

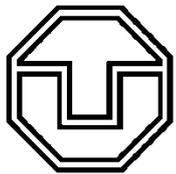
# Zemach and magnetic radius of the proton from the hyperfine splitting in hydrogen

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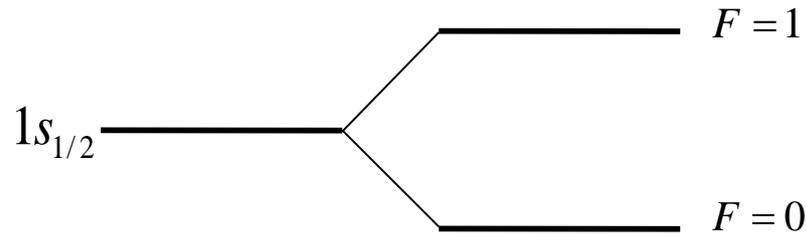
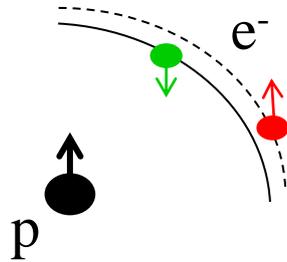
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## Hyperfine splitting



$$\Delta E^p = \langle FM_FIJ | e\boldsymbol{\alpha} \cdot \mathbf{A}_\mu | FM_FIJ \rangle$$

where  $A_\mu$  is potential induced by the magnetic moment  $\hat{\boldsymbol{\mu}}$  of the point nucleus

$$\mathbf{A}_\mu = \frac{1}{4\pi} \frac{[\hat{\boldsymbol{\mu}} \times \mathbf{r}]}{r^3}$$

In the nonrelativistic limit:

$$E_F^p = \frac{8}{3} \alpha (\alpha Z)^3 \frac{m_e^2 m_p^2}{(m_e + m_p)^3} \mu_p = 1\,418.840\,08 \text{ MHz}$$

$$\Delta E_{\text{theor}}^{\text{P}} = E_{\text{F}}^{\text{P}}(1 + \delta^{\text{Dirac}} + \delta^{\text{QED}} + \delta_{\text{p}}^{\text{structure}})$$

$$\delta^{\text{Dirac}} = \frac{3}{2}(\alpha Z)^2 + \frac{17}{8}(\alpha Z)^4 + \dots$$

$$\begin{aligned} \delta^{\text{QED}} = & \kappa_{\text{e}} + \frac{\alpha}{\pi} \left\{ \alpha Z \pi \left( \ln 2 - \frac{5}{2} \right) + (\alpha Z)^2 \left[ \ln(\alpha Z) \left( \frac{16}{3} \ln 2 - \frac{281}{180} - \frac{8}{3} \ln(\alpha Z) \right) - \frac{8}{15} \ln 2 \right. \right. \\ & \left. \left. + \frac{34}{225} + 17.122339 \dots \right] + (\alpha Z)^3 \pi \left[ \ln(\alpha Z) \left( \frac{547}{48} - 5 \ln 2 \right) + \frac{539}{288} - 4.55(35) \right] \right\} \\ & + \frac{\alpha^2}{\pi^2} \left\{ 0.7718(4) \alpha Z \pi + (\alpha Z)^2 \left[ \ln(\alpha Z) \left( 1.278001088 \dots - \frac{4}{3} \ln(\alpha Z) \right) + 10(2.5) \right] \right\} \end{aligned}$$

[Pachucki, Kinoshita, Nio, Karshenboim, Ivanov, Shabaev, Yerokhin, Jentschura]

$$\delta_p^{\text{structure}} = \delta_p^{\text{pol}} + \delta_p^{\text{hvp}} + \delta_p^{\mu\text{vp}} + \delta_p^{\text{weak}} + \delta_p^{\text{rigid}}$$

polarizability correction

$$\delta_p^{\text{pol}} = 1.4(6) \text{ ppm} \quad [\text{Faustov and Martynenko, Phys. At. Nucl. 2002}]$$

hadronic vacuum-polarization correction

$$\delta_p^{\text{hvp}} = 0.01 \text{ ppm} \quad [\text{Martynenko and Faustov, JETP 2004}]$$

muonic vacuum-polarization correction

$$\delta_p^{\mu\text{vp}} = 0.07(2) \text{ ppm} \quad [\text{Karshenboim, Phys. Lett. A 1997}]$$

weak interaction (Z-boson exchange) correction

$$\delta_p^{\text{weak}} = 0.06 \text{ ppm} \quad [\text{Beg and Feinberg, PRL 1975}]$$

