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- Computer codes `MB.m` and `MBresolve.m`

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- Simple one-loop examples
- General prescriptions for resolving singularities in ϵ in multiple Mellin-Barnes integrals. Two strategies
- Computer codes `MB.m` and `MBresolve.m`
- Various examples

Mellin transformation, Mellin integrals as a tool for Feynman integrals:

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Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ)
[V.A. Smirnov'99, J.B. Tausk'99]

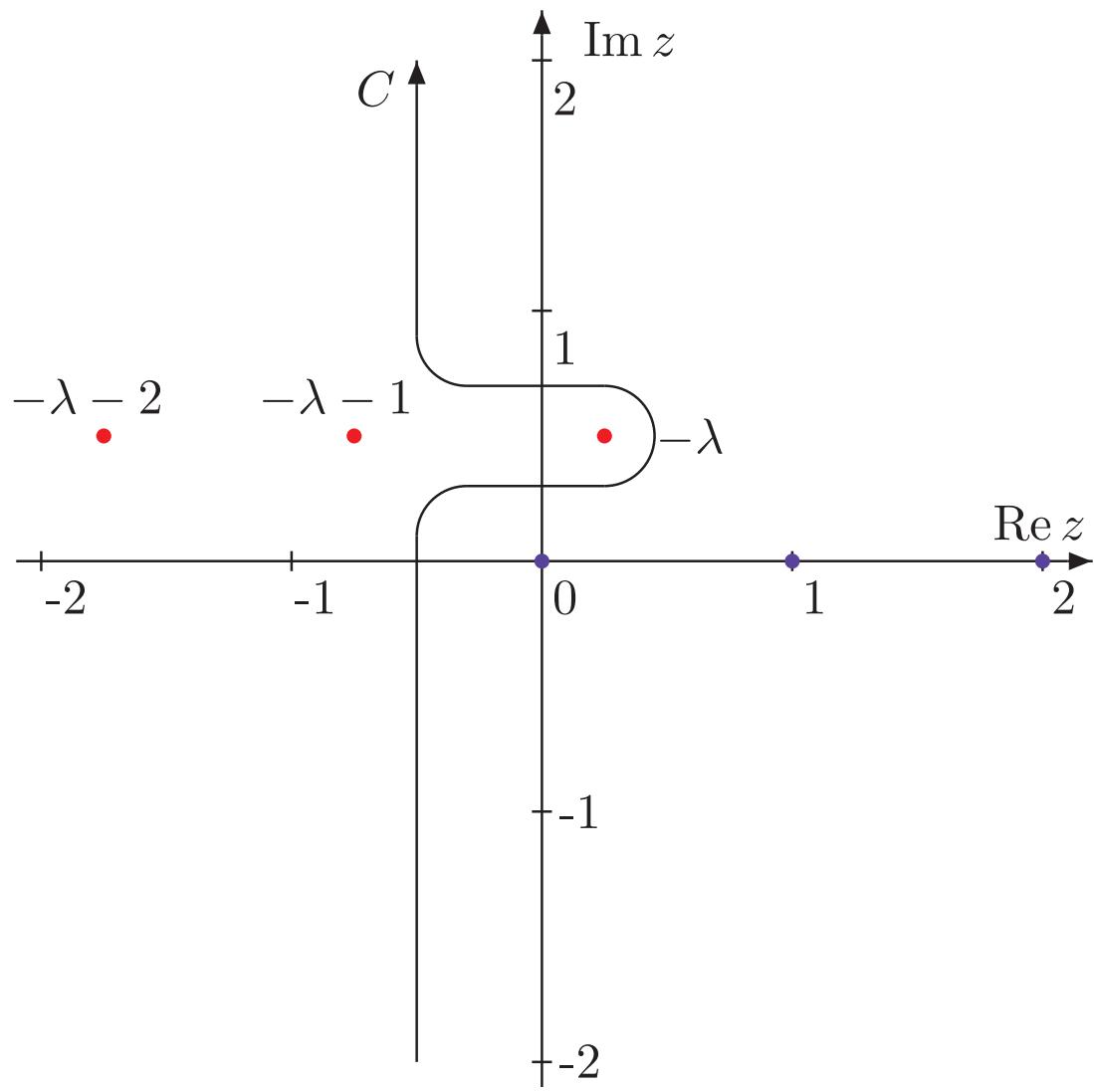
The basic formula:

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z) .$$

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The poles with a $\Gamma(\dots + z)$ dependence are to the left of the contour and the poles with a $\Gamma(\dots - z)$ dependence are to the right



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- Evaluate expanded MB integrals

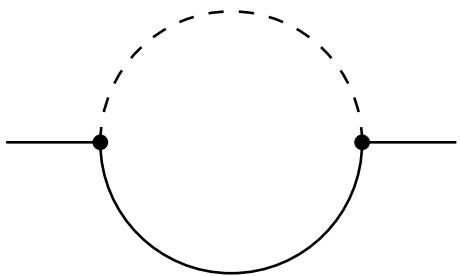
The simplest possibility:

$$\frac{1}{(m^2 - k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z)$$

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Example 1



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{d^d k}{(m^2 - k^2)^{a_1}(-(q - k)^2)^{a_2}}$$

$$\begin{aligned}
F_\Gamma = & \frac{1}{\Gamma(a_1)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz (m^2)^z \Gamma(a_1 + z) \Gamma(-z) \\
& \times \int \frac{d^d k}{(-k^2)^{a_1+z}(-(q-k)^2)^{a_2}}
\end{aligned}$$

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$$\int \frac{\mathbf{d}^d k}{(-k^2)^{a_1+z} [-(q-k)^2]^{a_2}} = i\pi^{d/2} \frac{G(a_1+z, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2+z}} \, ,$$

$$G(a_1, a_2) = \frac{\Gamma(a_1 + a_2 + \epsilon - 2)\Gamma(2 - \epsilon - a_1)\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4 - a_1 - a_2 - 2\epsilon)}$$

$$\begin{aligned}
F_\Gamma(q^2, m^2; a_1, a_2, d) &= \frac{i\pi^{d/2} \Gamma(2 - \epsilon - a_2)}{\Gamma(a_1) \Gamma(a_2) (-q^2)^{a_1 + a_2 + \epsilon - 2}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \left(\frac{m^2}{-q^2} \right)^z \Gamma(a_1 + a_2 + \epsilon - 2 + z) \\
&\times \frac{\Gamma(2 - \epsilon - a_1 - z) \Gamma(-z)}{\Gamma(4 - 2\epsilon - a_1 - a_2 - z)}
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&\times \frac{\Gamma(2 - \epsilon - a_1 - z)\Gamma(-z)}{\Gamma(4 - 2\epsilon - a_1 - a_2 - z)}
\end{aligned}$$

Unambiguous prescriptions for contours:

the poles with a $\Gamma(\dots + z)$ dependence are to the left and
 the poles with a $\Gamma(\dots - z)$ dependence are to the right of a
 contour

Strategy A

[V.A. Smirnov'99]

