Hydrodynamics and the quantum stress-energy and spin tensors

Outline

- Motivations and introduction
- Thermodynamical inequivalence of microscopic tensors: equilibrium
- Thermodynamical inequivalence of microscopic tensors: non equilibrium

Leonardo Tinti

UJK w Kielcach

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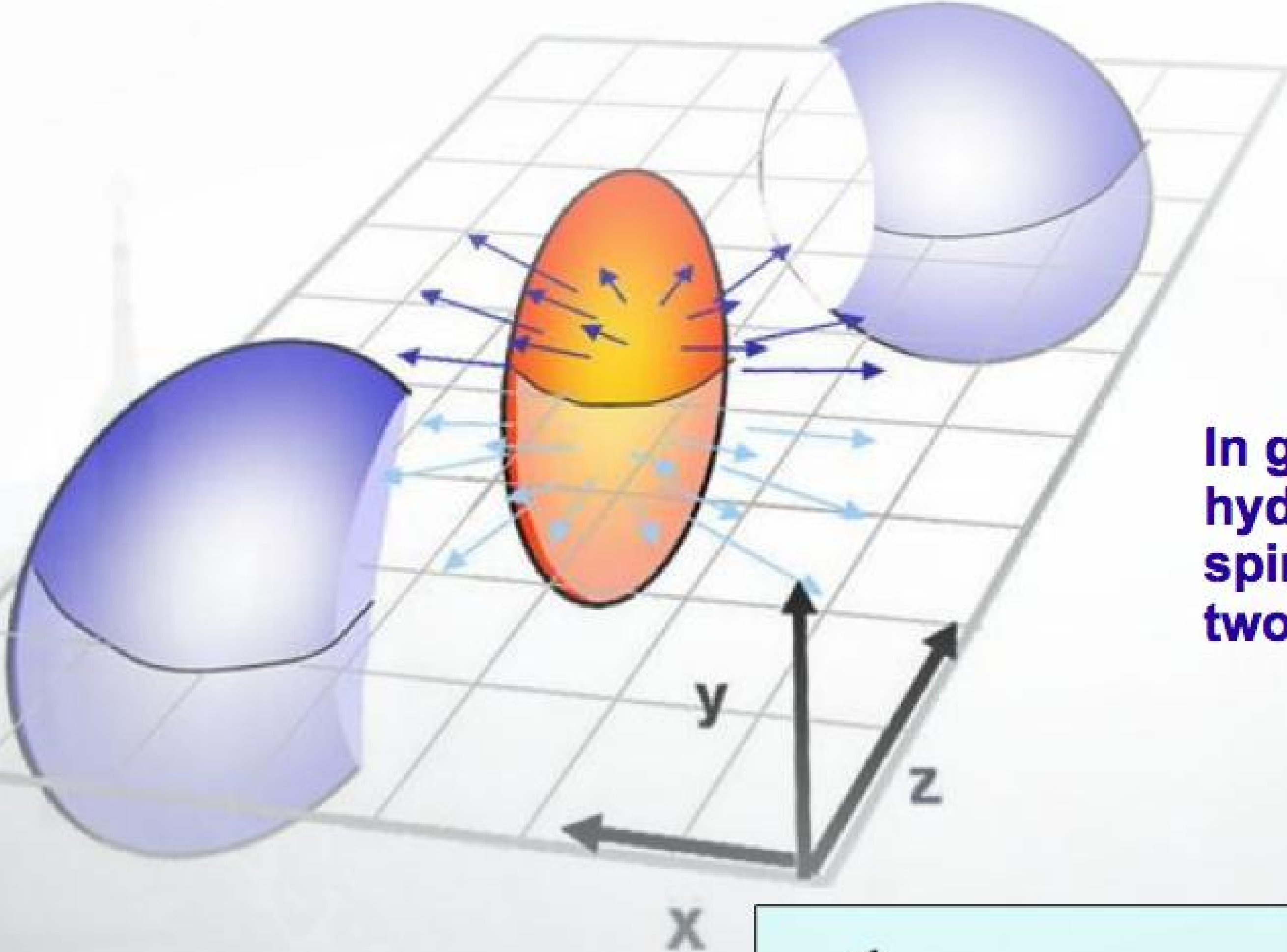
Motivations