$$\delta_p^{\text{rigid}} = \Delta E_{\text{exp}}^{\text{P}} / E_{\text{F}}^{\text{P}} - 1 - \delta^{\text{Dirac}} - \delta^{\text{QED}} - \delta_p^{\text{pol}} - \delta_p^{\mu\text{vp}} - \delta_p^{\text{hvp}} - \delta_p^{\text{weak}}$$

$$= -34.14(60)\text{ppm}$$

$$\delta_p^{\text{rigid}}(G_E, G_M) = \delta_p^{\text{recoil}}(G_E, G_M) + \delta_p^{\text{ps}}(G_E, G_M)$$

$$\delta_p^{\text{recoil}}(G_E, G_M) = \delta_p^{\text{recoil},0}(G_E, G_M) + \delta_p^{\text{recoil-rel}} + \delta_p^{\text{recoil-rad}}$$

[Bodwin and Yennie, PRD 1988; Karshenboim, Phys. Lett. A 1997]

$$\delta_p^{\text{ps}}(G_E, G_M) = \delta_p^{\text{Zemach}}(G_E, G_M) + \delta_p^{\text{ps-rel}} + \delta_p^{\text{ps-rad}}$$

[Zemach, Phys. Rev. 1956; Volotka, Shabaev, Plunien and Soff, Eur. Phys. J. D 2003; Karshenboim, Phys. Lett. A 1997]

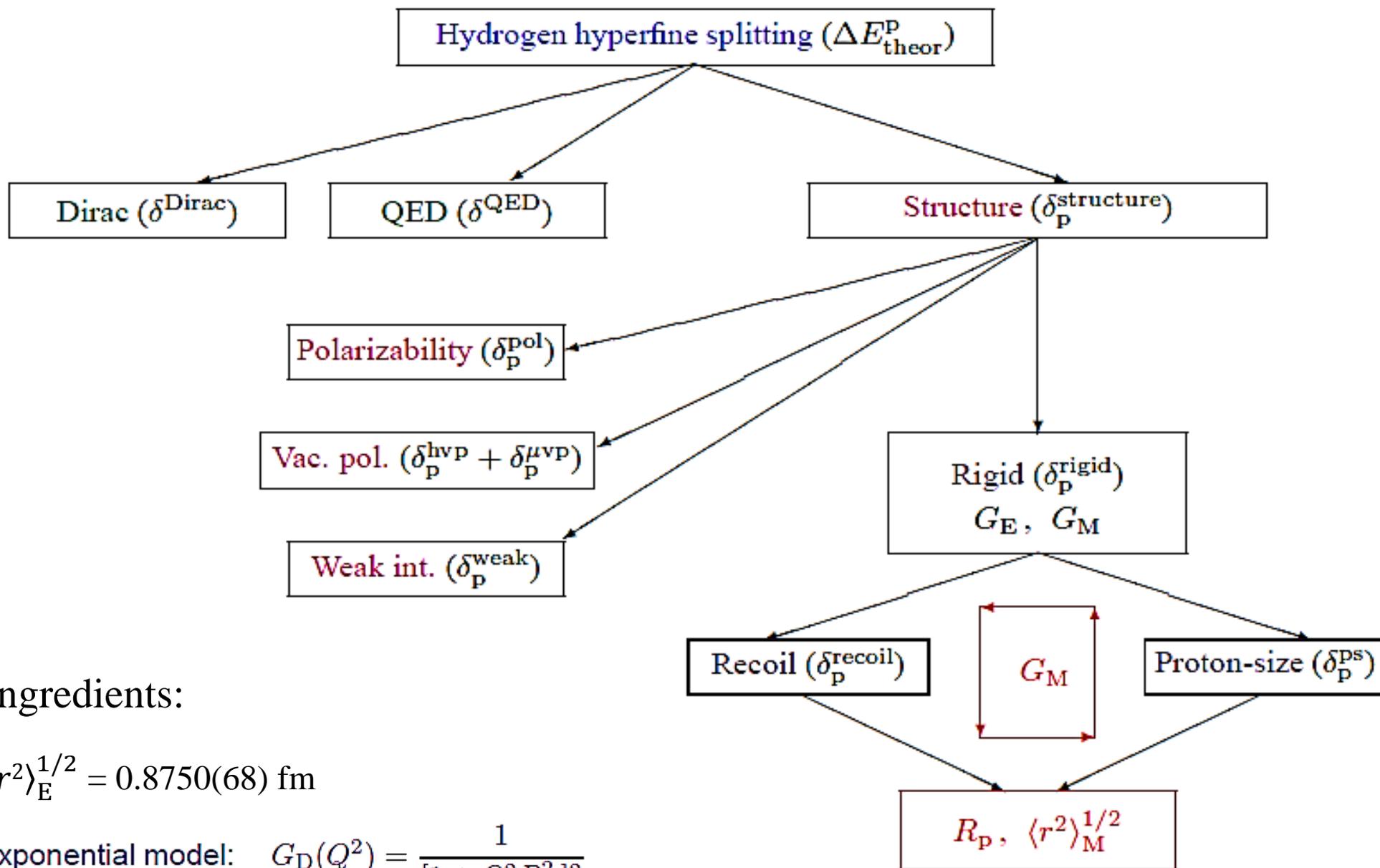
$$\delta_p^{\text{Zemach}} = -2\alpha Z \frac{m_e m_p}{m_e + m_p} R_p \quad [\text{Zemach, Phys. Rev. 1956}]$$