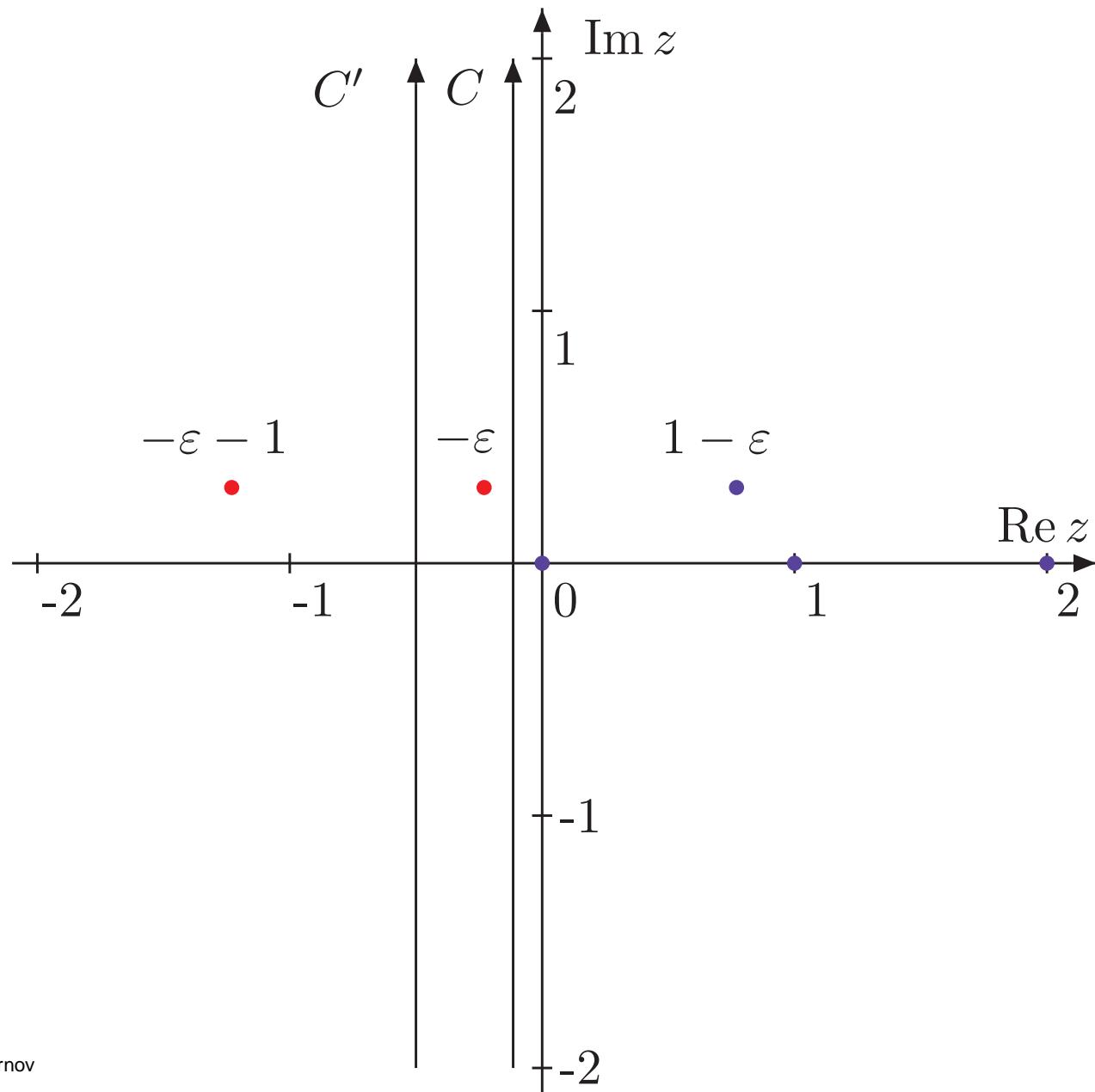
$$\begin{aligned} F_\Gamma(q^2, m^2; 1, 1, d) &= \frac{i\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \\ &\times \frac{1}{2\pi i} \int_C dz \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)} \end{aligned}$$

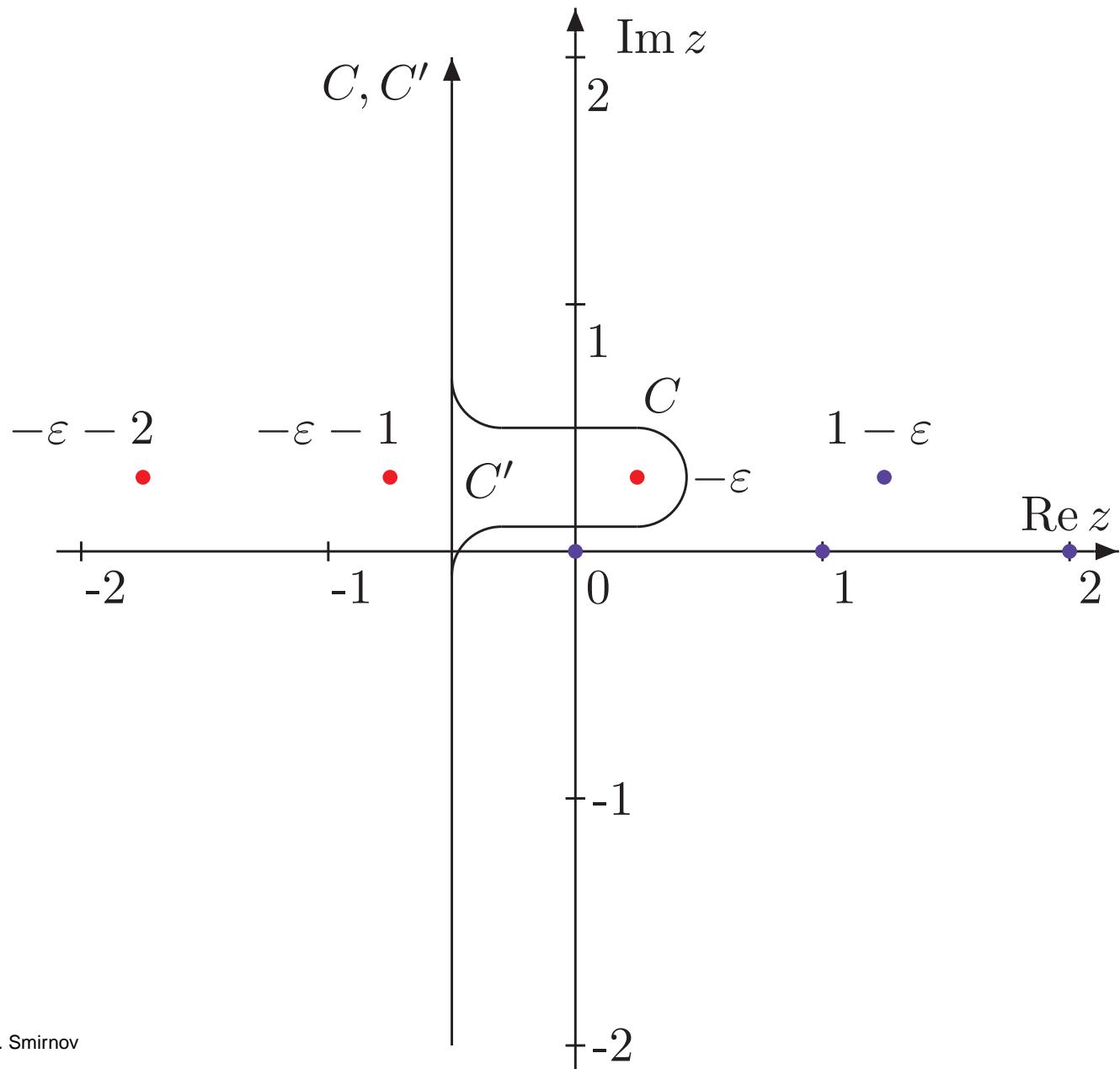
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$\Gamma(\epsilon+z)\Gamma(-z) \rightarrow$ a singularity in ϵ





Take a residue at $z = -\epsilon$:

$$i\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon(1-\epsilon)}$$

and shift the contour:

$$\frac{i\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_{C'} dz \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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NB:

$$\Gamma(\epsilon + z)\Gamma(1 - \epsilon - z) = -\Gamma(1 + \epsilon + z)\Gamma(-\epsilon - z)$$

The integral can be expanded in ϵ , e.g., the value at $\epsilon = 0$ is

$$\begin{aligned}\frac{1}{2\pi i} \int_{C'} f(z) dz &= - \sum_{n=0} \operatorname{res}_{z=n} f(z) \\ &= + \sum_{n=1} \operatorname{res}_{z=n} f(z) \\ &= 1 - \left(1 - \frac{m^2}{q^2}\right) \ln \left(1 - \frac{q^2}{m^2}\right)\end{aligned}$$

where

$$f(z) = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)\Gamma(1-z)}{\Gamma(2-z)} = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{(1-z)}$$

Strategy A in a modified form

[A.V. Smirnov and V.A. Smirnov'09]

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Choose a straight contour C_0 for which the gamma functions in the numerator of the integrand are spoiled at $\epsilon = 0$ in a minimal way, i.e. the initial rules for choosing a contour are changed in a minimal way

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Let us choose C_0 with $\operatorname{Re} z = -1/4$.

