. Dimensional regularization.

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- Symbols: attend Claude's lectures!

Perturbation theory. Feynman rules. A graph $\Gamma = \{\mathcal{V}, \mathcal{L}, \pi_{\pm}\}$ with vertices and lines (edges), where \mathcal{V} is the set of vertices, \mathcal{L} is the set of lines, and $\pi_{\pm} : \mathcal{L} \rightarrow \mathcal{V}$.

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$$F_{\Gamma}(a_1, a_2, \dots) = \int \dots \int \frac{\mathbf{d}^d k_1 \mathbf{d}^d k_2 \dots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \dots}$$

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Dimensional regularization: $d = 4 - 2\epsilon$; $d^4 k \rightarrow d^d k$
 $k = (k_0, \vec{k}) = (k_0, k_1, k_2, k_3)$
 k_1, k_2, \dots are loop momenta;
 p_1, p_2, \dots are momenta of the lines; they are linear combinations of k_1, k_2, \dots and external momenta q_1, q_2, \dots

The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0} = \lim_{\delta \rightarrow 0} \frac{1}{k^2 - m^2 + i\delta} ,$$
$$k^2 = k_0^2 - \vec{k}^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$$

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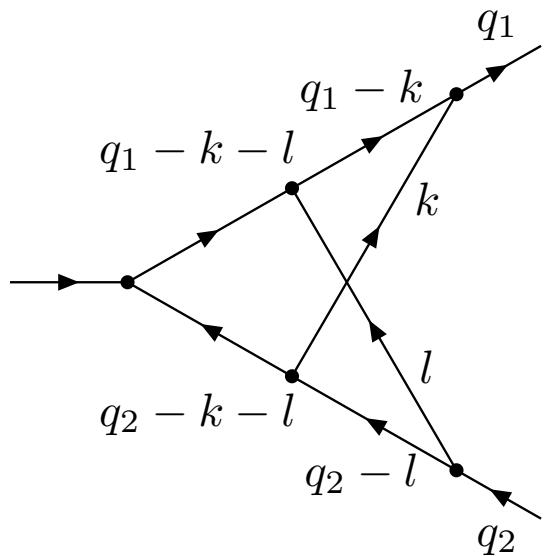
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HQET, NRQCD, ... → other types of propagators, e.g.

$$\frac{1}{v \cdot k \pm i0} , \quad v = (1, \vec{0})$$

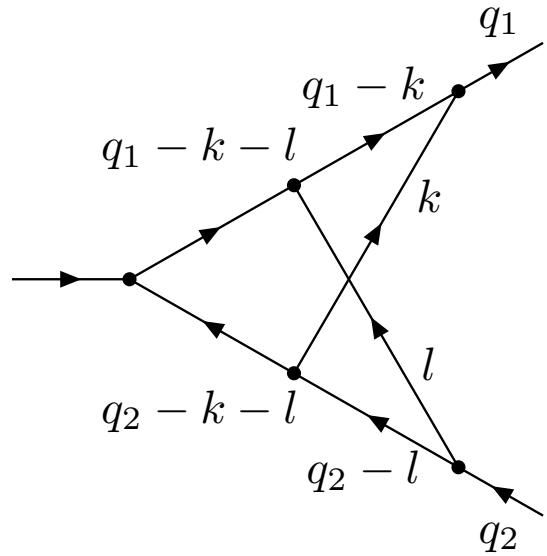
For example,

$$q_1^2 = q_2^2 = 0, Q^2 = -(q_1 - q_2)^2 = 2q_1 \cdot q_2$$



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$$\begin{aligned} F_\Gamma(Q^2; a_1, \dots, a_6, a_7, d) &= \int \int \frac{\mathbf{d}^d k \mathbf{d}^d l}{[-(k + l)^2 + 2q_1 \cdot (k + l)]^{a_1}} \\ &\times \frac{(2k \cdot l)^{-a_7}}{[-(k + l)^2 + 2q_2 \cdot (k + l)]^{a_2} (-k^2 + 2q_1 \cdot k)^{a_3} (-l^2 + 2q_2 \cdot l)^{a_4} (-k^2)^{a_5} (-l^2)^{a_6}} \end{aligned}$$

UV, IR and **collinear** divergences

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$$F_\Gamma = \int \frac{d^d k}{k^2 (k + p_1)^2 (k + p_2)^2}$$

at $p_1^2 = p_2^2 = 0$

Divergences → regularization

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Analytical regularization

[E. Speer'68]

$$\frac{1}{(-k^2 + m^2 - i0)^a} \rightarrow \frac{1}{(-k^2 + m^2 - i0)^{a+\lambda}}$$