where  $R_p$  is the Zemach radius:

$$R_p = \int d\mathbf{r} d\mathbf{r}' \rho_E(\mathbf{r}) \rho_M(\mathbf{r}') |\mathbf{r} - \mathbf{r}'| = -\frac{4}{\pi} \int_0^\infty \frac{dq}{q^2} \left[ \frac{1}{\mu_p} G_E(q^2) G_M(q^2) - 1 \right]$$

Alternatively, we derived the Zemach correction in terms of electric and magnetic moments

$$\delta_p^{\text{Zemach}} = 2\alpha Z m \langle r \rangle_e + \alpha Z m R_0 \left( \frac{\langle r^2 \rangle_m}{R_0^2} - \frac{1}{10} \frac{\langle r^4 \rangle_m}{R_0^4} \right)$$



Ingredients:

$$\langle r^2 \rangle_{\text{E}}^{1/2} = 0.8750(68) \text{ fm}$$

exponential model: 
$$G_{\text{D}}(Q^2) = \frac{1}{[1 + Q^2 R_{\text{D}}^2]^2}$$

JLab model: 
$$G_{\text{JLab}}(Q^2) = \left(1 - 0.13 \frac{Q^2}{\text{GeV}^2}\right) G_{\text{D}}(Q^2)$$

[Gayou et al., PRC, 2001]

Numerical values for various corrections to the hyperfine splitting in hydrogen.

The energies  $\Delta E_{\text{exp}}^{\text{P}}$  and  $E_{\text{F}}^{\text{P}}$  are given in units of MHz

	Value	Error
$\Delta E_{\text{exp}}^{\text{P}}$	1 420.405 751 767	0.000 000 001*
$E_{\text{F}}^{\text{P}}$	1 418.840 08	0.000 02
$\Delta E_{\text{exp}}^{\text{P}}/E_{\text{F}}^{\text{P}}$	1.001 103 49	0.000 000 01
$\delta^{\text{Dirac}}$	0.000 079 88	
$\delta^{\text{QED}}$	0.001 056 21	0.000 000 001
$\delta_{\text{P}}^{\text{ps}}$	− 0.000 040 11	0.000 000 61
$\delta_{\text{P}}^{\text{recoil}}$	0.000 005 97	0.000 000 06
$\delta_{\text{P}}^{\text{pol}}$	0.000 001 4	0.000 000 6
$\delta_{\text{P}}^{\text{hvp}}$	0.000 000 01	
$\delta_{\text{P}}^{\mu\text{vp}}$	0.000 000 07	0.000 000 02
$\delta_{\text{P}}^{\text{weak}}$	0.000 000 06	

\*Helwig et al., IEEE Trans. Instr. Meas. IM 1970

## Proton-structure parameters from hydrogen hyperfine splitting

Zemach radius:  $R_p = 1.045(16)$  fm

$R_p = 1.037(16)$  fm [Dupays et al., PRA, 2003]

Magnetic radius:  $\langle r^2 \rangle_M^{1/2} = 0.778(29)$  fm

Alternatively, Brodsky, Carlson, Hiller, and Hwang, PRL (2005) proposed to consider the muonium and hydrogen hyperfine splitting together:

$R_p = 1.019(16)$  fm

Considering a rescaled difference:

$$\frac{\Delta E_{\text{exp}}^{\text{P}}/E_{\text{F}}^{\text{P}}}{\Delta E_{\text{exp}}^{\mu}/E_{\text{F}}^{\mu}} = 1 + \delta^{\text{hfs}} \quad \Longrightarrow \quad \delta^{\text{hfs}} = 145.51(4) \text{ ppm}$$

$$: \Delta E_{\text{exp}}^{\mu} = 4\,463.302\,765(53) \text{ MHz} \quad [\text{Liu et al., PRL 1999}]$$

$$\delta_{\text{p}}^{\text{structure}} = \delta^{\text{hfs}} + \delta_{\mu}^{\text{recoil}} + \delta_{\mu}^{\text{hvp}} + \delta_{\mu}^{\text{weak}} + \delta^{\text{hfs}} (\delta^{\text{Dirac}} + \delta^{\text{QED}} + \delta_{\mu}^{\text{recoil}} + \delta_{\mu}^{\text{hvp}} + \delta_{\mu}^{\text{weak}})$$

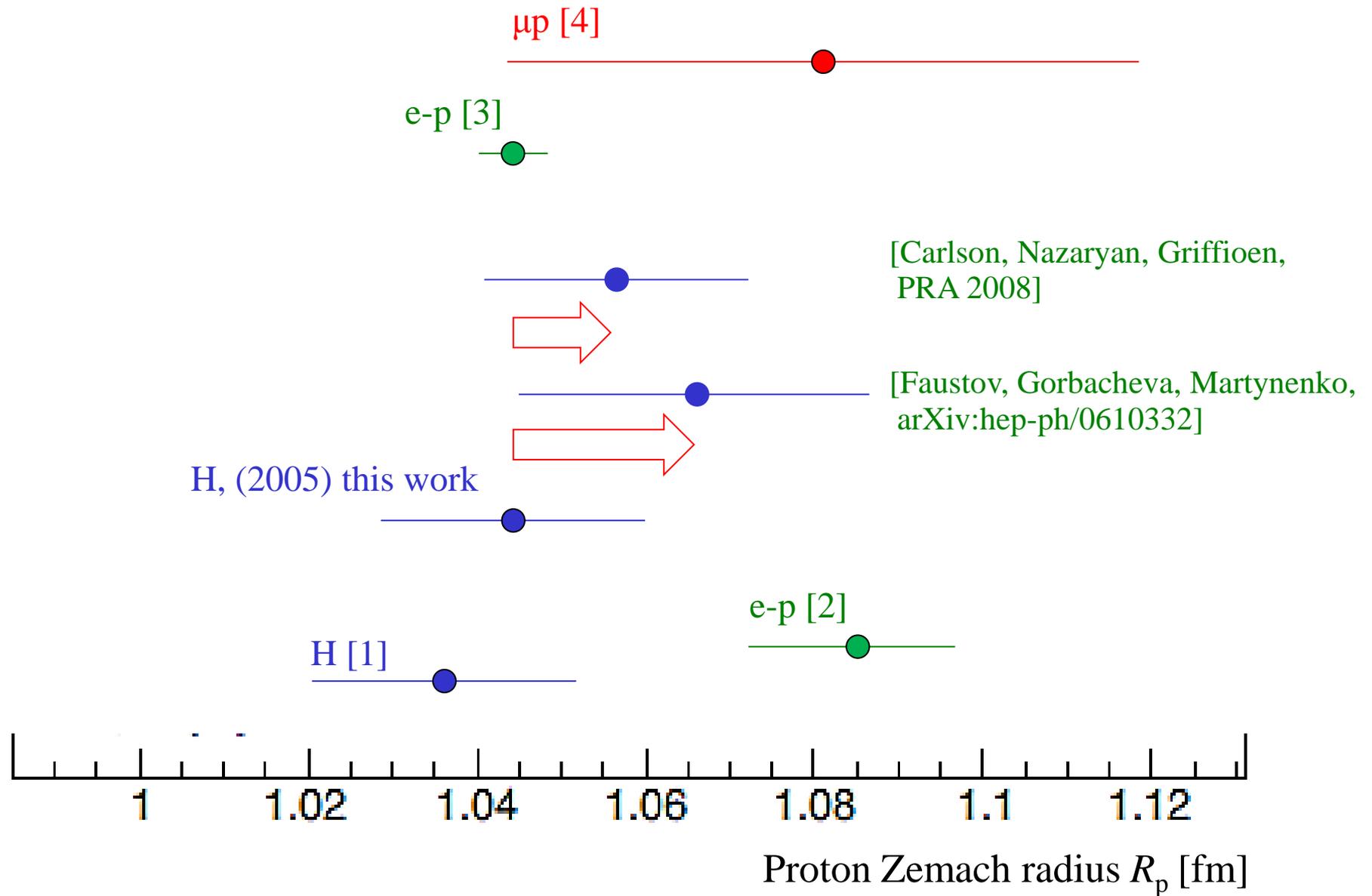
-> Dirac and QED corrections are almost subtracted

