Statistical properties of small systems

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The question of the issue of the application of statistical physics on the nano(femto)scale began to emerge after the nucleation reaction was discovered in the early 1930s and continues to be asked to this day! (the term ‘nano(femto)thermodynamics’ was being used as a synonym for small-systems thermodynamics as early as 2000). The first work in this regard was carried out by Hill about five decades ago. He used ordinary thermodynamic relations and added a correction term (called “ subdivision energy” and defined by $W=U-TS-PV-\mu N$) for each extra effect appearing in small systems. Another approach to femtothermodynamics is to focus on the fluctuations of thermodynamic functions, since fluctuations are not negligible in femtosystems. It has been shown that averaging the fluctuating quantities yields to Tsallis statistics for femtosystems. Our presentation differs from Hill’s approach by recognizing that for femtosystems , deviation from Boltzman formula become important.

in thermodynamics limit (large number N of costituents)

$$f(X) = \frac{1}{\sqrt{2\pi Var(X)}} \exp\left(- \frac{(X - <X>)^2}{2Var(X)} \right)$$

$N \to \infty$

$$\omega_X^2 = \frac{Var(X)}{<X>^2} = \frac{1}{N}$$

for relative small number of constituents

BG

$$U f_{T,N}(U) = N f_{T,U}(N) = \beta f_{U,N}(\beta) = \frac{(\beta U)^N}{\Gamma(N)} \exp(-\beta U)$$

$$\frac{Var(U)}{<U>^2} = \frac{Var(N)}{<N>^2} = \frac{Var(\beta)}{<\beta>^2} = \frac{1}{<N>}$$

$$\omega_U^2 = \omega_T^2 = \omega_N^2 = \frac{1}{<N>}$$

GW, ZW, Physica **A 390** (2011) 3566

relation $\omega_X^2 = \frac{Var(X)}{<X>^2} = \frac{1}{N}$ **is valid for Boltzman-Gibbs statistics**

$$P(E) \propto \exp(-\beta E)$$

$$E_r = U - E$$

$$P(E) \propto \Omega(U - E)$$

$$\ln \Omega(U - E) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{(k)} \ln \Omega}{\partial E_r^{(k)}} (-E)^k$$

first 2 terms
E << U

$$\ln P(E) \propto \ln \Omega(E) \propto -\beta E$$

$$\beta = \frac{1}{k_B T} = \frac{\partial \ln \Omega(E_r)}{\partial E_r}$$

in general

$$\ln\Omega(U-E)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^{(k)}\ln\Omega}{\partial E_r^{(k)}}(-E)^k$$

$\Omega(E_r)\propto(E_r/\nu)^{\alpha\nu-1}$ where $\alpha\propto 1$ (for particle in 3D box: $\Omega(E)\propto E^{1/2}$ i.e., $\alpha=1/2$)

$$\frac{\partial^k\beta}{\partial E_r^k}\propto(-1)^k k!\frac{\alpha\nu-1}{E_r^{k+1}}=(-1)^k k!\frac{\beta^{k+1}}{(\alpha\nu-1)^k}$$

$$\ln\Omega(U-E)-\ln\Omega(U)=\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^{(k)}\ln\Omega}{\partial E_r^{(k)}}(-E)^k=\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^{(k-1)}\beta}{\partial E_r^{(k-1)}}(-E)^k\propto$$

$$\propto\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{\beta^k}{(\alpha\nu-1)^{k-1}}(-E)^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\frac{1}{(\alpha\nu-1)^k}(-\beta E)^{k+1}$$

$$P(E)\propto\frac{\Omega(U-E)}{\Omega(U)}=\exp\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\frac{1}{(\alpha\nu-1)^k}(-\beta E)^{k+1}\right]$$

$$\ln x=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$$

$$P(E)=C\left(1-\frac{1}{\alpha\nu-1}\beta E\right)^{(\alpha\nu-1)}$$

$$1-q'=\frac{1}{\alpha\nu-1}$$

Tsallis distribution with $q'\le 1$

$$P(E)=\beta(2-q')\left(1-(1-q')\beta E\right)^{1/(1-q')}$$

Tsallis distribution

C. Tsallis, J.Stat.Phys. **52** (1988) 479

$$\frac{2-q}{T}\left[1-(1-q)\frac{E}{T}\right]^{\frac{1}{1-q}}$$

$q \to 1$

meaning of q ?

Examples of mechnisms leading to Tsallis distribution:

- Superstatistics
- Stochastic network approach
- Multiplicative noise
- MaxEnt (Shannon entropy)

BG

R. Hagedorn (1965)

$$\frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

more information:
arXiv:1307.7855

Superstatistics

Superstatistics which is a superposition of two different statistics relevant to driven nonequilibrium systems with a stationary state and intensive parameter fluctuations [C. Beck et al., Physica A322 (2003) 267]

$$h(E/T) = \int_0^\infty f(E/T)\, g(1/T)\, d(1/T)$$

Tsallis statistics as a special case of superstatistics

$$f(E) = \frac{1}{T} \exp\left(-\frac{E}{T}\right)$$ BG

$$g(1/T) = \frac{1}{\Gamma(\frac{1}{q-1} - s)} \frac{T_0}{q-1} \left(\frac{1}{q-1}\frac{T_0}{T}\right)^{\frac{1}{q-1}-1-s} \exp\left(-\frac{1}{q-1}\frac{T_0}{T}\right)$$ gamma distr.