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Let us choose C_0 with $\operatorname{Re} z = -1/4$.

Then $\Gamma(\epsilon + z)$ which transforms into $\Gamma(z)$ at $\epsilon = 0$ is spoiled.

$$\Gamma(\epsilon + z) \rightarrow \Gamma^{(1)}(\epsilon + z)$$

$\Gamma^{(1)}(\epsilon + z)$ means that the rule $\operatorname{Re}(\epsilon + z) > 0$ when crossing
the real axis is changed to $-1 < \operatorname{Re}(\epsilon + z) < 0$

We do not need to spoil it more, e.g., by

$\Gamma(\epsilon + z) \rightarrow \Gamma^{(2)}(\epsilon + z)$ with the rule

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$$f(z, \epsilon) = \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}$$

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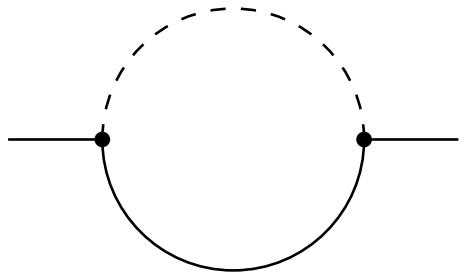
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Then

$$\begin{aligned} \frac{1}{2\pi i} \int_C f(z, \epsilon) dz &= \frac{1}{2\pi i} \int_{C_0} f(z, \epsilon) dz \\ &\quad + \left(\frac{1}{2\pi i} \int_C f(z, \epsilon) dz - \frac{1}{2\pi i} \int_{C_0} f(z, \epsilon) dz \right) \\ &= \frac{1}{2\pi i} \int_{C_0} f(z, \epsilon) dz + \operatorname{res}_{z=\epsilon} f(z, \epsilon) \end{aligned}$$

Strategy B Example 1

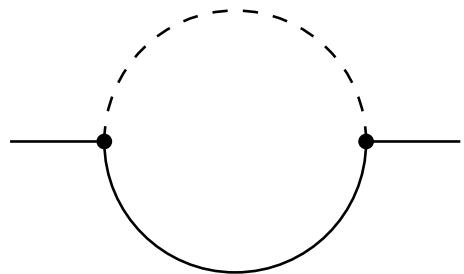
[J.B. Tausk'99, C. Anastasiou & A. Daleo'05, Czakon'05]



$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{i\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_C dz f(z, \epsilon)$$

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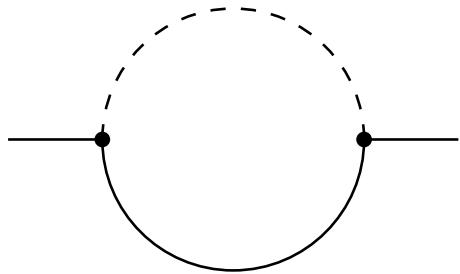


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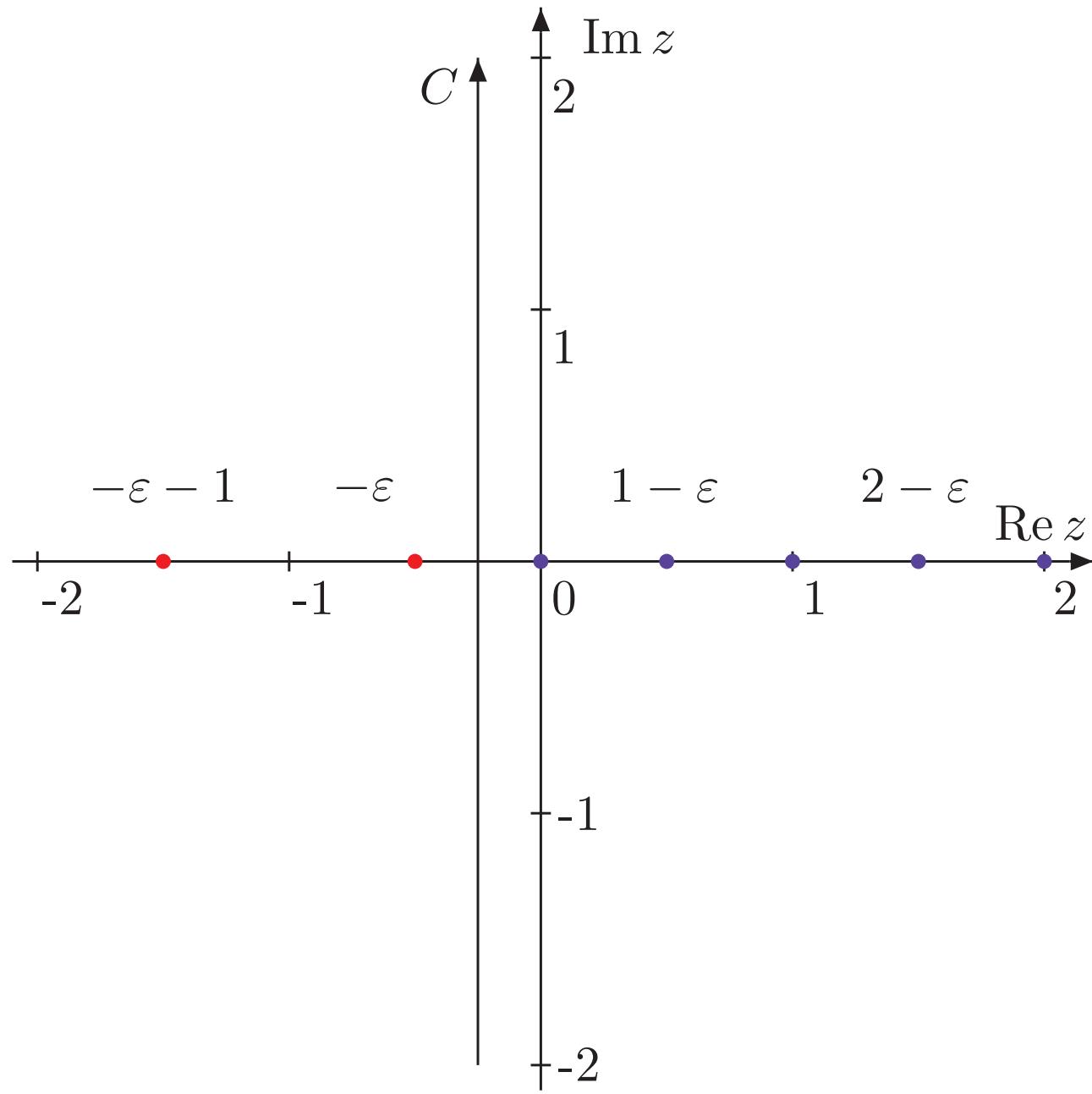
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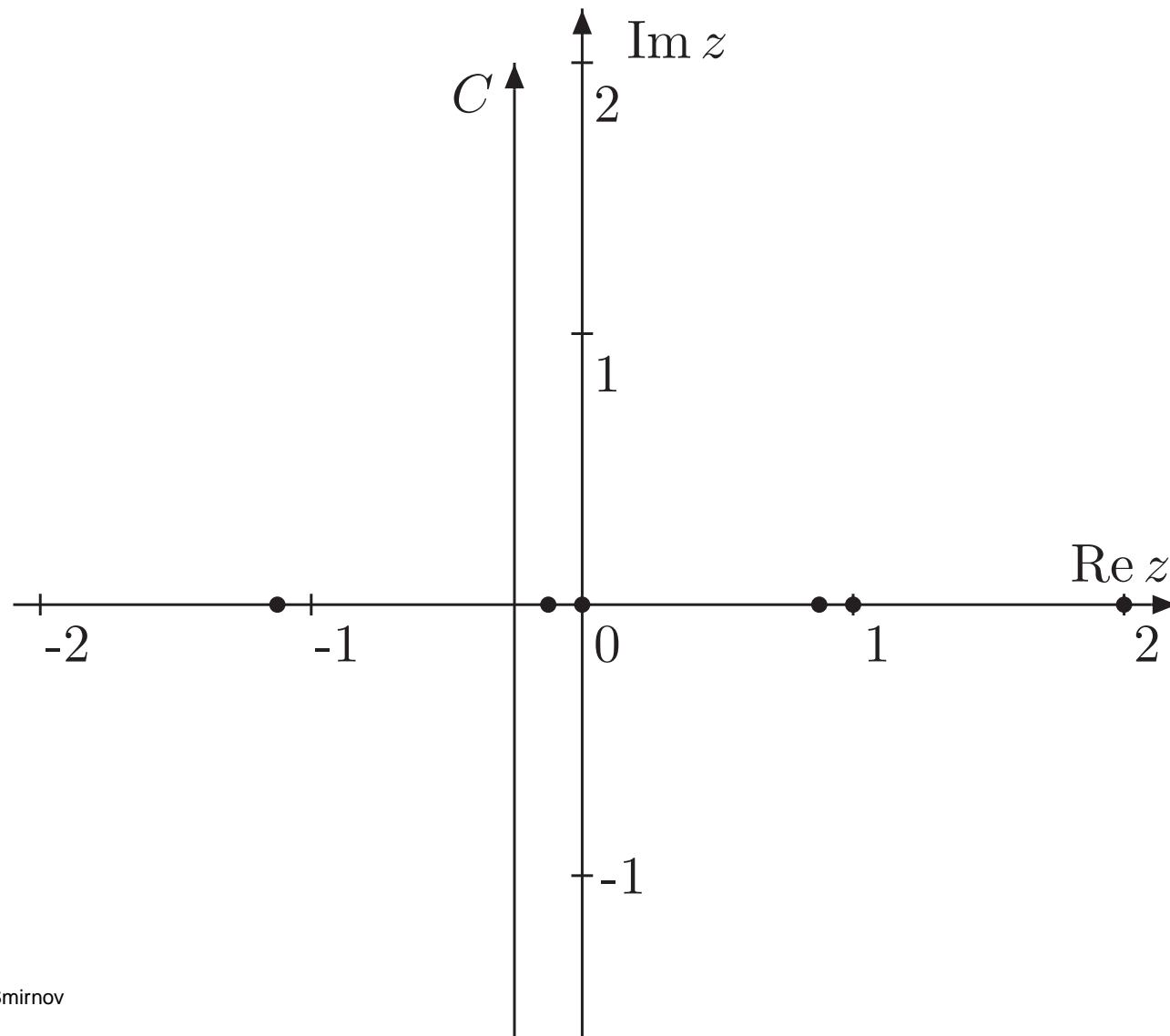
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Take ϵ real. Choose ϵ and a straight contour such that the arguments of the gamma functions are positive when crossing the real axis.