In general, relativistic hydrodynamics with spin is described by two equations:

SPIN TENSOR

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\lambda \mathcal{S}^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu} \end{cases}$$

Relation between macroscopic (classical) and microscopic quantum observables

$$T^{\mu\nu}(x) = \mathrm{tr}\left(\widehat{\rho}\ :\widehat{T}^{\mu\nu}(x):\right)$$

$$\frac{i}{2}\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\Psi$$

$$-\frac{1}{4m}\overline{\Psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\Psi$$

$$\frac{i}{4}\left[\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\Psi + (\mu\leftrightarrow\nu)\right]$$

And the spin tensor

$$\mathcal{S}^{\lambda,\mu\nu}(x) = \mathrm{tr}\left(\widehat{\rho}\ :\widehat{\mathcal{S}}^{\lambda,\mu\nu}(x):\right)$$

Noether's theorem give us canonical stress-energy and spin tensors

From space-time translation invariance:

$$\widehat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \widehat{\phi}_i \right)} \partial^\nu \widehat{\phi}_i - \mathcal{L} g^{\mu\nu} \qquad \partial_\mu \widehat{T}^{\mu\nu} = 0$$

and Lorentz group:

$$\widehat{\mathcal{J}}_{\lambda,\mu\nu} = -i \frac{\partial \mathcal{L}}{\partial \left(\partial^\lambda \widehat{\phi}_i \right)} \left(\Sigma_{\mu\nu} \right)_i^j \widehat{\phi}_j + x_\mu \widehat{T}_{\lambda\nu} - x_\nu \widehat{T}_{\lambda\mu}$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = 0$$

$$\widehat{\mathcal{S}}_{\lambda,\mu\nu} = -i \frac{\partial \mathcal{L}}{\partial \left(\partial^\lambda \widehat{\phi}_i \right)} \left(\Sigma_{\mu\nu} \right)_i^j \widehat{\phi}_j$$

Pseudo-gauge transformations with a *superpotential* $\widehat{\Phi}$

F.W. Hehl, Rep. Mat. Phys. 9 (1976) 55

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \widehat{Z}^{\alpha\lambda,\mu\nu}$$

With (we confine ourselves to a vanishing $\widehat{Z}$):

$$\int_{\partial V} \mathrm{d}S \, \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) n_i = 0$$

$$\int_{\partial V} \mathrm{d}S \, \left[x^\mu \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) - x^\nu \left(\widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) \right] n_i = 0$$

They leave the conservation equations and spatial integrals (generators, or total energy, momentum and angular momentum) invariant.

This seems to be enough for a quantum relativistic field theory. It is not in gravity but, as long as we disregard it, different couples of tensors related by a pseudo-gauge transformation cannot be distinguished.

