Reduction to Master Integrals

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- Solving IBP relations by hand: simple one-loop examples
- Laporta algorithm and its implementations
- Some other approaches to solve IBP relations

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices)

$$F_\Gamma(q_1,\ldots,q_n;d;a_1,\ldots,a_L)$$
$$= \int \ldots \int I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L)\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \mathrm{d}^d k_h$$

$$I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L) = \frac{1}{(p_1^2-m_1^2)^{a_1}(p_2^2-m_2^2)^{a_2}\ldots}$$

An old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Integral calculus:

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Feynman integral calculus:
Use IBP and neglect surface terms

[Chetyrkin & Tkachov'81]

$$\int \ldots \int \left[\left(q_i \cdot \frac{\partial}{\partial k_j} \right) \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0$$

$$\int \ldots \int \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0$$

An example

$$F(a) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a}$$

$F(a)$ for integer $a \leq 0$. We need $F(a)$ for positive integer a.

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Taking derivatives:

$$\frac{\partial}{\partial k} \cdot k = \frac{\partial}{\partial k_\mu} \cdot k_\mu = d$$

$$
\begin{aligned}
k \cdot \frac{\partial}{\partial k} \frac{1}{(k^2 - m^2)^a} &= -a \frac{2k^2}{(k^2 - m^2)^{a+1}} \\
&= -2a \left[\frac{1}{(k^2 - m^2)^a} + \frac{m^2}{(k^2 - m^2)^{a+1}} \right]
\end{aligned}
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IBP relation

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(d - 2a)F(a) - 2am^2 F(a+1) = 0
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IBP relation

$$(d - 2a) F(a) - 2am^2 F(a+1) = 0$$

Its solution

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1)$$

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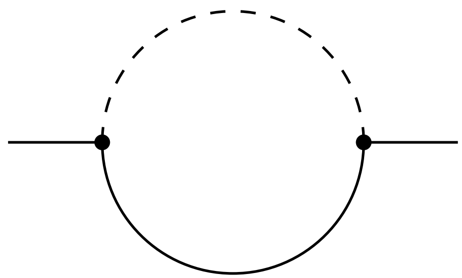
Explicitly,

$$F(a) = \frac{(-1)^a \, (1-d/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1 \ ,$$

where $(x)_a$ is the Pochhammer symbol

One more example



$$F_\Gamma(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

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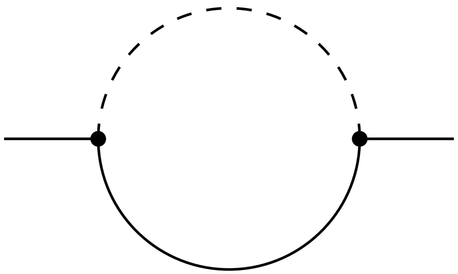

$$F_\Gamma(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(m^2 - k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

Apply IBP

$$\int \frac{\partial}{\partial k} \cdot k \left(\frac{1}{(m^2 - k^2)^{a_1}(-(q-k)^2)^{a_2}} \right) \mathrm{d}^d k \quad = \quad 0 \, ,$$

$$\int q \cdot \frac{\partial}{\partial k} \left(\frac{1}{(m^2 - k^2)^{a_1}(-(q-k)^2)^{a_2}} \right) \mathrm{d}^d k \quad = \quad 0 \, ,$$

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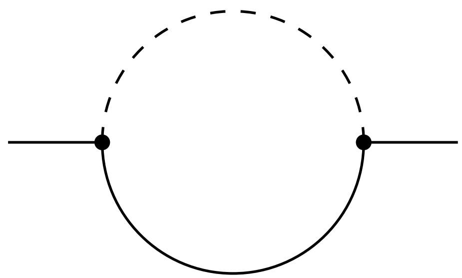

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use $2k \cdot (k-q) \to (k-q)^2 + (k^2 - m^2) - q^2 + m^2$ to obtain

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \quad (A)$$

$$a_2 - a_1 - a_1 \mathbf{1}^+ (q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \quad (B)$$

where, e.g., $\mathbf{1}^+ \mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1)$.

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$F_\Gamma(a_1, a_2) = 0$ for $a_1 \leq 0$

A manual solution.