Dimensional regularization

[G. 't Hooft & M. Veltman'72]

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$$\mathbf{d}^4 k = \mathbf{d}k_0 \mathbf{d}\vec{k} \rightarrow \mathbf{d}^d k, \quad d = 4 - 2\epsilon$$

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Informally, use alpha parameters

$$\frac{1}{(-k^2 + m^2 - i0)^a} = \frac{e^{i\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} e^{i(k^2 - m^2)\alpha} d\alpha$$
$$\frac{1}{(-v \cdot k - i0)^a} = \frac{e^{i\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} e^{i(v \cdot k)\alpha} d\alpha$$

Dimensional regularization:

when deriving alpha representations, apply this rule with
 $d = 4 - 2\epsilon$

$$\int d^4 k e^{i(\alpha k^2 - 2q \cdot k)} = -i\pi^2 \alpha^{-2} e^{-iq^2/\alpha}$$

→

$$\int d^d k e^{i(\alpha k^2 - 2q \cdot k)} = e^{i\pi(1-d/2)/2} \pi^{d/2} \alpha^{-d/2} e^{-iq^2/\alpha}$$

Graph $\Gamma \rightarrow$ dimensionally regularized Feynman integral

$$F_\Gamma(a_1 \dots, a_L; d) = \frac{e^{i\pi(a+h(1-d/2))/2} \pi^{h d/2}}{\prod_l \Gamma(a_l)} \\ \times \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{i\mathcal{V}/\mathcal{U} - i \sum m_l^2 \alpha_l},$$

where $a = \sum a_i$

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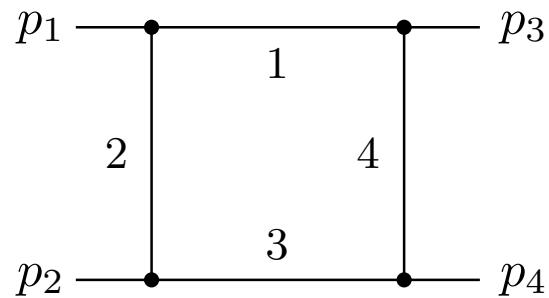
For a Feynman integral with $1/(m^2 - k^2 - i0)^{a_l}$ propagators,

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l,$$

$$\mathcal{V} = \sum_{2-\text{trees } T} \prod_{l \notin T} \alpha_l (q^T)^2.$$

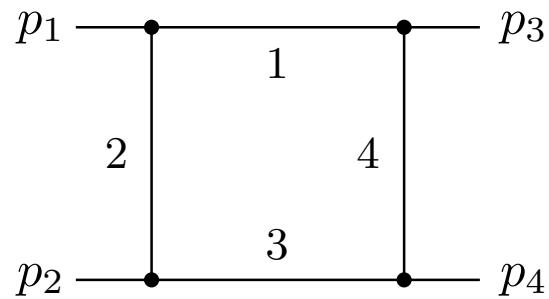
The massless box,

$$p_i^2 = 0, \quad i = 1, 2, 3, 4, \quad s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2$$



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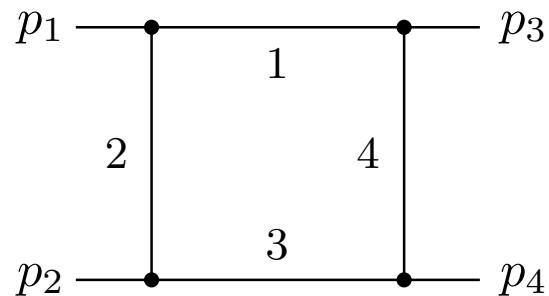


trees



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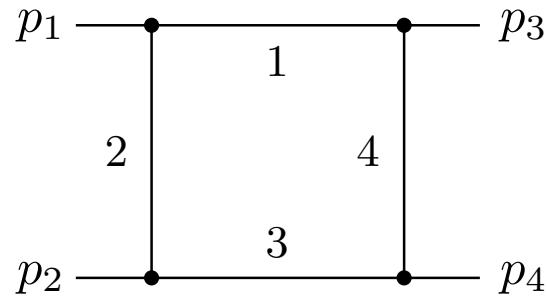