$$h_q(E) = \int_0^\infty f(E)\, g(1/T)\, d(1/T) = \frac{2-q}{T_0}\left[1-(1-q)\frac{E}{T_0}\right]^{\frac{1}{1-q}}$$ Tsallis

$$q = 1 + \frac{Var(T)}{<T>^2}$$

GW and ZW, PRL **84** (2000) 2770
C.Beck, PRL **87** (2001) 180601

Generalized thermodynamic uncertainty relations

BG

$$U f_{T,N}(U) = N f_{T,U}(N) = \beta f_{U,N}(\beta) = \frac{(\beta U)^N}{\Gamma(N)} \exp(-\beta U)$$

T=const., N=const. or
U=const., N=const. or
T=const., U=const.

$$\omega_U^2 = \omega_T^2 = \omega_N^2 = < N >^{-1}$$

$$\omega_X^2 = \frac{Var(X)}{< X >^2}$$

N=const.

$$\omega_U^2 + \omega_T^2 = \frac{1}{< N >}$$

Linhard (1986)

U=const.

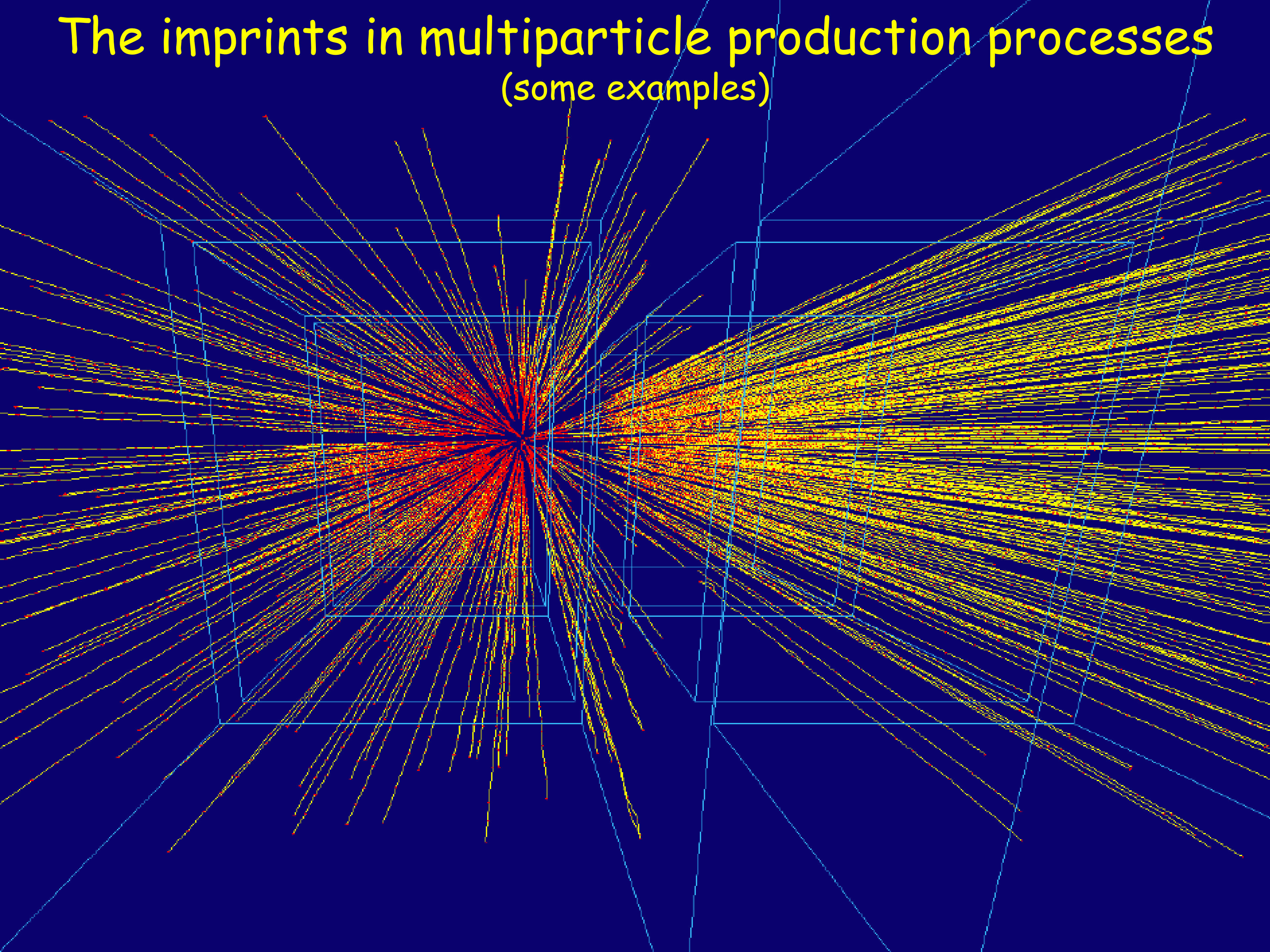
$$\omega_N^2 - \frac{1}{< N >} = \omega_T^2$$

GW, ZW (2007)

$$\left| \boldsymbol{\omega}_N^2 - \frac{\mathbf{1}}{\langle \mathbf{N} \rangle} \right| = \boldsymbol{\omega}_U^2 + \boldsymbol{\omega}_T^2 - 2\boldsymbol{\omega}_U \boldsymbol{\omega}_T \boldsymbol{\rho}$$