For example, take $\epsilon = 1/2$, $\text{Re } z = -1/4$. The contour is kept fixed. Tend ϵ to zero.



Whenever a pole of some gamma function is crossed add a residue and tend ϵ to zero further



$$\frac{1}{2\pi i} \int_C f(z, \epsilon) dz = \frac{1}{2\pi i} \int_{\mathbf{Re}_{z=-1/4}} f(z, \epsilon) dz + \mathbf{res}_{z=\epsilon} f(z, \epsilon)$$

General recipes for resolving the singularity structure in ϵ .

$$\frac{1}{(2\pi i)^n} \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \frac{\prod_i \Gamma(a_i + b_i\epsilon + \sum_j c_{ij}z_j)}{\prod_i \Gamma(a'_i + b'_i\epsilon + \sum_j c'_{ij}z_j)} \prod_k x_k^{d_k} \prod_{l=1}^n dz_l$$

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The goal is to represent a given MB integral as a sum of integrals where a Laurent expansion in ϵ becomes possible.

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Two strategies: Strategy A and Strategy B



Strategy B [J.B. Tausk'99, C. Anastasiou & A. Daleo'05, M. Czakon'05]

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Choose ϵ and $\text{Re}z_i$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis, i.e. the arguments of all the gamma functions are positive when crossing the real axis.

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Let $\epsilon \rightarrow 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

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Let $\epsilon \rightarrow 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

For every resulting residue, which involves one integration less, apply a similar procedure, etc.

Two algorithmic descriptions [C. Anastasiou & A. Daleo'05, M. Czakon'05]

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The Czakon's version **MB.m** implemented in Mathematica
is public.

<http://projects.hepforge.org/mbtools/>

- Strategy A in a modified form

[A.V. Smirnov & V.A. Smirnov'09]

- Strategy A in a modified form

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Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

- Strategy A in a modified form

[A.V. Smirnov & V.A. Smirnov'09]

Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

[MBresolve.m](#)

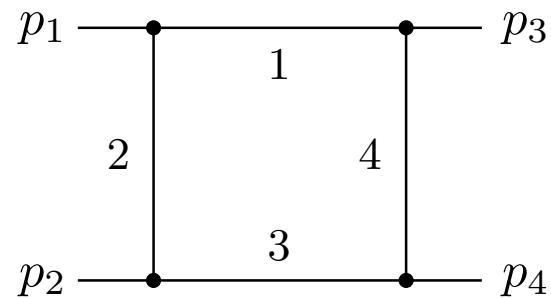
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<http://science.sander.su>

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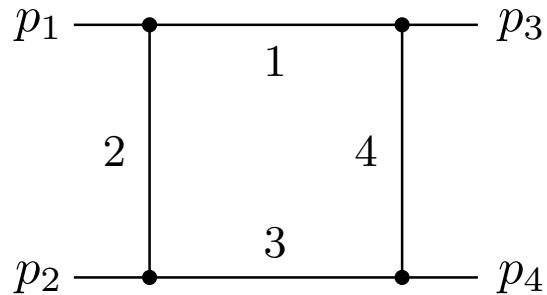
How to derive MB representations

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0$, $i = 1, 2, 3, 4$



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$$F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)$$

$$= \int \frac{d^d k}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]^{a_3} [-(k - p_3)^2]^{a_4}} ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 , \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 .$$

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$$\times \int_0^\infty \dots \int_0^\infty \frac{\delta\left(\sum_{l=1}^4 \alpha_l - 1\right)}{\left(-t\alpha_1\alpha_3 - s\alpha_2\alpha_4\right)^{a+\epsilon-2}} \prod_l \alpha_l^{a_l-1} \mathsf{d}\alpha_1 \dots \mathsf{d}\alpha_4 \ ,$$

$$a=a_1+\ldots+a_4.$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 , \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 .$$

$$F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) = i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)}{\prod \Gamma(a_l)}$$

$$\times \int_0^\infty \dots \int_0^\infty \frac{\delta \left(\sum_{l=1}^4 \alpha_l - 1 \right)}{(-t\alpha_1\alpha_3 - s\alpha_2\alpha_4)^{a+\epsilon-2}} \prod_l \alpha_l^{a_l-1} d\alpha_1 \dots d\alpha_4 ,$$

$$a = a_1 + \dots + a_4.$$

Introduce new variables by $\alpha_1 = \eta_1\xi_1$, $\alpha_2 = \eta_1(1 - \xi_1)$, $\alpha_3 = \eta_2\xi_2$, $\alpha_4 = \eta_2(1 - \xi_2)$, **with the Jacobian** $\eta_1\eta_2$

$$\begin{aligned}
& F_\Gamma(s, t; a_1, a_2, a_3, a_4, d) \\
&= i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)\Gamma(2 - \epsilon - a_1 - a_2)\Gamma(2 - \epsilon - a_3 - a_4)}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)} \\
&\times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - i0]^{a+\epsilon-2}} d\xi_1 d\xi_2
\end{aligned}$$

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& F_\Gamma(s, t; a_1, a_2, a_3, a_4, d) \\
&= i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)\Gamma(2 - \epsilon - a_1 - a_2)\Gamma(2 - \epsilon - a_3 - a_4)}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)} \\
&\times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - i0]^{a+\epsilon-2}} d\xi_1 d\xi_2
\end{aligned}$$

Apply the basic formula to separate
 $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