2-trees



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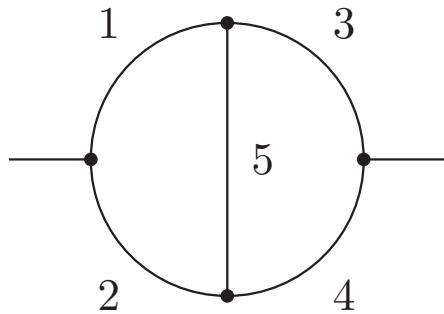
trees



2-trees



$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \mathcal{V} = s\alpha_1\alpha_3 + t\alpha_2\alpha_4.$$



trees



2-trees



$$\mathcal{U} = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\alpha_5 + (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) ,$$

$$\mathcal{V} = [(\alpha_1 + \alpha_2)\alpha_3\alpha_4 + \alpha_1\alpha_2(\alpha_3 + \alpha_4) + (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)\alpha_5]q^2$$

The code UF.m to evaluate \mathcal{U} and \mathcal{V}

<http://science.sander.su>

Alpha representation →

- Mathematical proofs (for Feynman integrals at Euclidean external momenta, $(\sum q_i)^2 < 0$)
Analysis of convergence.

[K. Hepp'66; P. Breitenlohner & D. Maison'77; E. Speer'68,'77]

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Speer's sectors

[E. Speer'77, A. Smirnov& VS'2009]

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 & \quad \times \int \mathbf{d}^d k e^{i[\alpha_1 k^2 + \alpha_2 (k^2 + 2q \cdot k + q^2)]} \\
 &= \frac{e^{i\pi(a_1+a_2+1-d/2)/2} \pi^{d/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \frac{\alpha_1^{a_1-1} \alpha_2^{a_2-1}}{(\alpha_1 + \alpha_2)^{d/2}} e^{i\alpha_1 \alpha_2 q^2 / (\alpha_1 + \alpha_2)}
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 \end{aligned}$$

$\alpha_1 = \eta\xi$, $\alpha_2 = \eta(1 - \xi)$, with the Jacobian η , integrate over η and ξ

$$\int \frac{\mathbf{d}^d k}{(-k^2)^{a_1} [-(q-k)^2]^{a_2}} = i\pi^{d/2} \frac{\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)} \frac{\Gamma(a_1+a_2+\epsilon-2)}{(-q^2)^{a_1+a_2+\epsilon-2}} .$$

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$$\int \frac{\mathbf{d}^d k}{(-k^2)^{a_1}(2v \cdot k + \omega - i0)^{a_2}} = i\pi^{d/2} \frac{\Gamma(2-a_1-\epsilon)\Gamma(2a_1+a_2+2\epsilon-4)}{\Gamma(a_1)\Gamma(a_2)\omega^{2a_1+a_2+2\epsilon-4}} (v^2)^{a_1+\epsilon-2} .$$

$$= i\pi^{d/2} G(\lambda_1, \lambda_2) \times \frac{\lambda_1 + \lambda_2 - d/2}{\bullet \text{---} \bullet}$$

A diagram showing a circle with two points on its boundary. The top point is labeled λ_2 and the bottom point is labeled λ_1 . Two horizontal lines extend from these points towards each other, meeting at a central point.

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$$= i\pi^{d/2} \bar{G}(\lambda_1, \lambda_2) (v^2)^{\lambda_1 - d/2} \times \frac{2\lambda_1 + \lambda_2 - d}{\bullet \text{---} \bullet}$$

$\alpha = \eta\alpha'_l$, $l = 1, 2, \dots, L - 1$, $\eta = \sum_{l=1}^L \alpha_l$, integrate over η , introduce $\alpha'_L = 1 - \sum_{l=1}^{L-1} \alpha'_l$ by inserting an integration over α'_L with $\delta\left(\sum_{l=1}^L \alpha_l - 1\right)$, replace α'_l by α_l :

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Cheng–Wu theorem:

$$\delta\left(\sum_{l=1}^L \alpha_l - 1\right) \rightarrow \delta\left(\sum_{l \in \nu} \alpha_l - 1\right) \rightarrow \delta(\alpha_l - 1)$$