GW, ZW, Physica **A 390** (2011) 3566

The imprints in multiparticle production processes
(some examples)



$$\left|\omega_N^2 - \langle N \rangle^{-1}\right| = \omega_U^2 + \omega_T^2 - 2\rho\,\omega_U\,\omega_T$$

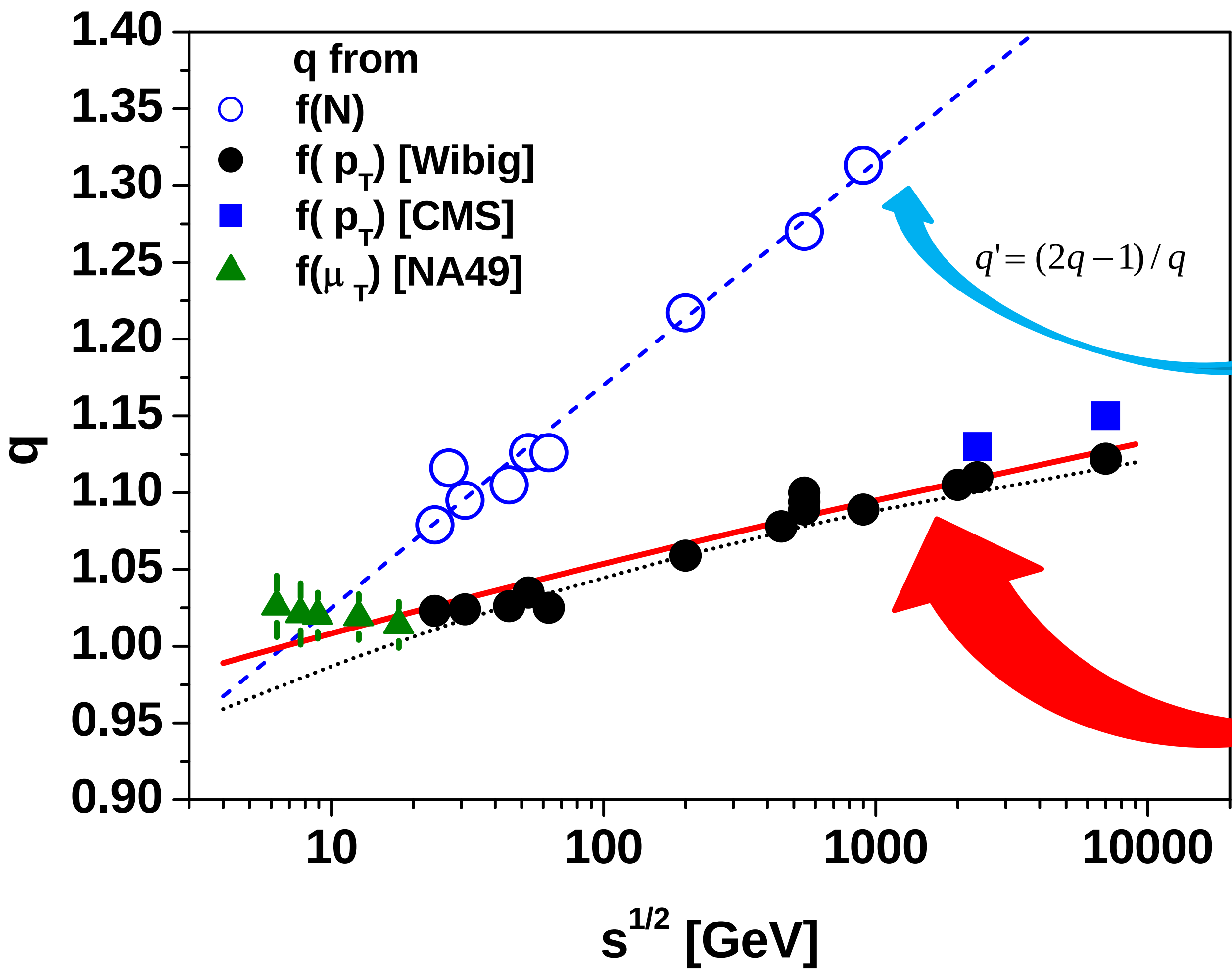
$\rho = 0$

$$q - 1 = \omega_U^2 + \omega_T^2$$

for

$$\omega_U^2 = \omega_T^2$$

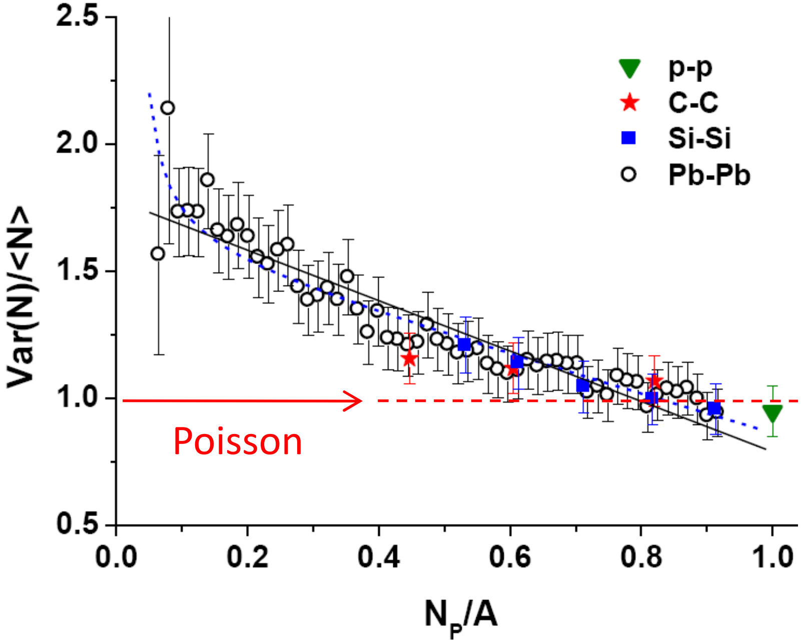
$$q_T - 1 = \frac{2}{3}\omega_T^2$$

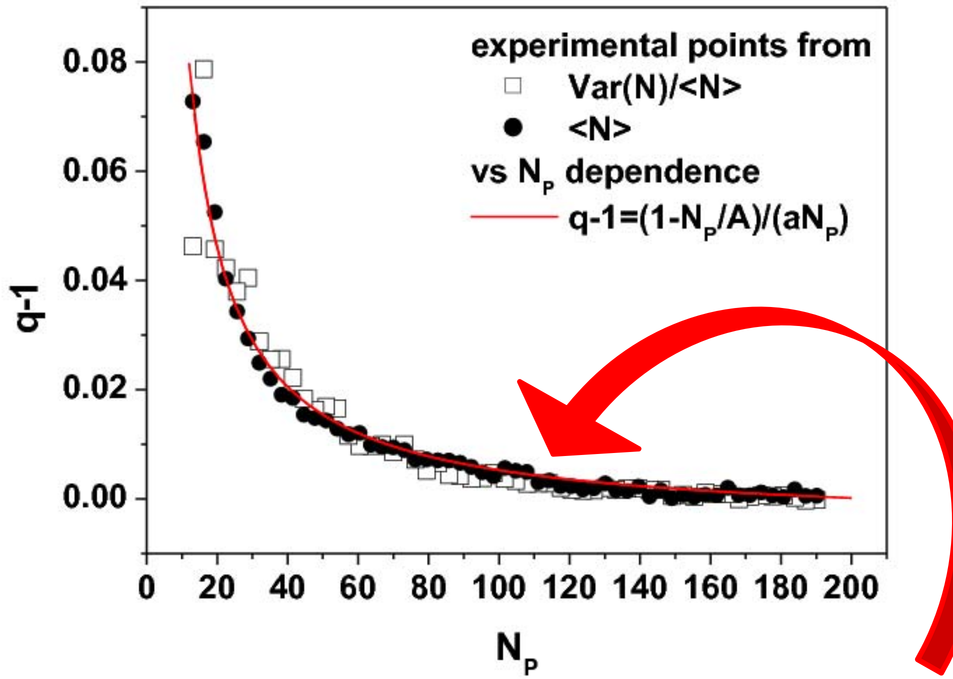


$$q' = (2q - 1)/q$$

$$q - 1 = 3(q_T - 1)$$

Multiplicity fluctuations in A+A collisions: *- system size dependence*





$$\frac{Var(N)}{\langle N\rangle} = 1 + \langle N\rangle (q-1) = 1 + \frac{n_0\left(1-\frac{N_P}{A}\right)}{a - \frac{c}{A}\left(\frac{A}{N_P}-1\right)}$$