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Apply the basic formula to separate
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Change the order of integration over z and ξ -parameters,
evaluate parametric integrals in terms of gamma functions

$$\begin{aligned}
F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) &= \frac{i\pi^{d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)(-s)^{a+\epsilon-2}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \left(\frac{t}{s}\right)^z \Gamma(a + \epsilon - 2 + z) \Gamma(a_2 + z) \Gamma(a_4 + z) \Gamma(-z) \\
&\times \Gamma(2 - a_1 - a_2 - a_4 - \epsilon - z) \Gamma(2 - a_2 - a_3 - a_4 - \epsilon - z)
\end{aligned}$$

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F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) &= \frac{i\pi^{d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)(-s)^{a+\epsilon-2}} \\
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&\times \Gamma(2 - a_1 - a_2 - a_4 - \epsilon - z) \Gamma(2 - a_2 - a_3 - a_4 - \epsilon - z) \\
F_{\Gamma}(s, t; 1, 1, 1, 1, d) &= \frac{i\pi^{d/2}}{\Gamma(-2\epsilon)(-s)^{2+\epsilon}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \left(\frac{t}{s}\right)^z \Gamma(2 + \epsilon + z) \Gamma(1 + \textcolor{red}{z})^2 \Gamma(-1 - \epsilon - \textcolor{blue}{z})^2 \Gamma(-z)
\end{aligned}$$

The massless box diagram with two legs on shell,
 $p_3^2 = p_4^2 = 0$, and two legs off shell, $p_1^2, p_2^2 \neq 0$

$$\begin{aligned}
B_{1100} &= i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)}{\prod \Gamma(a_l)} \\
&\times \int_0^\infty \dots \int_0^\infty \left(\prod_{l=1}^4 \alpha_l^{a_l-1} d\alpha_l \right) \delta \left(\sum_{l=1}^4 \alpha_l - 1 \right) \\
&\times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - i0)^{2-a-\epsilon}
\end{aligned}$$

Apply

$$\frac{1}{(X_1 + \dots + X_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{+i\infty} \dots \int_{-i\infty}^{+i\infty} dz_2 \dots dz_n \prod_{i=2}^n X_i^{-\lambda - z_2 - \dots - z_n} \Gamma(\lambda + z_2 + \dots + z_n) \prod_{i=2}^n \Gamma(-z_i)$$

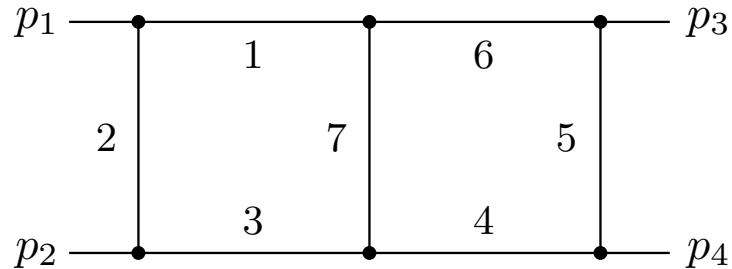
Separate terms with p_1^2 and p_2^2 , turn to new variables by

$$\alpha_1 = \eta_1 \xi_1, \quad \alpha_2 = \eta_1 (1 - \xi_1), \quad \alpha_3 = \eta_2 \xi_2, \quad \alpha_4 = \eta_2 (1 - \xi_2)$$

and evaluate integrals over parameters to obtain a three fold MB representation

$$\begin{aligned}
B_{1100} &= \frac{i\pi^{d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l) (-s)^{a + \epsilon - 2}} \\
&\times \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} dz_2 dz_3 dz_4 \frac{(-p_1^2)^{z_2} (-p_2^2)^{z_3} (-t)^{z_4}}{(-s)^{z_2 + z_3 + z_4}} \\
&\times \Gamma(a + \epsilon - 2 + z_2 + z_3 + z_4) \Gamma(a_2 + z_2 + z_3 + z_4) \Gamma(a_4 + z_4) \\
&\times \Gamma(2 - \epsilon - a_{234} - z_3 - z_4) \Gamma(2 - \epsilon - a_{124} - z_2 - z_4) \\
&\times \Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_4) .
\end{aligned}$$

Double box with irreducible numerator $(k + p_1 + p_2 + p_4)^2$



$$\begin{aligned}
 B_2(s, t; a_1, \dots, \epsilon) &= \int \int \frac{\mathbf{d}^d k \mathbf{d}^d l}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]} \\
 &\times \frac{[-(k + p_1 + p_2 + p_4)^2]^{-a_8}}{[-(l + p_1 + p_2)^2]^{a_4} [-(l + p_1 + p_2 + p_4)^2]^{a_5} (-l^2)^{a_6} [-(k - l)^2]^{a_7}}
 \end{aligned}$$

$$B_2(s, t; a_1, \dots, a_8, \epsilon) = \int \frac{\mathbf{d}^d k [-(k + p_1 + p_2 + p_4)^2]^{-a_8}}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]} \\ \times B_{1100}(s, (k + p_1 + p_2 + p_4)^2, k^2, (k + p_1 + p_2)^2; a_6, a_7, a_4, a_5, d)$$

$$B_2(s, t; a_1, \dots, a_8, \epsilon) = \int \frac{d^d k [-(k + p_1 + p_2 + p_4)^2]^{-a_8}}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]} \\ \times B_{1100}(s, (k + p_1 + p_2 + p_4)^2, k^2, (k + p_1 + p_2)^2; a_6, a_7, a_4, a_5, d)$$

After using the threefold MB representation for B_{1100} and changing the order of integration we obtain an on-shell box integral with indices shifted by z -variables. Apply then the onefold representation for the this box.

$$B_2(s, t; a_1, \dots, a_8, \epsilon) = \int \frac{d^d k [-(k + p_1 + p_2 + p_4)^2]^{-a_8}}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]} \\ \times B_{1100}(s, (k + p_1 + p_2 + p_4)^2, k^2, (k + p_1 + p_2)^2; a_6, a_7, a_4, a_5, d)$$

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The loop by loop derivation of MB representations.

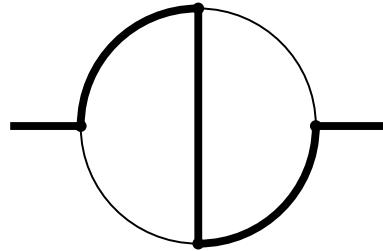
[A. Davydychev & N. Ussyukina'93]

The loop by loop derivation made automatic: AMBRE

[J. Gluza, K. Kajda & T. Riemann'07]

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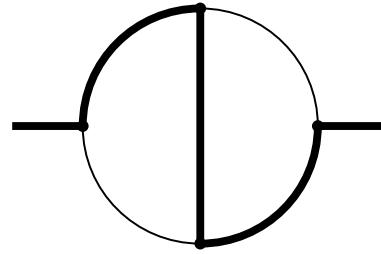


$$p^2 = m^2$$

The loop by loop derivation made automatic:

AMBRE

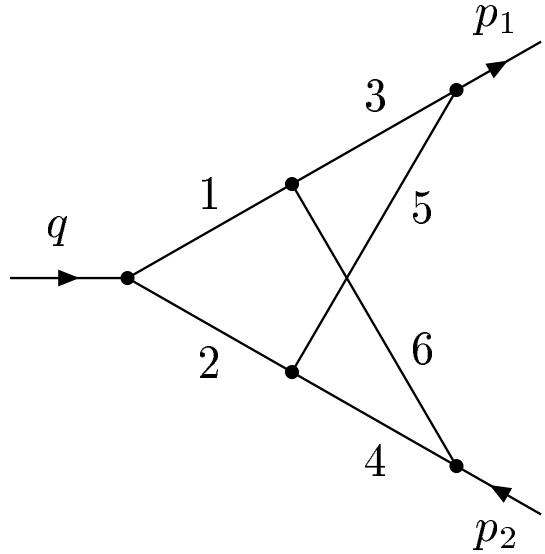
[J. Gluza, K. Kajda & T. Riemann'07]



$$p^2 = m^2$$

Start from the right subgraph (triangle). The upper left external kinematic invariant has to be chosen $(-p_1^2 + m^2)$ and the lower left external kinematic invariant has to be chosen $(-p_2^2)$ where $p_{1,2}$ are the momenta of the left upper and lower lines.