Inequivalence, equilibrium

Necessary condition for equivalence

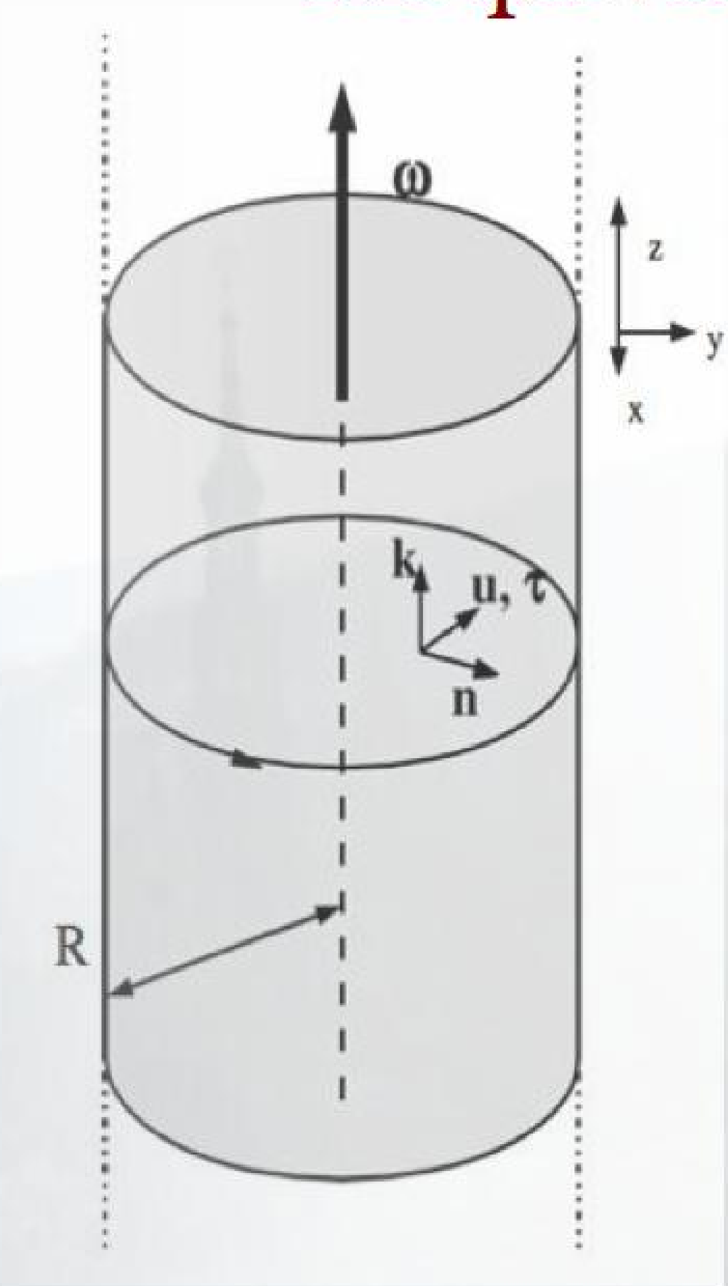
Four momentum density and angular momentum density are in principle observable

$$
\begin{aligned}
T'^{0\mu}(x) &= T^{0\mu}(x) \\
\mathcal{J}'^{0,ij}(x) &= \mathcal{J}^{0,ij}(x)
\end{aligned}
$$

to be true in any inertial frame, it means

$$
\begin{aligned}
&T'^{\mu\nu} = T^{\mu\nu} \\
&\mathcal{J}'^{\lambda,\mu\nu} = \mathcal{J}^{\lambda,\mu\nu} + g^{\lambda\mu}K^{\nu} - g^{\lambda\nu}K^{\mu}
\end{aligned}
$$

Inequivalence, equilibrium



The *grand-canonical ensemble*, because of its symmetries, always fulfills the equivalence conditions

$$\widehat{\rho} = \frac{1}{Z}\exp(-\widehat{H}/T + \mu\widehat{Q}/T)$$
$$Z = \mathrm{tr}[\exp(-\widehat{H}/T + \mu\widehat{Q}/T)]$$

The situation is very different in a less symmetric case

$$\widehat{\rho} = \frac{1}{Z_\omega}\mathrm{P}_V\exp(-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\boldsymbol{J}}/T + \mu\widehat{Q}/T)$$
$$Z_\omega = \mathrm{tr}[\mathrm{P}_V\exp(-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\boldsymbol{J}}/T + \mu\widehat{Q}/T)]$$

- F.Becattini, L. Tinti, Phys. Rev. D 84, 025013 (2011)

An example: the free Dirac field in a cylinder

A comparison between the canonical and Belinfante couple of tensors

$$\mathcal{S}^{0,ij} = D(r)\epsilon_{ijk}\hat{k}^k$$

$$T^{0i}_{\text{Belinfante}} = T^{0i}_{\text{canonical}} - \frac{1}{2}\frac{\mathrm{d}D(r)}{\mathrm{d}r}\hat{v}^i$$

$$\boldsymbol{\mathcal{J}}_{\text{Belinfante}} = \boldsymbol{\mathcal{J}}_{\text{canonical}} - \left(\frac{1}{2}r\frac{\mathrm{d}D(r)}{\mathrm{d}r} + D(r)\right)\hat{\mathbf{k}}$$

The momentum density and/or angular momentum density differ in the canonical or Belinfante case if D(r) is non-vanishing

To prove the inequivalence we calculated analytically $\mathrm{tr}[\widehat{\rho} : \widehat{\mathcal{S}} :]$ in a cylinder with finite radius

$$D(r) = \mathrm{tr}_V[\widehat{\rho} : \Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}) :] \neq 0$$