$$q-1 = \frac{1}{aN_P}\left(1-\frac{N_P}{A}\right)$$

GW and WW, Phys.Rev.C 79 (2009) 054903

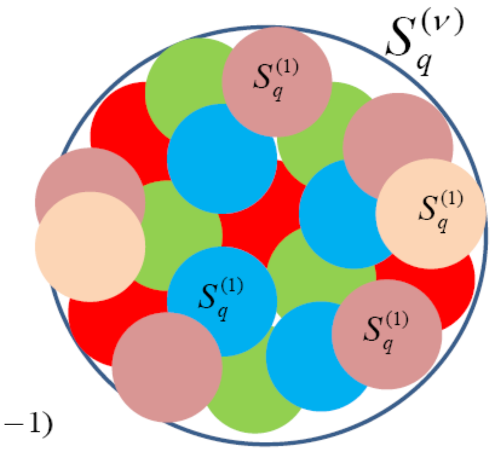
Non-additivity in nuclear collisions

Nucleus-nucleus collisions is treated as quasi-superposition of nucleon-nucleon collision

$$s_{\tilde{q}}^{(N)}=\sum_{k=1}^{N}\frac{N!}{(N-k)!k!}(1-\tilde{q})^{k-1}(s_{\tilde{q}}^{(1)})^{k}=\frac{[1+(1-\tilde{q})s_{\tilde{q}}^{(1)}]^{N}-1}{1-\tilde{q}}$$ N **particles**

$$S_{q}^{(\nu)}=\sum_{k=1}^{\nu}\frac{\nu!}{(\nu-k)!k!}(1-q)^{k-1}(S_{q}^{(1)})^{k}=\frac{[1+(1-q)S_{q}^{(1)}]^{\nu}-1}{1-q}$$ ν **nucleons**

$S_{\tilde{q}}^{(N)}=S_{q}^{(\nu)}$ **for** $\tilde{q}=q$ **superposition** $(N=\nu\cdot n)$