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Proof. Use $\eta = \sum_{l \in \nu} \alpha_l$ instead of $\eta = \sum_{l=1}^L \alpha_l$

$$\begin{aligned}
& \int \int \frac{\mathbf{d}^d k \mathbf{d}^d l}{(-k^2 + m^2)^{\lambda_1} [-(k+l)^2]^{\lambda_2} (-l^2 + m^2)^{\lambda_3}} \\
&= \left(i\pi^{d/2} \right)^2 \frac{\Gamma(\lambda_1 + \lambda_2 + \epsilon - 2)\Gamma(\lambda_2 + \lambda_3 + \epsilon - 2)\Gamma(2 - \epsilon - \lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_3)} \\
&\times \frac{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + 2\epsilon - 4)}{\Gamma(\lambda_1 + 2\lambda_2 + \lambda_3 + 2\epsilon - 4)\Gamma(2 - \epsilon)(m^2)^{\lambda_1 + \lambda_2 + \lambda_3 + 2\epsilon - 4}}
\end{aligned}$$

choose $\delta(\alpha_1 + \alpha_3 - 1)$

Feynman parameters:

$$\frac{1}{(m_1^2 - p_1^2)^{a_1} (m_2^2 - p_2^2)^{a_2}} = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \frac{d\xi \xi^{a_1-1} (1-\xi)^{a_2-1}}{[(m_1^2 - p_1^2)\xi + (m_2^2 - p_2^2)(1-\xi)]^{a_1+a_2}}$$

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$$\frac{1}{\prod A_l^{a_l}} = \frac{\Gamma(\sum a_l)}{\prod \Gamma(a_l)} \int_0^1 d\xi_1 \dots \int_0^1 d\xi_L \prod_l \xi_l^{a_l-1} \frac{\delta(\sum \xi_l - 1)}{(\sum A_l \xi_l)^{\sum a_l}}$$

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Then one can expand the integrand in parametric
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[F. Brown'08]

*Perform integrations in a given alpha-parametric integral
one by one, in some order.*

Presumably, results are expressed in terms of multiple
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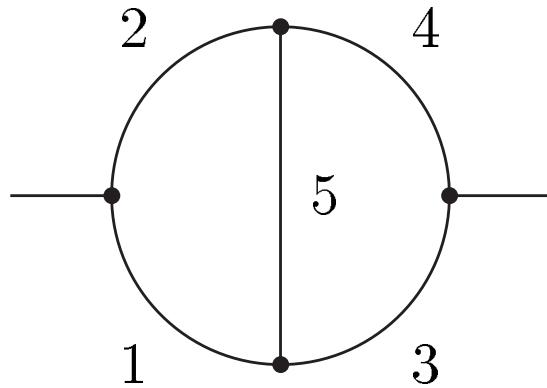
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$$F(q^2; 1, 1, 1, 1, 1; 4) = \frac{(\mathrm{i}\pi^2)^2}{q^2} \int_0^\infty \mathrm{d}\alpha_1 \dots \int_0^\infty \mathrm{d}\alpha_5 \frac{\delta(\sum \alpha_l - 1)}{\mathcal{U}\bar{\mathcal{V}}} .$$

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Apply the Cheng–Wu theorem by choosing the delta function $\delta(\alpha_5 - 1)$, with the integration over the rest of the four variables from zero to infinity and integrate in Mathematica

$$F(q^2; 1, 1, 1, 1, 1; 4) = \frac{(\mathrm{i}\pi^2)^2}{q^2} 6\zeta(3)$$

- A tool to evaluate Feynman integrals numerically.
Modern sector decompositions

[T. Binoth & G. Heinrich'00; C. Bogner & S. Weinzierl'07; A.V. Smirnov & M.N. Tentyukov'08;
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Public computer codes:

SecDec, sector_decomposition, FIESTA

The factorization can be always achieved in Hepp sectors:

$$\mathcal{U} = \prod_l t_l^{h(\gamma_l)} [1 + P_f]$$

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For example, $\mathcal{U} = \alpha_1 + \alpha_2$.

Two sectors: $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$.

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Hepp and Speer sectors are applicable only at external Euclidean momenta, i.e. when $(\sum_{i \in \nu} p_i)^2 < 0$ for any nonempty subset ν . The second function of alpha parameters is not generally proper factorized.

**Example: the massless on-shell box in the sector
 $\alpha_2 \leq \alpha_1 \leq \alpha_3 \leq \alpha_4 = 1$, with**

$$\mathcal{U} = 1 + \alpha_1 + \alpha_2 + \alpha_3, \quad \mathcal{V} = s\alpha_1\alpha_3 + t\alpha_2.$$

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In the sector variables $\alpha_2 = t_1t_2t_3$, $\alpha_1 = t_2t_3$, $\alpha_3 = t_3$,
we have

$$\mathcal{V} = t_1t_3(st_3 + t t_1)$$

so that a further sector decomposition is desirable.