Example Non-planar two-loop massless vertex diagram
with $p_1^2 = p_2^2 = 0$, $Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$



$$F_\Gamma(Q^2; a_1, \dots, a_6, d) = \int \int \frac{\mathbf{d}^d k \mathbf{d}^d l}{[(k+l)^2 - 2p_1 \cdot (k+l)]^{a_1}} \\ \times \frac{1}{[(k+l)^2 - 2p_2 \cdot (k+l)]^{a_2} (k^2 - 2p_1 \cdot k)^{a_3} (l^2 - 2p_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

$$\frac{1}{(k^2 - 2p_1 \cdot k)^{a_3} (k^2)^{a_5}} = \frac{(-1)^{a_3+a_5} \Gamma(a_3 + a_5)}{\Gamma(a_3) \Gamma(a_5)} \\ \times \int_0^1 \frac{d\xi_1 \xi_1^{a_3-1} (1-\xi_1)^{a_5-1}}{[-(k - \xi_1 p_1)^2 - i0]^{a_3+a_5}}$$

and, similarly, for the second pair, with the replacements

$$\xi_1 \rightarrow \xi_2, \quad p_1 \rightarrow p_2, \quad k \rightarrow l, \quad a_3 \rightarrow a_4, \quad a_5 \rightarrow a_6$$

Change the integration variable $l \rightarrow r = k + l$ and integrate over k by means of our massless one-loop formula

$$\int \frac{dk}{[-(k - \xi_1 p_1)^2]^{a_3+a_5} [-(r - \xi_2 p_2 - k)^2]^{a_4+a_6}} = i\pi^{d/2} \frac{G(a_3 + a_5, a_4 + a_6)}{[-(r - \xi_1 p_1 - \xi_2 p_2)^2]^{a_3+a_4+a_5+a_6+\epsilon-2}}$$

Apply Feynman parametric formula to the propagators 1 and 2 and the propagator arising from the previous integration, with a resulting integral over r evaluated in terms of gamma functions:

$$\int \frac{d^d r}{[-(r^2 - Q^2 A(\xi_1, \xi_2, \xi_3, \xi_4))]^{a+\epsilon-2}} = i\pi^{d/2} \frac{\Gamma(a + 2\epsilon - 4)}{\Gamma(a + \epsilon - 2)} \frac{1}{(Q^2)^{a+2\epsilon-4} A(\xi_1, \xi_2, \xi_3, \xi_4)^{a+2\epsilon-4}}$$

where $a = a_1 + \dots + a_6$ and

$$A(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_3 \xi_4 + (1 - \xi_3 - \xi_4)[\xi_2 \xi_3 (1 - \xi_1) + \xi_1 \xi_4 (1 - \xi_2)]$$

Gonsalves'83:

$$\begin{aligned} F_\Gamma(Q^2; a_1, \dots, a_6, d) &= \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35}) \Gamma(2 - \epsilon - a_{46})}{(Q^2)^{a+2\epsilon-4} \prod \Gamma(a_l) \Gamma(4 - 2\epsilon - a_{3456})} \\ &\times \Gamma(a + 2\epsilon - 4) \int_0^1 d\xi_1 \dots \int_0^1 d\xi_4 \xi_1^{a_3-1} (1 - \xi_1)^{a_5-1} \xi_2^{a_4-1} (1 - \xi_2)^{a_6-1} \\ &\times \xi_3^{a_1-1} \xi_4^{a_2-1} (1 - \xi_3 - \xi_4)_+^{a_{3456}+\epsilon-3} A(\xi_1, \xi_2, \xi_3, \xi_4)^{4-2\epsilon-a} \end{aligned}$$

$$\begin{aligned}
& \frac{\Gamma(a + 2\epsilon - 4)}{[\eta\xi(1 - \xi) + (1 - \eta)(\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2))]^{a+2\epsilon-4}} \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_1 \Gamma(-z_1) \eta^{z_1} \xi^{z_1} (1 - \xi)^{z_1}}{(1 - \eta)^{a+2\epsilon-4+z_1}} \\
&\times \frac{\Gamma(a + 2\epsilon - 4 + z_1)}{[\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2)]^{a+2\epsilon-4+z_1}}
\end{aligned}$$

The last line →

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2 \Gamma(a + 2\epsilon - 4 + z_1 + z_2) \Gamma(-z_2) \xi^{z_2} \xi_2^{z_2} (1 - \xi_1)^{z_2}}{(1 - \xi)^{a+2\epsilon-4+z_1+z_2} \xi_1^{a+2\epsilon-4+z_1+z_2} (1 - \xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

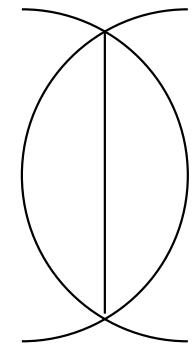
$$\begin{aligned}
F_\Gamma(Q^2; a_1, \dots, a_6, d) &= \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35})}{(Q^2)^{a+2\epsilon-4} \Gamma(6 - 3\epsilon - a) \prod \Gamma(a_l)} \\
&\times \frac{\Gamma(2 - \epsilon - a_{46})}{\Gamma(4 - 2\epsilon - a_{3456})} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} dz_1 dz_2 \Gamma(a + 2\epsilon - 4 + z_1 + z_2) \\
&\quad \times \Gamma(-z_1) \Gamma(-z_2) \Gamma(a_4 + z_2) \Gamma(a_5 + z_2) \Gamma(a_1 + z_1 + z_2) \\
&\quad \times \frac{\Gamma(2 - \epsilon - a_{12} - z_1) \Gamma(4 - 2\epsilon + a_2 - a - z_2)}{\Gamma(4 - 2\epsilon - a_{1235} - z_1) \Gamma(4 - 2\epsilon - a_{1246} - z_1)} \\
&\quad \times \Gamma(4 - 2\epsilon + a_3 - a - z_1 - z_2) \Gamma(4 - 2\epsilon + a_6 - a - z_1 - z_2) ,
\end{aligned}$$

where $a_{3456} = a_3 + a_4 + a_5 + a_6$, etc.

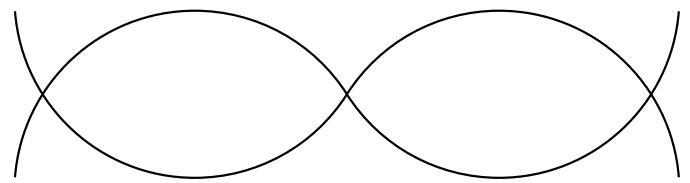
How to check a given MB representation

	a_1		a_6	
a_2		a_7		a_5
	a_3		a_4	

$$a_1, a_3, a_4, a_6 \rightarrow 0$$



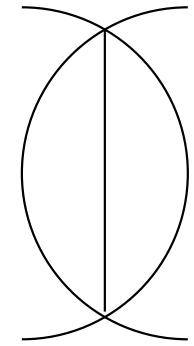
$$a_2, a_5, a_7 \rightarrow 0$$



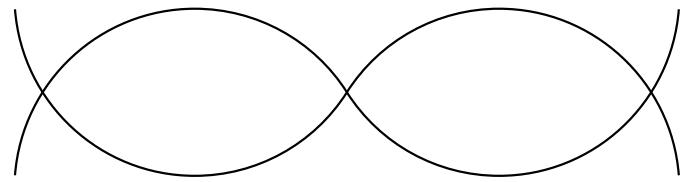
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Typically, $\frac{1}{\Gamma(a)} \Gamma(a+z) \Gamma(-z)$ at $a \rightarrow 0$

How to evaluate MB integrals after expanding in ϵ

The first Barnes lemma

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z) \\ &= \frac{\Gamma(\lambda_1 + \lambda_3) \Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_3) \Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \end{aligned}$$

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Multiple corollaries, e.g.,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z) \Gamma^*(\lambda_2 + z) \Gamma(-\lambda_2 - z) \Gamma(\lambda_3 - z) \\ &= \Gamma(\lambda_1 - \lambda_2) \Gamma(\lambda_2 + \lambda_3) [\psi(\lambda_1 - \lambda_2) - \psi(\lambda_1 + \lambda_3)] \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz}{z} \Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z) \\
&= \frac{\Gamma(2 - \lambda_1 - \lambda_3) \Gamma(1 - \lambda_2 - \lambda_3) \Gamma(\lambda_1 + \lambda_3 - 1) \Gamma(\lambda_2 + \lambda_3)}{\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2)} \\
&\times [\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2) - \Gamma(2 - \lambda_1 - \lambda_2 - \lambda_3) \Gamma(\lambda_3)]
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&\quad \times [\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2) - \Gamma(2 - \lambda_1 - \lambda_2 - \lambda_3) \Gamma(\lambda_3)]
\end{aligned}$$

The second Barnes lemma

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 + z) \Gamma(\lambda_4 - z) \Gamma(\lambda_5 - z)}{\Gamma(\lambda_6 + z)} \\
&= \frac{\Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_4) \Gamma(\lambda_3 + \lambda_4) \Gamma(\lambda_1 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5) \Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)} \\
&\quad \times \frac{\Gamma(\lambda_2 + \lambda_5) \Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}, \quad \lambda_6 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5
\end{aligned}$$

Transform a given multiple MB integral originating after expanding in ϵ into multiple series.