1. Apply $(q^2+m^2)(A) - 2m^2(B)$,

$$(q^2-m^2)^2 a_2 \mathbf{2}^+ = (q^2-m^2) a_2 \mathbf{1}^-\mathbf{2}^+$$
$$-(d-2a_1-a_2)q^2 - (d-3a_2)m^2 + 2m^2 a_1 \mathbf{1}^+\mathbf{2}^-$$

to reduce a_2 to 1 or 0.

```
F[a1_, a2_ /; a2 > 1] :=
1/(a2 - 1)/(qq - mm)^2 (
(a2 - 1) (qq - mm) F[a1 - 1, a2]
- ((d - 2 a1 - a2 + 1) qq
+ (d - 3 a2 + 3) mm) F[a1, a2 - 1]
+ 2 mm a1 F[a1 + 1, a2 - 2]);
```

2. Suppose that $a_2 = 1$. Apply $(A) - (B)$, i.e.

$$(q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^-$$

to reduce a_1 to 1 or a_2 to 0.

```
F[a1_ /; a1 > 1, 1] :=
1/(a1 - 1)/(qq - mm) ((a1 - 1) F[a1, 0]
-(d - a1 - 1)  F[a1 - 1, 1]);
```

Therefore, any $F(a_1, a_2)$ can be reduced to $I_1 = F(1,1)$ and integrals with $a_2 \leq 0$ (which can be evaluated in terms of gamma functions for general d).

3. Let $a_2 \leq 0$. Apply (A) to reduce a_1 to one.

```
F[a1_ /; a1 > 1, a2_ /; a2 <= 0] :=
1/(a1 - 1)/2/mm ((d - 2 a1 - a2 + 2) F[a1 - 1, a2]
-a2 F[a1 - 2, a2 + 1] + a2 (qq - mm) F[a1 - 1, a2 + 1]
```

4. Let $a_1 = 1$. Apply the following corollary of (A) and (B)

$$(d - a_2 - 1)\mathbf{2}^- = (q^2 - m^2)^2 a_2 \mathbf{2}^+ + (q^2 + m^2)(d - 2a_2 - 1)$$

to increase a_2 to zero or one starting from negative values.

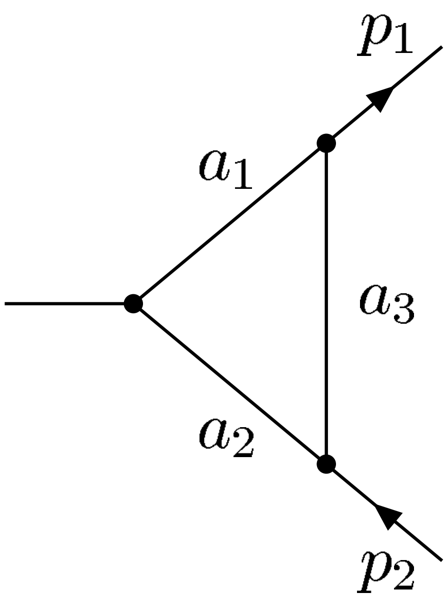
```
F[1, a2_ /; a2 < 0] := 1/(d - a2 - 2) (
(a2 + 1) (qq - mm)ˆ2 F[1, a2 + 2] +
(qq + mm) (d - 2 a2 - 3) F[1, a2 + 1] );
```

Any $F(a_1, a_2)$ is a linear combination of the two master integrals $I_1 = F(1,1)$ and $I_2 = F(1,0)$.

For example,

```
F[3, 2]=
(-(((-5 + d) (-3 + d) (-4 mm + d mm - 8 qq + d qq))/(
2 (mm - qq)^4)) I1
+ ((-2 + d) (96 mm^2 - 39 d mm^2 + 4 d^2 mm^2
+ 28 mm qq - 6 d mm qq - 4 qq^2 + d qq^2))/
(8 mm^2 (mm - qq)^4) I2)
```