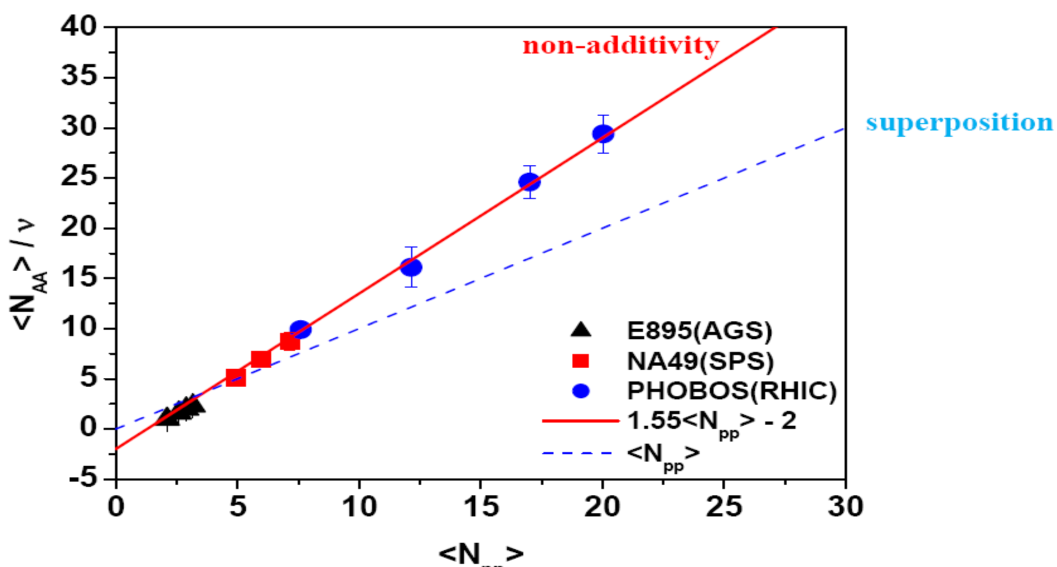


$$\ln\left(1+(1-q)S_{q}^{(\nu)}\right)=\nu\ln\left(1+(1-q)S_{q}^{(1)}\right)$$

$$1+(1-q)S_q=e^{\alpha(1-q)(N-1)}$$

$$S_q(N)=\frac{e^{\alpha(1-q)(N-1)}-1}{1-q}\xrightarrow[q\to 1]{}\alpha(N-1)$$

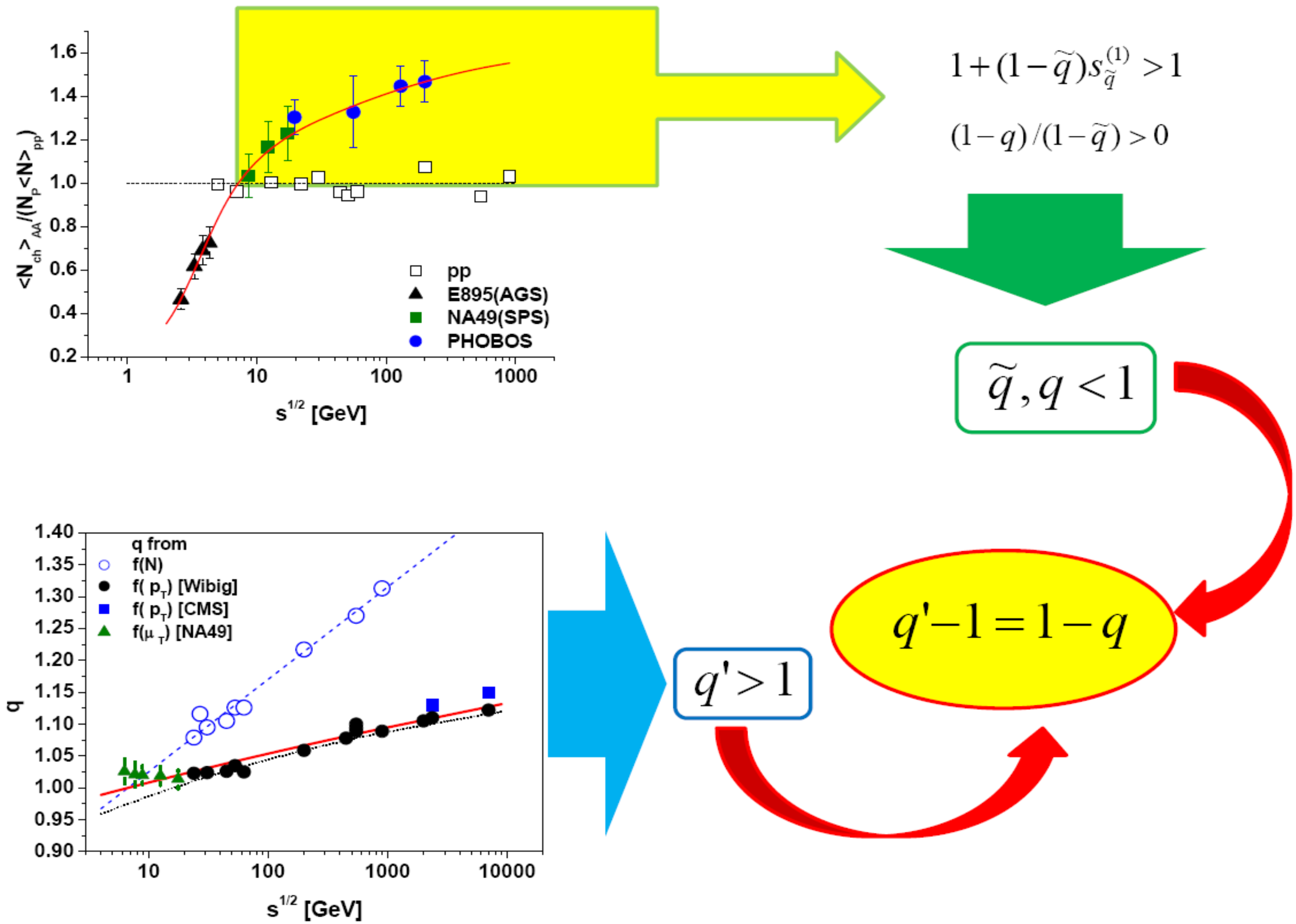
$$\frac{N_{(\nu)}}{\nu}=\frac{\alpha_{(1)}}{\alpha_{(\nu)}}N_{(1)}-\frac{\alpha_{(1)}}{\alpha_{(\nu)}}+\frac{1}{\nu}\approx aN_{(1)}-b$$