An example: the free Dirac field in a cylinder

Free Dirac equation in a cylinder (with MIT boundary conditions) solved analytically recently:
E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian and A. S. Tarloyan, Phys. Rev. D 78, 105007 (2008).

$$i\gamma^\mu \partial_\mu \Psi - m\Psi = 0$$

$$\Sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathrm{tr}_V[\widehat{\rho} : \Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}) :] = D(r)$$
$$= \sum_M \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \left[\frac{1}{\mathrm{e}^{(\varepsilon - M\omega+\mu)/T}+1} + \frac{1}{\mathrm{e}^{(\varepsilon - M\omega-\mu)/T}+1}\right] \frac{p_{Tl}^2\left[J^2_{|M-\frac{1}{2}|}(p_{Tl}r) - b_\xi^{(+)^2} J^2_{|M+\frac{1}{2}|}(p_{Tl}r)\right]}{4\pi R J^2_{|M-\frac{1}{2}|}(p_{Tl}R)(2Rm_{Tl}^2 + 2\xi M m_{Tl} + m)}$$

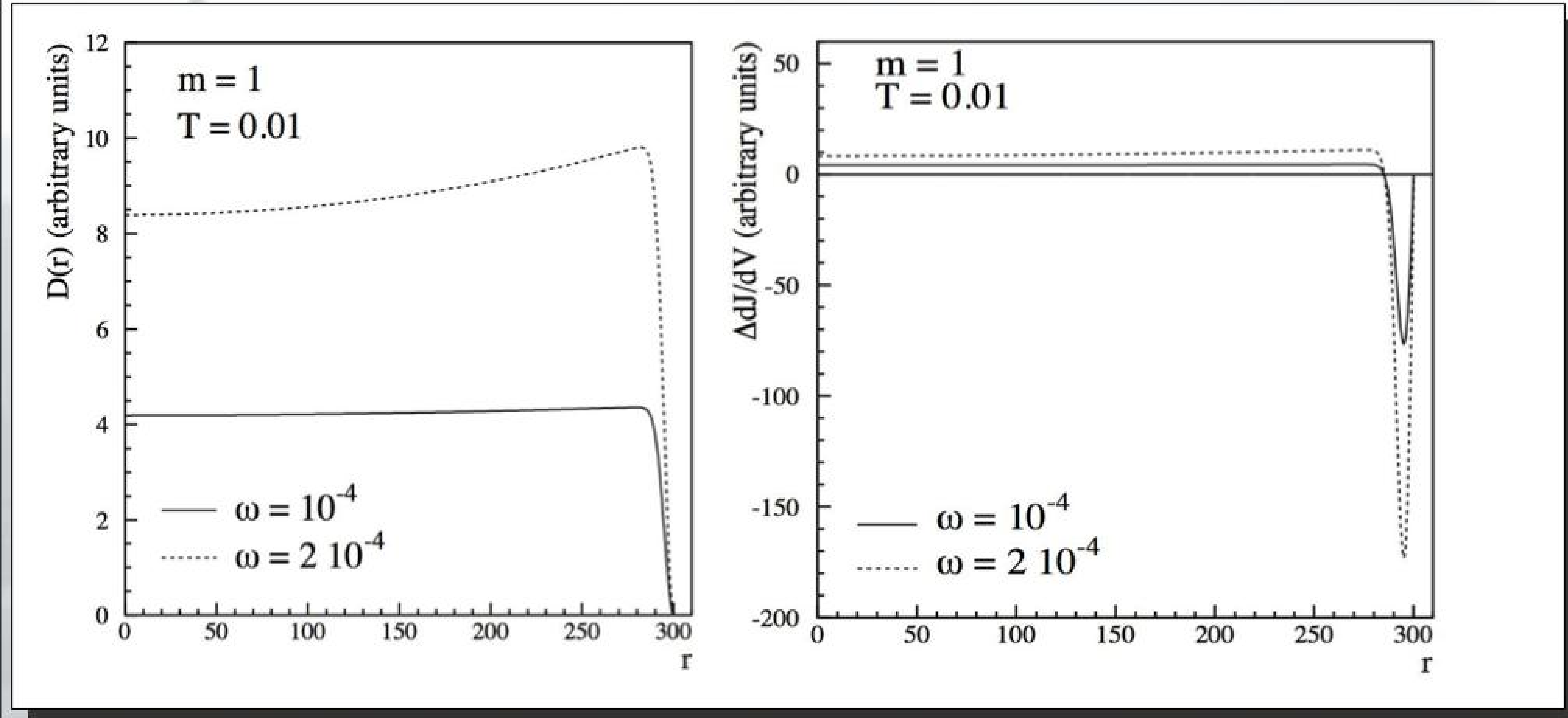
We have proved that this expression is non-vanishing