Triangle rule



$m_3 = 0$

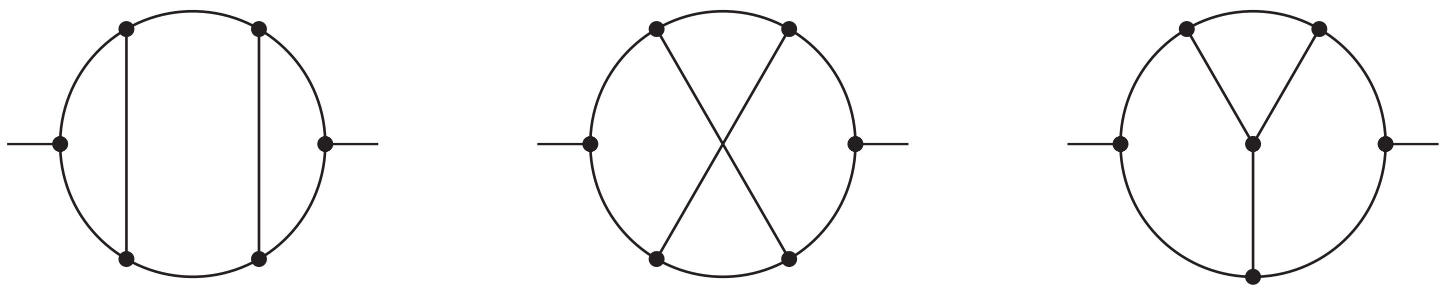
$$F(a_1, a_2, a_3) = \int \frac{\mathrm{d}^d k}{[(k+p_1)^2 - m_1^2]^{a_1} [(k+p_2)^2 - m_2^2]^{a_2} (k^2)^{a_3}}$$

The IBP identity with the operator $(\partial/\partial k)\cdot k \to$

$$
\begin{aligned}
1 \;&=\; \frac{1}{d-a_1-a_2-2a_3} \\
&\quad \times \left[a_1\mathbf{1}^+\left(\mathbf{3}^- -(p_1^2-m_1^2)\right)+a_2\mathbf{2}^+\left(\mathbf{3}^- -(p_2^2-m_2^2)\right)\right]
\end{aligned}
$$

A manual solution of IBP relations for massless three-loop propagator diagrams

[K.G. Chetyrkin & F.V. Tkachov'81]



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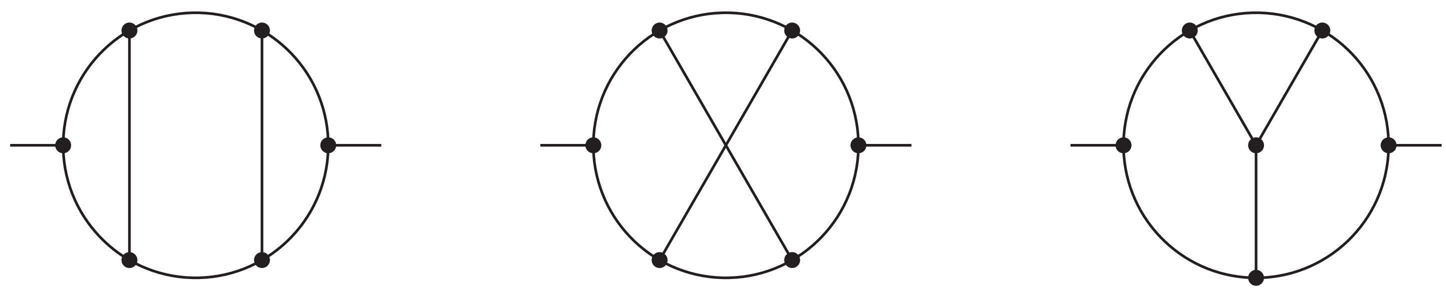

MINCER:

[S.G. Gorishny, S.A. Larin, L.R. Surguladze & F.V. Tkachov'89]

[S.A. Larin, F.V. Tkachov & J. Vermaseren'91]

(implemented in `FORM`)

Solving IBP relations algorithmically:

- Laporta's algorithm

  [Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

  Use IBP relations written at points $(a_1, \ldots, a_L)$ with $\sum |a_i| \leq N$ and solve them for the Feynman integrals involved.

  (A Gauss elimination)

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  Feynman integrals on the right-hand sides of such solutions are master integrals.

  When N increases, the situation stabilizes, in the sense that the number of the master integrals becomes stable starting from sufficiently large N.

Experience: the number of master integrals is always finite.

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A sector σ_ν is defined by a subset $\nu \subseteq \{1, \ldots, N\}$ for which indices are positive:
$\sigma_\nu = \{(a_1, \ldots, a_N) : a_i > 0 \text{ if } i \in \nu\,, \;\; a_i \leq 0 \text{ if } i \notin \nu\}$.

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The problem was reduced to evaluating the volume of the Newton polytope associated with the polynomials in the alpha representation.