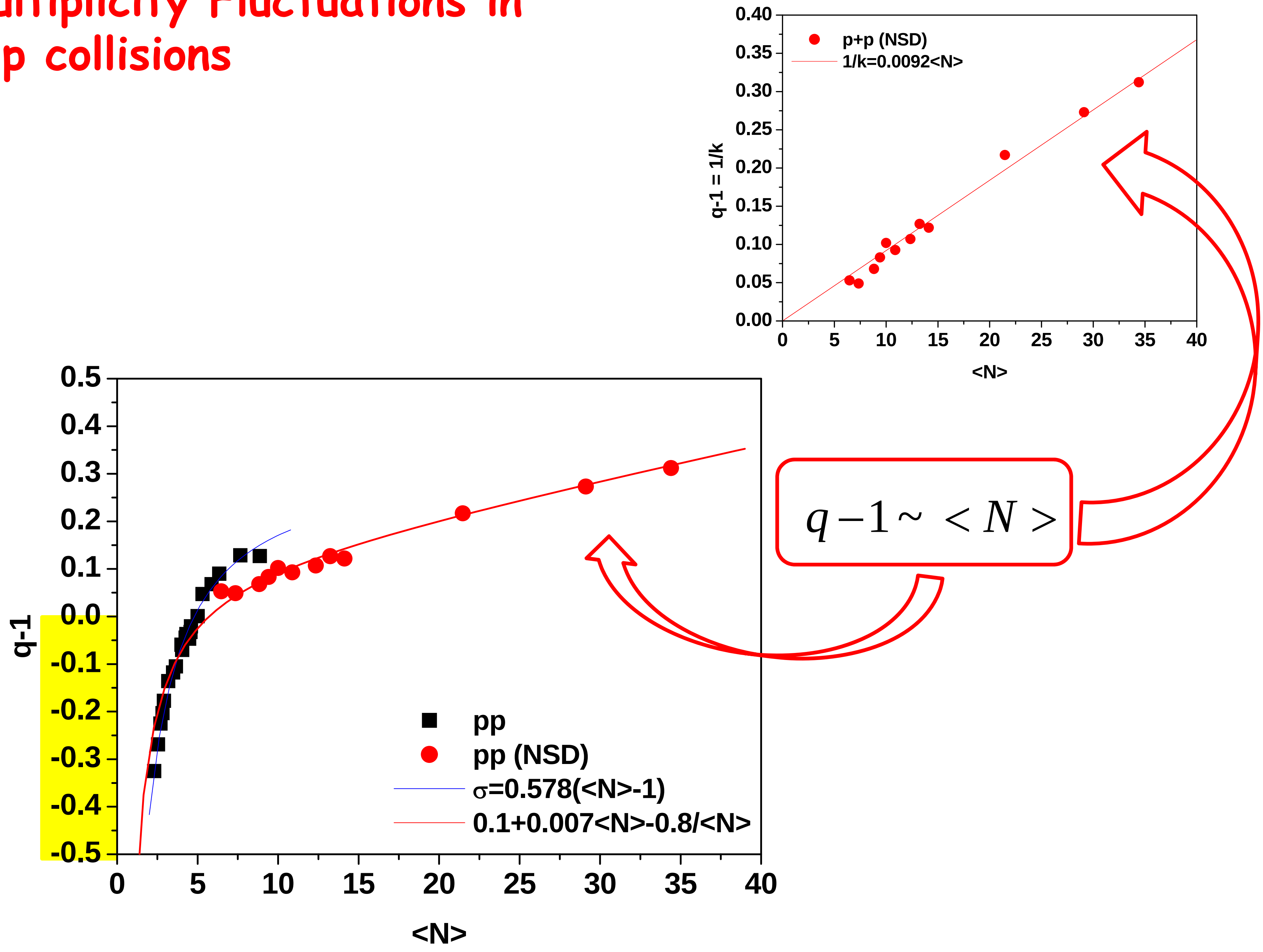
GW, ZW, CEJP **10** (2012)568

Non-additivity in nuclear collisions
(on the q puzzle in nonextensive statistics)