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Summing up series with nested sums

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For example, with one index:

$$\begin{aligned}\psi(n) &= S_1(n-1) - \gamma_E, \\ \psi^{(k)}(n) &= (-1)^k k! (S_{k+1}(n-1) - \zeta(k+1)), \quad k = 1, 2, \dots,\end{aligned}$$

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SUMMER

[J.A.M. Vermaseren'00]

XSummer

[S. Moch and P. Uwer'00]

Sigma, EvaluateMultiSums, SumProduction,

HarmonicSums

[C. Schneider, J. Ablinger, ...]

Harmonic polylogarithms (HPL)

$H_{a_1, a_2, \dots, a_n}(x) \equiv H(a_1, a_2, \dots, a_n; x)$, with $a_i = 1, 0, -1$

[E. Remiddi & J.A.M. Vermaseren'00]

are generalizations of the usual polylogarithms $\text{Li}_a(z)$ and
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$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x,$$

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Nielsen polylogarithms $S_{a,b}(z)$

$$H(a_1, a_2, \dots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \dots, a_n; t) dt,$$

where $f(\pm 1; t) = 1/(1 \mp t)$, $f(0; t) = 1/t$,

$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x,$$

with $a_i = 1, 0, -1$.

HPL implemented in Mathematica

[D. Maitre'06]

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MB representations for non-planar diagrams?

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Poles in ϵ can arise not only locally but also from an
integration over large z .

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z} (-1)^z dz$$

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$$z = x + iy; \epsilon \text{ real}$$

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$$= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(1+2\epsilon+x+iy)\Gamma(-x-iy)}{1+\epsilon+x+iy} e^{-i\pi x + \pi y} dy$$

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$$\Gamma(x \pm iy) \sim \sqrt{2\pi} e^{\pm i\frac{\pi}{4}(2x-1)} e^{\pm iy(\ln y - 1)} e^{-\frac{\pi}{2}y}$$

when $y \rightarrow +\infty$

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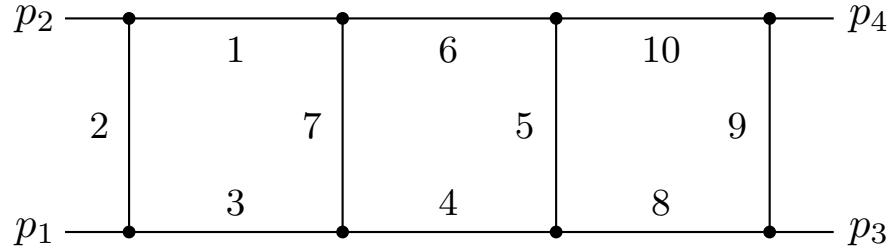
$$= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(1+2\epsilon+x+iy)\Gamma(-x-iy)}{1+\epsilon+x+iy} e^{-i\pi x + \pi y} dy$$

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when $y \rightarrow +\infty$

The integrand behaves like

$$2\pi \frac{1}{y^{1-2\epsilon}}$$



The general planar triple box Feynman integral

$$\begin{aligned}
 T(a_1, \dots, a_{10}; s, t; \epsilon) &= \int \int \int \frac{d^d k d^d l d^d r}{[k^2]^{a_1} [(k + p_2)^2]^{a_2}} \\
 &\times \frac{1}{[(k + p_1 + p_2)^2]^{a_3} [(l + p_1 + p_2)^2]^{a_4} [(r - l)^2]^{a_5} [l^2]^{a_6} [(k - l)^2]^{a_7}} \\
 &\times \frac{1}{[(r + p_1 + p_2)^2]^{a_8} [(r + p_1 + p_2 + p_3)^2]^{a_9} [r^2]^{a_{10}}}
 \end{aligned}$$

General 7fold MB representation:

$$\begin{aligned}
T(a_1, \dots, a_{10}; s, t, m^2; \epsilon) &= \frac{\left(i\pi^{d/2}\right)^3 (-1)^a}{\prod_{j=2,5,7,8,9,10} \Gamma(a_j) \Gamma(4 - a_{589(10)} - 2\epsilon) (-s)^{a-6+3\epsilon}} \\
&\times \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} dw \prod_{j=2}^7 dz_j \left(\frac{t}{s}\right)^w \frac{\Gamma(a_2 + w)\Gamma(-w)\Gamma(z_2 + z_4)\Gamma(z_3 + z_4)}{\Gamma(a_1 + z_3 + z_4)\Gamma(a_3 + z_2 + z_4)} \\
&\times \frac{\Gamma(2 - a_1 - a_2 - \epsilon + z_2)\Gamma(2 - a_2 - a_3 - \epsilon + z_3)\Gamma(a_7 + w - z_4)}{\Gamma(4 - a_1 - a_2 - a_3 - 2\epsilon + w - z_4)\Gamma(a_6 - z_5)\Gamma(a_4 - z_6)} \\
&\times \Gamma(+a_1 + a_2 + a_3 - 2 + \epsilon + z_4)\Gamma(w + z_2 + z_3 + z_4 - z_7)\Gamma(-z_5)\Gamma(-z_6) \\
&\times \Gamma(2 - a_5 - a_9 - a_{10} - \epsilon - z_5 - z_7)\Gamma(2 - a_5 - a_8 - a_9 - \epsilon - z_6 - z_7) \\
&\times \Gamma(a_4 + a_6 + a_7 - 2 + \epsilon + w - z_4 - z_5 - z_6 - z_7)\Gamma(a_9 + z_7) \\
&\times \Gamma(4 - a_4 - a_6 - a_7 - 2\epsilon + z_5 + z_6 + z_7) \\
&\times \Gamma(2 - a_6 - a_7 - \epsilon - w - z_2 + z_5 + z_7)\Gamma(2 - a_4 - a_7 - \epsilon - w - z_3 + z_6 + z_7) \\
&\times \Gamma(a_5 + z_5 + z_6 + z_7)\Gamma(a_5 + a_8 + a_9 + a_{10} - 2 + \epsilon + z_5 + z_6 + z_7),
\end{aligned}$$