The same example

$$F_\Gamma(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(m^2 - k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

The left-hand sides of the two primary IBP relations:

```
ibp1[a1_, a2_] := (d - 2 a1 - a2) F[a1, a2]
- 2 mm a1 F[a1 + 1, a2] - a2 (F[a1 - 1, a2 + 1]
+ (mm - qq)  F[a1, a2 + 1]);
ibp2[a1_, a2_] := (a2 - a1) F[a1, a2] -
a1 ((qq + mm) F[a1 + 1, a2] - F[a1 + 1, a2 - 1]) -
a2 (F[a1 - 1, a2 + 1] + (mm - qq)  F[a1, a2 + 1]);
```

Let us consider the sector $a_1 > 0, a_2 \leq 0$
Use IBP at various (a_1, a_2) with $a_1 + |a_2| \leq N$
Solve the corresponding linear system of equation with respect to $F(a_1, a_2)$ involved.
Increase N.

$N = 1$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0},
{F[2, 0], F[2, -1]}]


{F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
 F[2, 0] -> ((-2 + d) F[1, 0])/(2 mm)}
```

$N = 2$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0,
ibp1[2, 0] == 0, ibp2[2, 0] == 0,
ibp1[1, -1] == 0,  ibp2[1, -1] == 0 },
{F[2, 0], F[3, 0], F[1, -1],
F[2, -1], F[3, -1],  F[2, -2]}]

{F[2, -2] -> (((2 + d) mm^2 + 2 (2 + d) mm qq
+ (-2 + d) qq^2) F[1,0])/(2 mm),
F[3, -1] -> ((-2 + d) (-4 qq + d (mm + qq))
F[1, 0])/(8 mm^2),
F[3, 0] -> ((-4 + d) (-2 + d) F[1, 0])/(8 mm^2),
F[1, -1] -> (mm + qq) F[1, 0],
F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
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Private versions

