Resurgence in quantum field theory: dealing with the Devil's invention

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The dark side of perturbation theory

In QFT with small coupling λ, observables computable as

$$\mathcal{O}(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots$$

But in interesting QFTs like QCD, $c_n \sim n!$ for large n

Dyson 1952

Perturbation theory yields divergent asymptotic series!

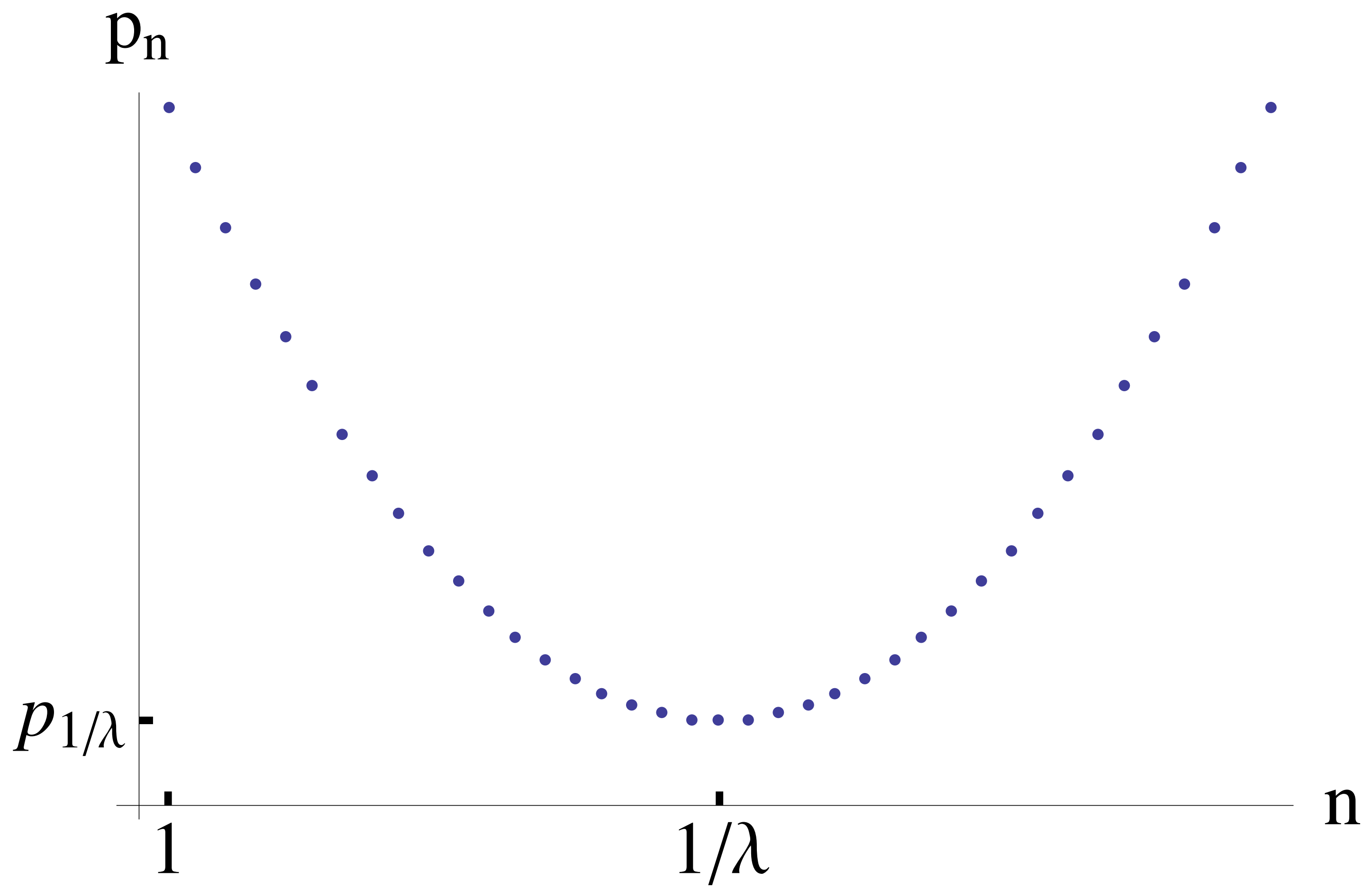
"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever… For the most part, the results are valid, it is true, but it is a curious thing. I am looking for the reason, a most interesting problem."