Recursively sector decompositions

[T. Binoth & G. Heinrich'00]

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Primary sectors Δ_l

$$\alpha_i \leq \alpha_l , \quad l \neq i = 1, 2, \dots, L ,$$

with new variables

$$\alpha'_i = \begin{cases} \alpha_i / \alpha_l & \text{if } i \neq l \\ \alpha_l & \text{if } i = l \end{cases}$$

The contribution of a primary sector for Δ_L :

$$F^{(L)} = (-1)^L \frac{(\mathrm{i}\pi^{d/2})^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \int_0^1 \dots \int_0^1 \prod_l \alpha_l^{a_l - 1} \\ \times \hat{\mathcal{U}}_{\Gamma}^{a - (h+1)d/2} \mathcal{W}^{hd/2 - a} \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_{L-1},$$

where

$$\mathcal{W}_{\Gamma} = -\hat{\mathcal{V}}_{\Gamma} + \hat{\mathcal{U}}_{\Gamma} \left(\sum_{l=1}^{L-1} m_l^2 \prod_{l'=l'}^{L-1} \alpha_{l'} + m_L^2 \right),$$

$$\hat{\mathcal{U}}_{\Gamma} = \mathcal{U}(\alpha_1, \dots, \alpha_{L-1}, 1),$$

$$\hat{\mathcal{V}}_{\Gamma} = \mathcal{V}_{\Gamma}(\alpha_1, \dots, \alpha_{L-1}, 1)$$

Let us choose a subset $I = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, with $n \equiv L - 1$.

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The unit hypercube $\{(t_1, \dots, t_n) | 0 \leq t_i \leq 1 \forall i \in (1, \dots, n)\}$ is then decomposed into k sectors

$$S_l = \{(t_1, \dots, t_n) | t_i \leq t_{i_l} \forall i \in I\},$$

for $l = 1, \dots, k$, and the new (sector) variables are introduced as follows:

$$t_i = t'_i \quad \forall i \notin I$$

$$t_{i_l} = t'_{i_l}$$

$$t_{i_r} = t'_{i_l} t'_{i_r} \quad \forall i_r \in I, r \neq l$$

The integration region in the new variables t'_i is again a unit hypercube.

Then for each of the k resulting sectors subsets of the indices are chosen and new sectors are introduced in a similar way.

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This process is terminated when the contribution of each of the final sectors takes a proper factorized form

$$\int_0^1 \dots \int_0^1 f(t_1, \dots, t_n; \epsilon) \prod_{i=1}^n t_i^{a_i + b_i \epsilon} dt_i$$

with a function f which is regular near the origin.

To make the poles in ϵ manifest, the integrations over the final sector variables t_i are analyzed one by one.

$$G(\epsilon) = \int_0^1 t^{a+b\epsilon} g(t, \epsilon) dt$$

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If $a < 0$, subtract first terms of the Taylor series of $g(t)$ in t at the origin up to order $-1 - a$ and obtain, after explicitly integrating, the subtracted terms,

$$G = \sum_{k=0}^{-1-a} \frac{g^{(k)}(0, \epsilon)}{k!(a+k+b\epsilon+1)} + \int_0^1 t^{a+b\epsilon} \left[g(t) - \sum_{k=0}^{-1-a} \frac{g^{(k)}(0, \epsilon)}{k!} t^k \right] dt$$

A first implementation [T. Binoth & G. Heinrich'00]

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Strategy S

[A.V. Smirnov & M.N. Tentyukov'08]

FESTA

(Feynman Integral Evaluation by a Sector decompositiOn Approach)

<http://science.sander.su>

Fiesta 2

[A.V. Smirnov, VS & M.N. Tentyukov'09]

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data bases (Kyoto cabinet)

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integration in c++ (cuba library)

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Speer sectors are implemented.
They are reproduced within Strategy S.

[A.V. Smirnov & VS'09]

SecDec

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An important new feature of SecDec 2.1 is the possibility to apply it at physical values of kinematic invariants, i.e. where the second function of alpha parameters has terms of different sign.

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Based on a contour deformation in parametric integrals

[D. Soper'99, C. Anastasiou, S. Beerli & A. Daleo'07]

Fiesta 3

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mpi parallelization

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A geometrical strategy based on computational geometry

[T. Kaneko & T. Ueda'10]

is implemented.