[Gehrmann & Remiddi, Laporta, Czakon, Schröder, Pak, Sturm, Marquard & Seidel, Velizhanin, . . .]

Solving reduction problems algorithmically in other ways:

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- Baikov's method [P.A. Baikov'96–. . .]

  [V.A. Smirnov & M. Steinhauser'03]

  An Ansatz for coefficient functions at master integrals

$$\int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{a_1} \ldots x_N^{a_N}} \left[P(\underline{x}')\right]^{(d-h-1)/2}$$

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- Gröbner bases [O.V. Tarasov'98]

  An alternative approach [A. Smirnov & V. Smirnov, '05–08]

- Lee's approach (based on Lie algebras) [R.N. Lee'08–13]

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the IBP relations have the structure of a Lie algebra.
The generators are $O_{ij} = \frac{\partial}{\partial k_i} \cdot p_j$
The commutation relations are

$$[O_{ik}, O_{i'j'}] = \delta_{ij'} O_{i'j} - \delta_{i'j} O_{ij'} \; .$$

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It is sufficient to use IBP corresponding to

$$\frac{\partial}{\partial k_i} \cdot k_{i+1}, \quad i = 1, \ldots, h, \quad k_{h+1} \equiv k_1 \ ;$$

$$\frac{\partial}{\partial k_1} \cdot p_j, \quad j = 1, \ldots, n \ ; \quad \sum_{i=1}^{h} \frac{\partial}{\partial k_i} \cdot k_i \ .$$

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A code to reveal independent IBP relations [P. Kant'13]