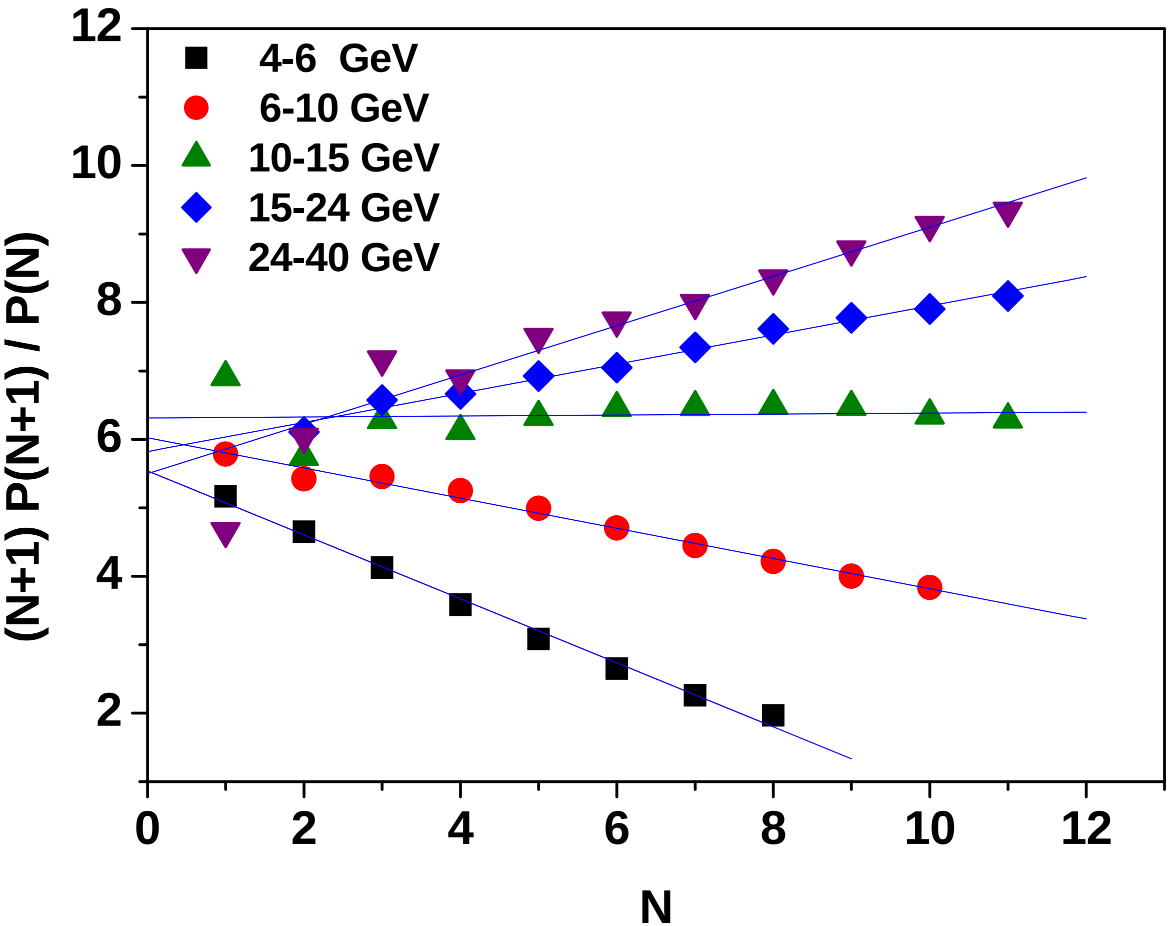
We advocate that for the standard ***Tsallis entropy*** *with degree of nonadditivity* $q<1$
the corresponding standard ***Tsallis distribution*** *is described by* $q' = 2-q > 1$
Nucleus-nucleus collisions (treated as quasi-superposition of nucleon-nucleon collision) confirm such conclusion.



Multiplicity Fluctuations in p+p collisions



Multiplicity distribution in jet events [ATLAS,7 TeV]