General 7fold MB representation:

$$\begin{aligned}
T(a_1, \dots, a_{10}; s, t, m^2; \epsilon) &= \frac{\left(i\pi^{d/2}\right)^3 (-1)^a}{\prod_{j=2,5,7,8,9,10} \Gamma(a_j) \Gamma(4 - a_{589(10)} - 2\epsilon) (-s)^{a-6+3\epsilon}} \\
&\times \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} dw \prod_{j=2}^7 dz_j \left(\frac{t}{s}\right)^w \frac{\Gamma(a_2 + w)\Gamma(-w)\Gamma(z_2 + z_4)\Gamma(z_3 + z_4)}{\Gamma(a_1 + z_3 + z_4)\Gamma(a_3 + z_2 + z_4)} \\
&\times \frac{\Gamma(2 - a_1 - a_2 - \epsilon + z_2)\Gamma(2 - a_2 - a_3 - \epsilon + z_3)\Gamma(a_7 + w - z_4)}{\Gamma(4 - a_1 - a_2 - a_3 - 2\epsilon + w - z_4)\Gamma(a_6 - z_5)\Gamma(a_4 - z_6)} \\
&\times \Gamma(+a_1 + a_2 + a_3 - 2 + \epsilon + z_4)\Gamma(w + z_2 + z_3 + z_4 - z_7)\Gamma(-z_5)\Gamma(-z_6) \\
&\times \Gamma(2 - a_5 - a_9 - a_{10} - \epsilon - z_5 - z_7)\Gamma(2 - a_5 - a_8 - a_9 - \epsilon - z_6 - z_7) \\
&\times \Gamma(a_4 + a_6 + a_7 - 2 + \epsilon + w - z_4 - z_5 - z_6 - z_7)\Gamma(a_9 + z_7) \\
&\times \Gamma(4 - a_4 - a_6 - a_7 - 2\epsilon + z_5 + z_6 + z_7) \\
&\times \Gamma(2 - a_6 - a_7 - \epsilon - w - z_2 + z_5 + z_7)\Gamma(2 - a_4 - a_7 - \epsilon - w - z_3 + z_6 + z_7) \\
&\times \Gamma(a_5 + z_5 + z_6 + z_7)\Gamma(a_5 + a_8 + a_9 + a_{10} - 2 + \epsilon + z_5 + z_6 + z_7),
\end{aligned}$$

$$\begin{aligned}
& T(1, 1, \dots, 1; s, t; \epsilon) \\
&= \frac{(i\pi^{d/2})^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} dw \prod_{j=2}^7 dz_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
&\times \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
&\times \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
&\times \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
&\times \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)
\end{aligned}$$

Result

[V.A. Smirnov'03]

$$T(1, 1, \dots, 1; s, t; \epsilon) = -\frac{(i\pi^{d/2} e^{-\gamma_E \epsilon})^3}{s^3 (-t)^{1+3\epsilon}} \sum_{i=0}^6 \frac{c_j(x, L)}{\epsilon^j},$$

where $x = -t/s$, $L = \ln(s/t)$, and

$$c_6 = \frac{16}{9}, \quad c_5 = -\frac{5}{3}L, \quad c_4 = -\frac{3}{2}\pi^2,$$

$$c_3 = 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2 + \pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3,$$

$$\begin{aligned} c_2 &= -3(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)) \\ &\quad - L(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)) - \frac{3}{2}(L^2 + \pi^2)H_{1,1}(x) \\ &\quad - \left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4, \end{aligned}$$

$$\begin{aligned}
c_1 = & 3(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x) \\
& + 33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x) \\
& + H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)) + L(177H_{0,0,0,1}(x) + 85H_{0,0,1,1}(x) \\
& + 73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) + 61H_{1,0,0,1}(x) + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)) \\
& + \left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right)H_{0,1,1}(x) \\
& + \left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,0,1}(x) + \frac{3}{2}(L^2 + \pi^2)H_{1,1,1}(x) \\
& + \left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2L - 96\zeta_3\right)H_{0,1}(x) + \left(\frac{3}{2}L^3 + \pi^2L - 3\zeta_3\right)H_{1,1}(x) \\
& + \left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2L^2 - 58\zeta_3L + \frac{13}{8}\pi^4\right)H_1(x) - \frac{503}{1440}\pi^4L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5,
\end{aligned}$$

$$\begin{aligned}
c_0 = & - (951H_{0,0,0,0,0,1}(x) + 819H_{0,0,0,0,1,1}(x) + 699H_{0,0,0,1,0,1}(x) + 195H_{0,0,0,1,1,1}(x) \\
& + 547H_{0,0,1,0,0,1}(x) + 231H_{0,0,1,0,1,1}(x) + 159H_{0,0,1,1,0,1}(x) + 3H_{0,0,1,1,1,1}(x) \\
& + 363H_{0,1,0,0,0,1}(x) + 267H_{0,1,0,0,1,1}(x) + 195H_{0,1,0,1,0,1}(x) + 3H_{0,1,0,1,1,1}(x) \\
& + 123H_{0,1,1,0,0,1}(x) + 3H_{0,1,1,0,1,1}(x) + 3H_{0,1,1,1,0,1}(x) + 147H_{1,0,0,0,0,1}(x) \\
& + 303H_{1,0,0,0,1,1}(x) + 231H_{1,0,0,1,0,1}(x) + 3H_{1,0,0,1,1,1}(x) + 159H_{1,0,1,0,0,1}(x) \\
& + 3H_{1,0,1,0,1,1}(x) + 3H_{1,0,1,1,0,1}(x) + 87H_{1,1,0,0,0,1}(x) + 3H_{1,1,0,0,1,1}(x) \\
& + 3H_{1,1,0,1,0,1}(x) + 3H_{1,1,1,0,0,1}(x)) \\
& - L (729H_{0,0,0,0,1}(x) + 537H_{0,0,0,1,1}(x) + 445H_{0,0,1,0,1}(x) + 133H_{0,0,1,1,1}(x) \\
& + 321H_{0,1,0,0,1}(x) + 169H_{0,1,0,1,1}(x) + 97H_{0,1,1,0,1}(x) + 3H_{0,1,1,1,1}(x) \\
& + 165H_{1,0,0,0,1}(x) + 205H_{1,0,0,1,1}(x) + 133H_{1,0,1,0,1}(x) + 3H_{1,0,1,1,1}(x) \\
& + 61H_{1,1,0,0,1}(x) + 3H_{1,1,0,1,1}(x) + 3H_{1,1,1,0,1}(x)) \\
& - \left(\frac{531}{2}L^2 + \frac{89}{4}\pi^2 \right) H_{0,0,0,1}(x) - \left(\frac{311}{2}L^2 + \frac{619}{12}\pi^2 \right) H_{0,0,1,1}(x) \\
& - \left(\frac{247}{2}L^2 + \frac{307}{12}\pi^2 \right) H_{0,1,0,1}(x) - \left(\frac{71}{2}L^2 + 32\pi^2 \right) H_{0,1,1,1}(x)
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{151}{2}L^2 - \frac{197}{12}\pi^2 \right) H_{1,0,0,1}(x) - \left(\frac{107}{2}L^2 + 50\pi^2 \right) H_{1,0,1,1}(x) \\
& - \left(\frac{35}{2}L^2 + 14\pi^2 \right) H_{1,1,0,1}(x) - \frac{3}{2} \left(L^2 + \pi^2 \right) H_{1,1,1,1}(x) \\
& - \left(\frac{119}{2}L^3 + \frac{317}{12}\pi^2 L - 455\zeta_3 \right) H_{0,0,1}(x) - \left(\frac{47}{2}L^3 + \frac{179}{12}\pi^2 L \right. \\
& \quad \left. - 120\zeta_3 \right) H_{0,1,1}(x) - \left(\frac{35}{2}L^3 + \frac{35}{12}\pi^2 L - 156\zeta_3 \right) H_{1,0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L \right. \\
& \quad \left. - 3\zeta_3 \right) H_{1,1,1}(x) - \left(\frac{69}{8}L^4 + \frac{101}{8}\pi^2 L^2 - 291\zeta_3 L + \frac{559}{90}\pi^4 \right) H_{0,1}(x) \\
& - \left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4 \right) H_{1,1}(x) \\
& - \left(\frac{27}{40}L^5 + \frac{25}{8}\pi^2 L^3 - \frac{183}{2}\zeta_3 L^2 + \frac{131}{60}\pi^4 L - \frac{37}{12}\pi^2 \zeta_3 + 57\zeta_5 \right) H_1(x) \\
& + \left(\frac{223}{12}\pi^2 \zeta_3 + 149\zeta_5 \right) L + \frac{167}{9}\zeta_3^2 - \frac{624607}{544320}\pi^6.
\end{aligned}$$

‘Inverse Feynman parameters’

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \rightarrow \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

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Integrating over a MB variable (not by a Barnes lemma)

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(-z)x^z dz \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)} {}_2F_1(a; b; a+b+c; 1-x) \\ &= \Gamma(a)\Gamma(b+c) \int_0^1 t^{b-1}(1-t)^{a+c-1}(1-t+tx)^{-a} dt \end{aligned}$$

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AMBRE [J. Gluza, K. Kajda & T. Riemann'07]