LiteRed [R.N. Lee'12]

A package which provides the possibility to find reduction rules in a given sector. It includes various additional tools.

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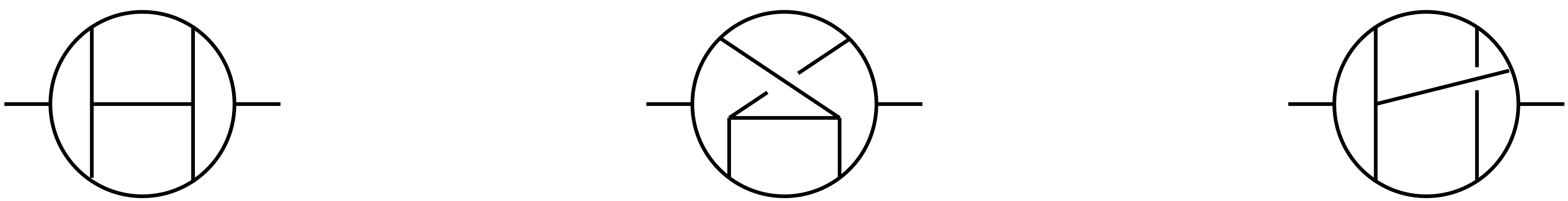
A package which provides the possibility to find reduction rules in a given sector. It includes various additional tools. For example, reduction rules for massless four-loop propagator integrals

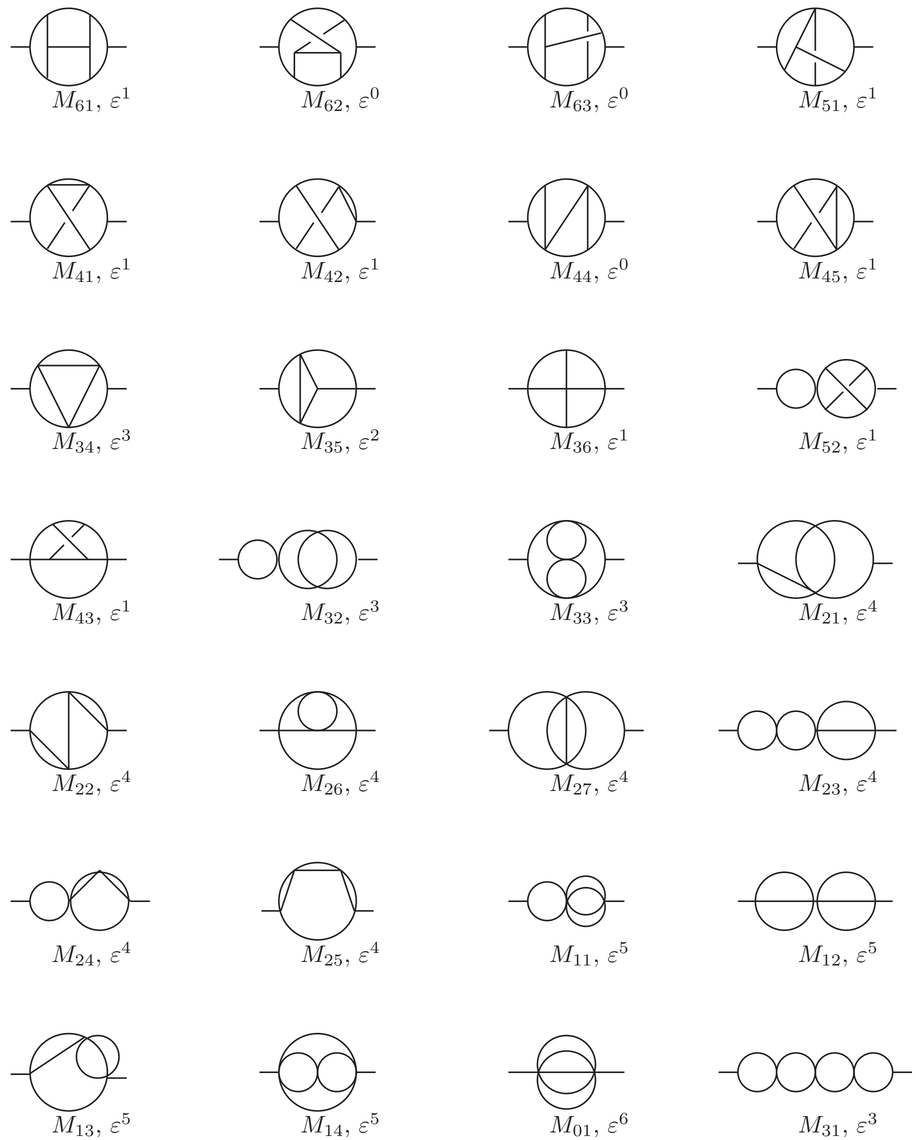


M_{61} M_{62} M_{63}

have been constructed in all the sectors.
A four-loop variant of `MINCER`!

The existence of the explicit reduction rules shows that any four-loop massless propagator integral can be represented as a linear combination, with coefficients rational in d, of the twenty eight master integrals revealed by Baikov and Chetyrkin.

The existence of the explicit reduction rules shows that any four-loop massless propagator integral can be represented as a linear combination, with coefficients rational in d, of the twenty eight master integrals revealed by Baikov and Chetyrkin.
This statement has the status of a mathematical theorem.


M_{61}, ε^1
M_{62}, ε^0
M_{63}, ε^0
M_{51}, ε^1
M_{41}, ε^1
M_{42}, ε^1
M_{44}, ε^0
M_{45}, ε^1
M_{34}, ε^3
M_{35}, ε^2
M_{36}, ε^1
M_{52}, ε^1
M_{43}, ε^1
M_{32}, ε^3
M_{33}, ε^3
M_{21}, ε^4
M_{22}, ε^4