-> Recoil correction consists of relativistic and radiative parts

-> Including omitted radiative-recoil correction we obtained

$$R_{\text{p}} = 1.047(16) \text{ fm}$$

This was later confirmed by Brodsky, Carlson, Hiller, and Hwang, Err. (2005)

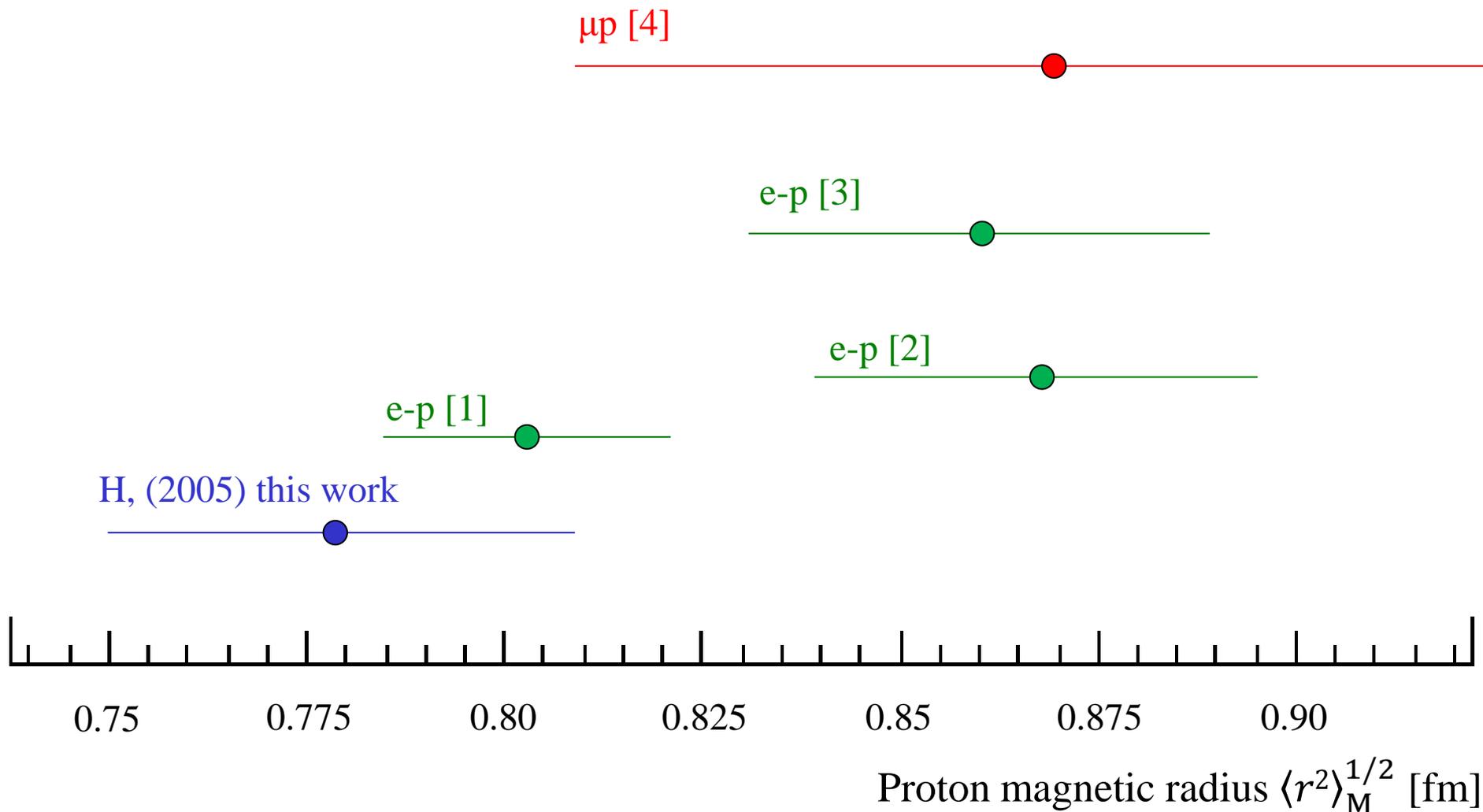


[1] Dupays et al., PRA (2003)

[2] Friar and Sick, Phys. Lett. B (2004)

[3] Distler, Bernauer, and Walcher, Phys. Lett. B (2011)

[4] Antognini et al., Science (2013)



[1] Bernauer et al., PRL (2011)

[2] Zhan et al., Phys. Lett. B (2011)

[3] Lorenz, Hammer, and Meißner, Eur. Phys. J. A (2012)

[4] Antognini et al., Science (2013)