$$\frac{(N+1)P(N+1)}{P(N)} = a + bN$$

NBD: $a = \frac{<N> k}{k + <N>}$ $b = \frac{a}{k}$

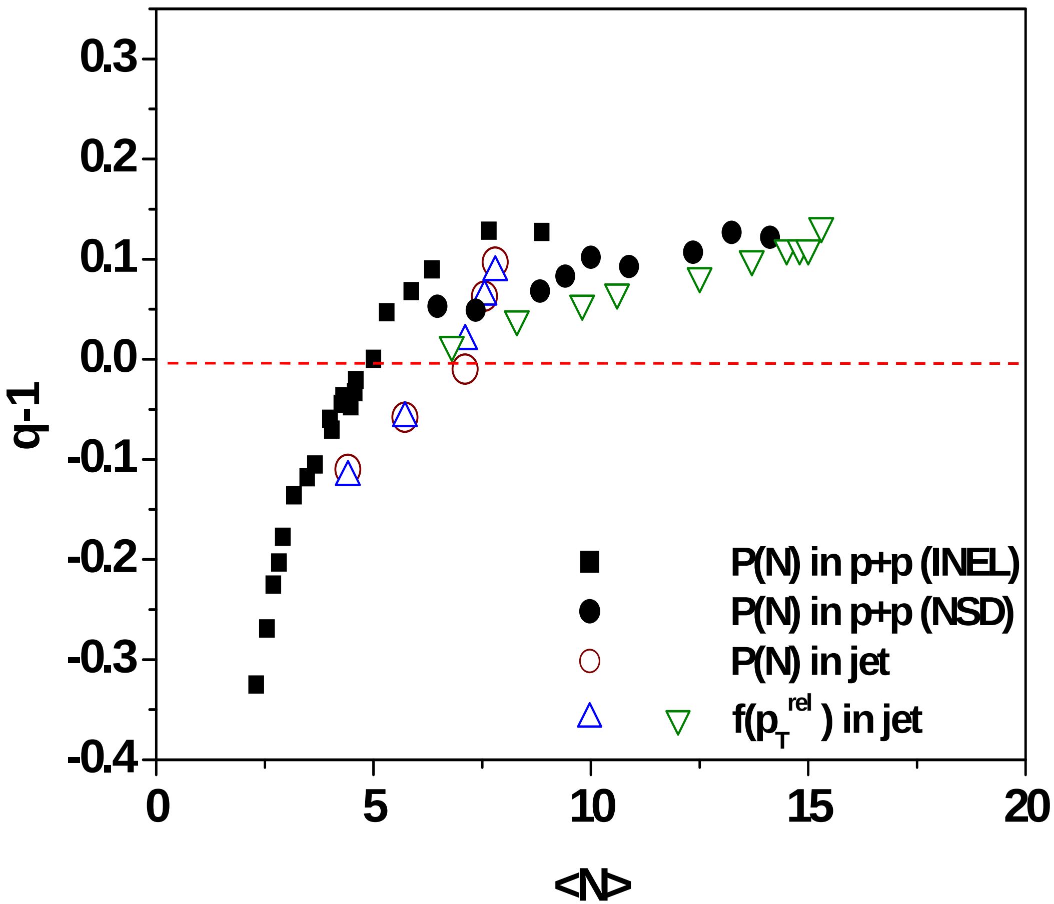
Poisson: $a = <N>$ $b = 0$

BD: $a = \frac{<N> \kappa}{\kappa - <N>}$ $b = \frac{a}{\kappa}$

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} (<N>/k)^N (1 + <N>/k)^{-k-N} \qquad 1/k = q - 1$$

$$P(N) = \frac{\Gamma(\kappa+1)}{\Gamma(N+1)\Gamma(\kappa - N + 1)} (<N>/\kappa)^N (1 - <N>/\kappa)^{\kappa-N} \qquad 1/\kappa = 1 - q$$

Self-similarity in jet events following from p-p collisions at LHC



The self-similarity of the scattering process was already recognized by Hagedorn [R. Hagedorn and R. Ranft, Suppl. Nuovo Cim. 6, 169 (1968)], who described the various possible particle states as a 'fireball' and who defined a fireball as follows:

A fireball is

***... a statistical equilibrium of an undetermined number of all kinds of fireballs, each of which in turn is considered to be...**

(back to *)

Clearly, nowadays we would call this a self-similarity assumption.

Also [G. Gustafson and A. Nilsson, Nucl. Phys. **B355**, 106, (1991)]

QCD predicts that parton fragmentation into final state hadrons proceeds through multiple sub-jet production. This cascade of jets to sub-jets to sub-sub-jets (et cetera) to final state hadrons should demonstrate self-similar behavior.

and [J.D. Bjorken, Phys. Rev. **D45**, 4077 (1992)]

In QCD extra gluons of lower-pt, scales can also be radiated. This provides new populations of jets, which again extend the entire lego plot, including the extensions we have exhibited. The self-similar character of this extension should be evident.

to be published in Phys.Lett.B (arXiv:1310.0671)

Concluding Remarks

For small systems: $P(E) \propto (1-(1-q')\beta E)^{1/(1-q')}$ → large N → $P(E) \propto \exp(-\beta E)$

with $q' = q'(N) = 1 - \frac{1}{\alpha N - 1} \leq 1$

if $q = 1 + \frac{Var(T)}{<T>^2}$ then $q = q(N) = 1 + \frac{\kappa}{N} \geq 1$

$$\frac{Var(U)}{<U>^2} = \boxed{\frac{1}{N}} + \boxed{\frac{q(N)-1}{3-2q(N)}\left(1+\frac{1}{N}\right)}$$

$$\frac{Var(T)}{<T>^2} = \boxed{0} + \boxed{[q(N)-1]}$$

$$\frac{Var(N)}{<N>^2} = \boxed{\frac{1}{<N>}} + \boxed{[q(N)-1]}$$

$$\frac{1}{C_V} = \boxed{\frac{1}{C_V^{(B)}}} + \boxed{\frac{1}{C^{(f)}}}$$

$C_V^{(B)} = kN$

$C^{(f)} = kN/(N+1)\frac{3-2q}{q-1}$

BG + **N dependent term** $\xrightarrow[N\to\infty]{} 0$

$q(N) - 1 = \frac{\kappa}{N}$

$\exp_q$ is fruitful description of the femtosystems

- describe deviation from Boltzman formula
- fluctuations of the scale parameters are fully accounted by *q* value
- successfully connect fluctuations observed in U, T and N
- desribe deviation from $Var(X)/\langle X\rangle^2 \sim 1/N$ relation for $q>1$ as well as $q'<1$

Grazie molte

Thank you for you attention