M_{26}, ε^4
M_{27}, ε^4
M_{23}, ε^4
M_{24}, ε^4
M_{25}, ε^4
M_{11}, ε^5
M_{12}, ε^5
M_{13}, ε^5
M_{14}, ε^5
M_{01}, ε^6
M_{31}, ε^3

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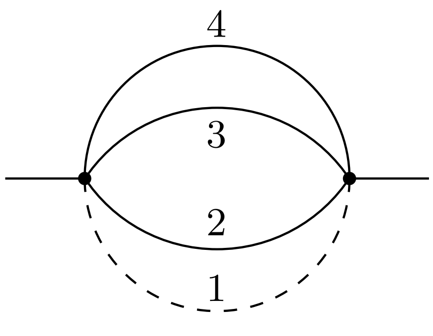
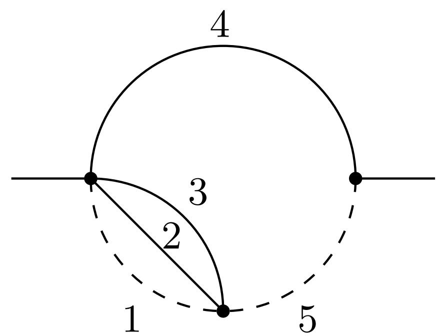
- to run reduction in the Laporta mode if reduction rules are unknown in some sectors.

Using symmetries to find extra relations

I_{11} [S. Laporta & E. Remiddi'96]



$(\mathbf{a})$



$(\mathbf{b})$

(at $p^2 = m^2$) is the integral with the numerator is $k \cdot p$ where k is the momentum of the massless line, in addition to the corresponding master integral I_{10} without numerator (for the three-loop $g-2$ factor).

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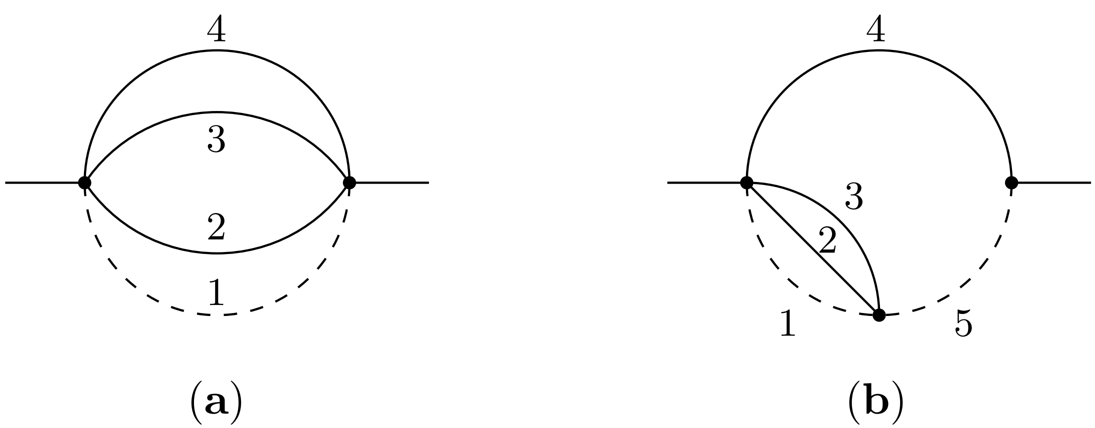

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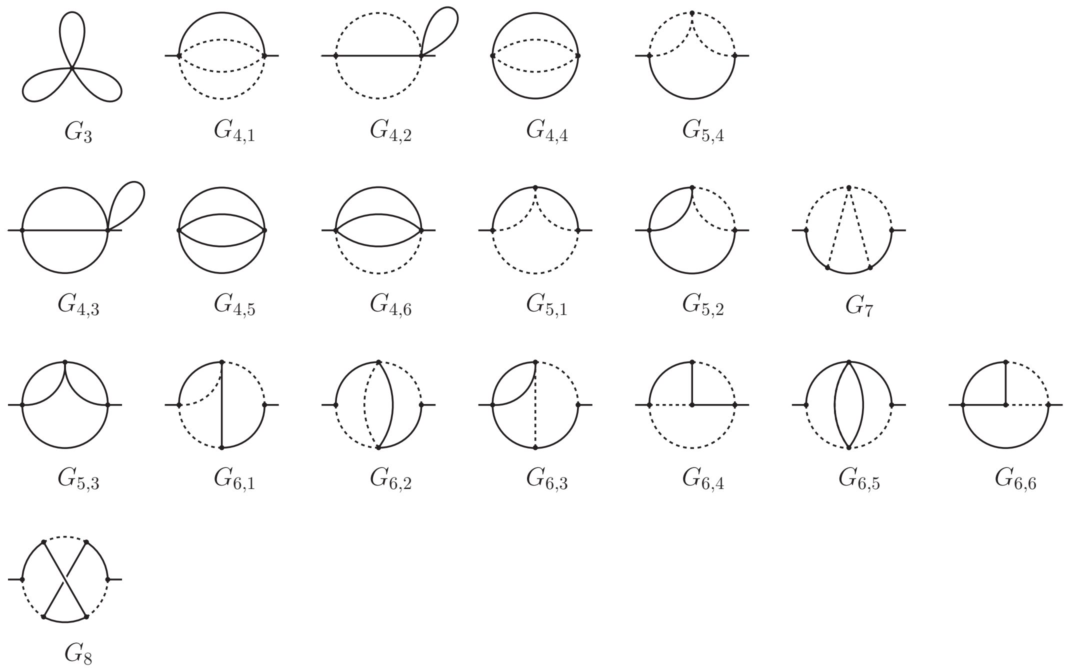
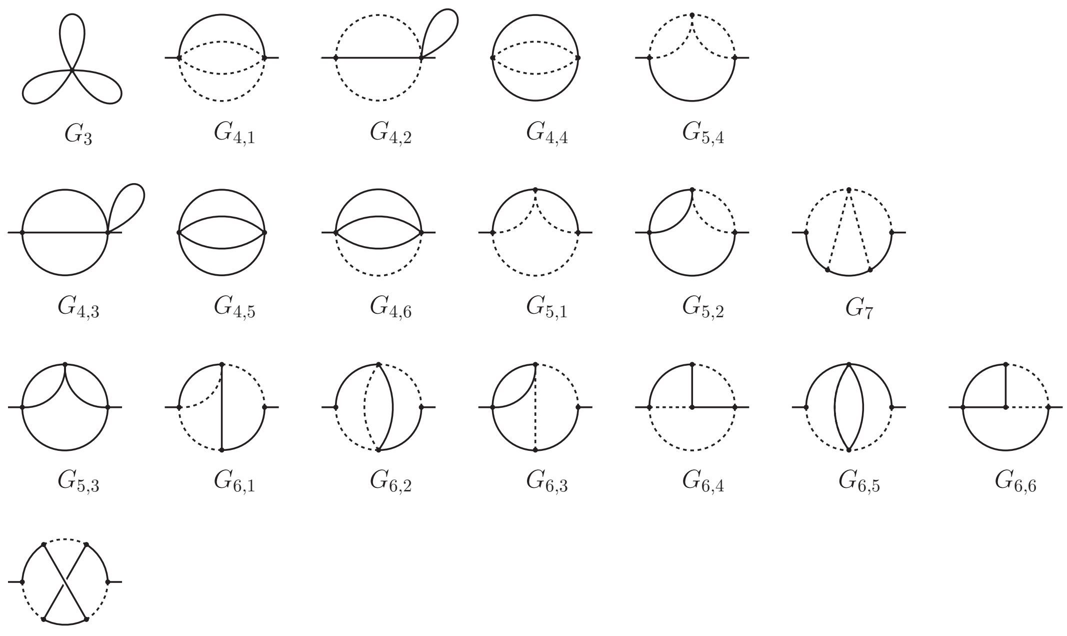
Indeed, the IBP reduction shows that there are two master integrals in the highest sector.



$$I_{11} = \frac{2d-5}{2(d-2)} G_{4,4} - \frac{1}{4} G_3 \; .$$



$$I_{11} = \frac{2d-5}{2(d-2)} G_{4,4} - \frac{1}{4} G_3 \ .$$

can be obtained from the IBP reduction of

$F(1,2,1,2,1) = F(1,1,2,2,1)$

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- R. Lee's method [R. Lee'09, R. Lee & VS'12]
  based on the use of dimensional recurrence relations

  [O. Tarasov'96]