D(r) in the non-relativistic limit

It is the sum of a particle and antiparticle term:

$$D(r)^{\pm} = \hbar \mathrm{tr}\left[\widehat{\rho}\left(:\Psi^{\dagger}\Sigma_z\Psi:\right)^{\pm}\right] \simeq \frac{1}{2}\frac{\hbar\omega}{KT}\hbar \mathrm{tr}\left[\widehat{\rho}\left(:\Psi^{\dagger}\Psi:\right)^{\pm}\right] = \hbar\frac{1}{2}\frac{\hbar\omega}{KT}\left(\frac{\mathrm{d}n}{\mathrm{d}^3\mathbf{x}}\right)^{\pm}$$

we can make a numerical computation of the D(r) function:



Transport coefficients are determined with the method of Linear response Theory applied to the Non-Equilibrium Stationary Density Operator (Zubarev) which lead to relativistic Kubo formulae

Zubarev D N 1974 *Nonequilibrium Statistical Thermodynamics* (New York: Plenum Press)

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{\Upsilon}] = \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}\left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)\right]$$

Additional term, necessary if we consider a generic pair of stress-energy and spin tensors

$$\eta = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}\ \mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\,\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\left[\widehat{T}_S^{12}(x),\widehat{T}_S^{12}(0)\right]\rangle_0$$

- F.Becattini, L. Tinti, Phys.Rev. D87 (2013) 025029

Zubarev operator is NOT invariant under a change of the quantum tensors

$$\Delta\langle\widehat{O}\rangle \simeq -\lim_{\varepsilon\to 0}\frac{T}{2i}\int_{-\infty}^{t'} \mathrm{d}t\; \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}\; \langle[\widehat{\Phi}^{\lambda,0\nu}(x),\widehat{O}]\rangle_0(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x))$$

The mean values of ALL observables in non-equilibrium depend on the choice of the stress energy and spin tensor.

Entropy production rate is not invariant:

$$\begin{aligned}S'(t) \simeq S(t) \; &+ \frac{1}{2}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'\; \left(\langle\widehat{\Phi}^{\lambda,0\nu}(x)\,\widehat{T}^{0\mu}(x')\rangle_{\widehat{\Upsilon}} - \langle\widehat{\Phi}^{\lambda,0\nu}(x)\rangle_{\widehat{\Upsilon}}\langle\widehat{T}^{0\mu}(x')\rangle_{\widehat{\Upsilon}}\right)\beta_\mu(x')(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x))\\ &- \frac{1}{2}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'\; \left(\langle\widehat{\Phi}^{\lambda,0\nu}(x)\,\widehat{j}^{0}(x')\rangle_{\widehat{\Upsilon}} - \langle\widehat{\Phi}^{\lambda,0\nu}(x)\rangle_{\widehat{\Upsilon}}\langle\widehat{j}^{0}(x')\rangle_{\widehat{\Upsilon}}\right)\xi(x')(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x))\\ &- \frac{1}{4}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'\; \left(\langle\widehat{\Phi}^{\lambda,0\nu}(x)\,\widehat{\mathcal{S}}^{0,\rho\sigma}(x')\rangle_{\widehat{\Upsilon}} - \langle\widehat{\Phi}^{\lambda,0\nu}(x)\rangle_{\widehat{\Upsilon}}\langle\widehat{\mathcal{S}}^{0,\rho\sigma}(x')\rangle_{\widehat{\Upsilon}}\right)\omega_{\rho\sigma}(x')(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x))\end{aligned}$$

Conserved charges however don't change at the lowest order, as long the commutator with the equilibrium density matrix is vanishing

$$\mathrm{tr}(\widehat{\rho}_0[\widehat{\Phi}^{\lambda,0\nu},\widehat{O}]) = \mathrm{tr}(\widehat{\Phi}^{\lambda,0\nu}[\widehat{O},\widehat{\rho}_0])$$

Transport coefficients: shear vscosity

- ***See also*** Y. Nakayama, Int. J. Mod. Phys. A 27 (2012) 1250125

A change of the stress-energy tensor:

$$\widehat{T}'^{\mu\nu}_S = \widehat{T}^{\mu\nu}_S - \frac{1}{2}\partial_\lambda(\widehat{\Phi}^{\mu,\lambda\nu} + \widehat{\Phi}^{\nu,\lambda\mu}) = \widehat{T}^{\mu\nu}_S - \partial_\lambda \widehat{\Xi}^{\lambda\mu\nu}$$

Reflects into a change of shear viscosity

$$\begin{aligned}\Delta\eta = \eta' - \eta = & - \lim_{k\to 0} \int_V \mathrm{d}^3\mathrm{x}\, \cos kx^1 \langle \Big[\widehat{\Xi}^{012}(0,\mathbf{x}), \widehat{\Xi}^{012}(0,\mathbf{0})\Big]\rangle_0 \\ & - 2\lim_{\varepsilon\to 0}\lim_{k\to 0} \,\mathrm{Im} \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{e}^{\varepsilon t} \int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{ikx^1} \langle \Big[\widehat{\Xi}^{012}(x), \widehat{T}^{12}_S(0,\mathbf{0})\Big]\rangle_0\end{aligned}$$

Note that the change of shear viscosity is not compensated by a change of another transport coefficient so as to maintain the same entropy production rate: also entropy changes.

An example for spinor electrodynamics

Symmetrized interacting gauge-invariant stress-energy tensor:

$$\widehat{T}^{\mu\nu} = \frac{i}{4}\left(\overline{\Psi}\gamma^{\mu}\overset{\leftrightarrow}{\nabla}^{\nu}\Psi + \overline{\Psi}\gamma^{\nu}\overset{\leftrightarrow}{\nabla}^{\mu}\Psi\right) + \widehat{F}^{\mu}{}_{\lambda}\widehat{F}^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}\widehat{F}^{2}$$

Add a gauge-invariant superpotential (De Groot Relativistic kinetic theory):

$$\widehat{\Phi}^{\lambda,\mu\nu} = \frac{1}{8m}\overline{\Psi}\left(\gamma^{\mu}\overset{\leftrightarrow}{\nabla}^{\nu} - \gamma^{\nu}\overset{\leftrightarrow}{\nabla}^{\mu}\right)\gamma^{\lambda}\Psi + \mathrm{h.c} = \frac{1}{8m}\overline{\Psi}\left([\gamma^{\mu},\gamma^{\lambda}]\overset{\leftrightarrow}{\nabla}^{\nu} - [\gamma^{\nu},\gamma^{\lambda}]\overset{\leftrightarrow}{\nabla}^{\mu}\right)\Psi$$

Then:

$$\widehat{\Xi}^{\lambda\mu\nu} = \frac{1}{16m}\overline{\Psi}\left([\gamma^{\lambda},\gamma^{\mu}]\overset{\leftrightarrow}{\nabla}^{\nu} + [\gamma^{\lambda},\gamma^{\nu}]\overset{\leftrightarrow}{\nabla}^{\mu}\right)\Psi$$

The resulting variation of the shear viscosity:

$$\frac{\Delta\eta}{\eta} \approx \mathcal{O}\left(\frac{\hbar}{mc^{2}\tau}\right)$$

τ is the microscopic time scale

Summary & outlook

- Thermodynamics implies an inequivalence of the stress-energy and spin tensors of relativistic quantum field theories
- The differences between observable physical quantities which are dependent on the particular form of these tensors are a quantum effect which, at least in principle, could be measured
- An experiment aimed at measuring with great accuracy these observable (transport coefficients, entropy production rate etc.) could rule out a particular stress-energy tensor
- NEXT: devise a thought experiment and investigate possible phenomenological consequences