- Niels Henrik Abel 1828

N. H. Abel, 1802-1829

Common approach to asymptotic series

$$\sum_{n=1}^{\infty} p_n \lambda^n$$



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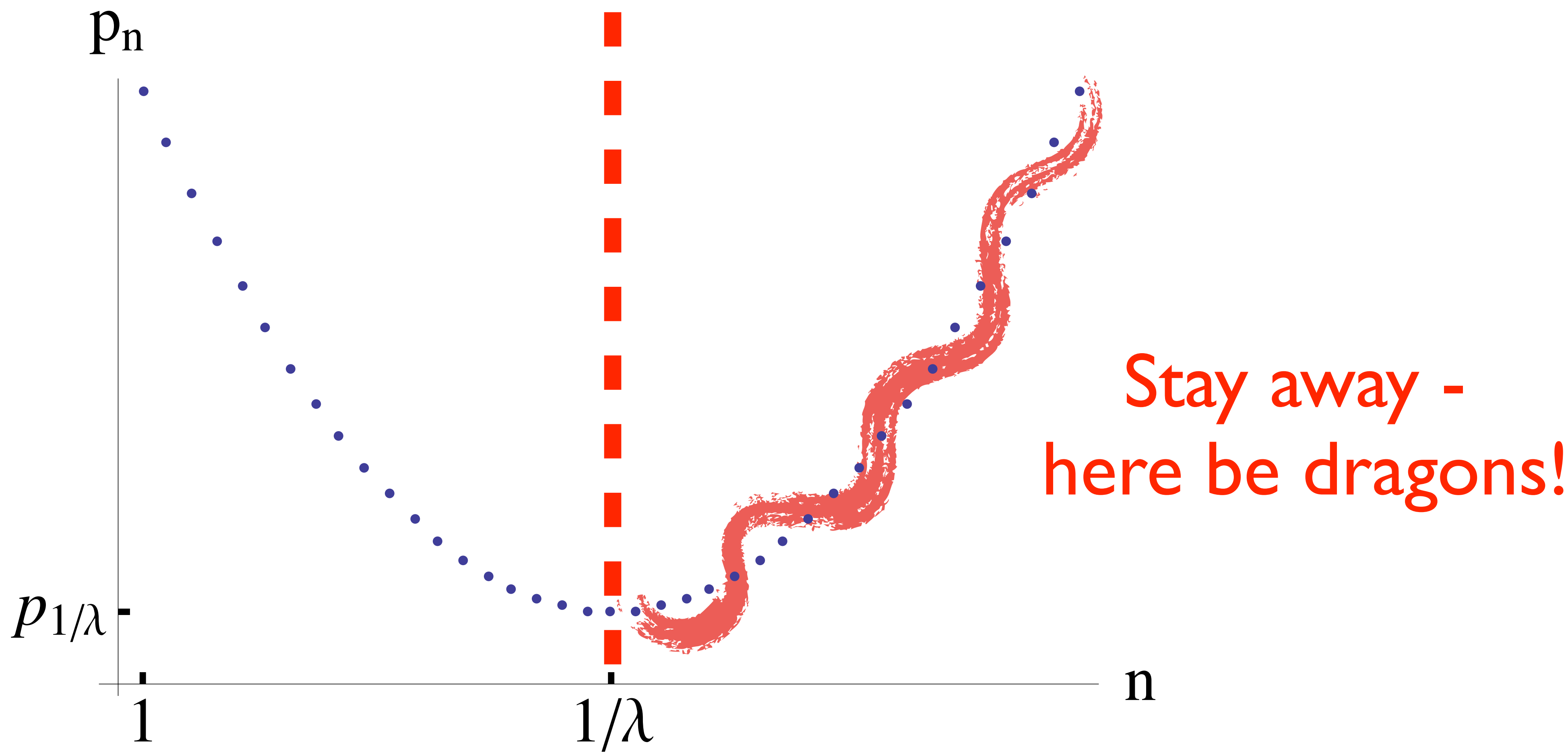

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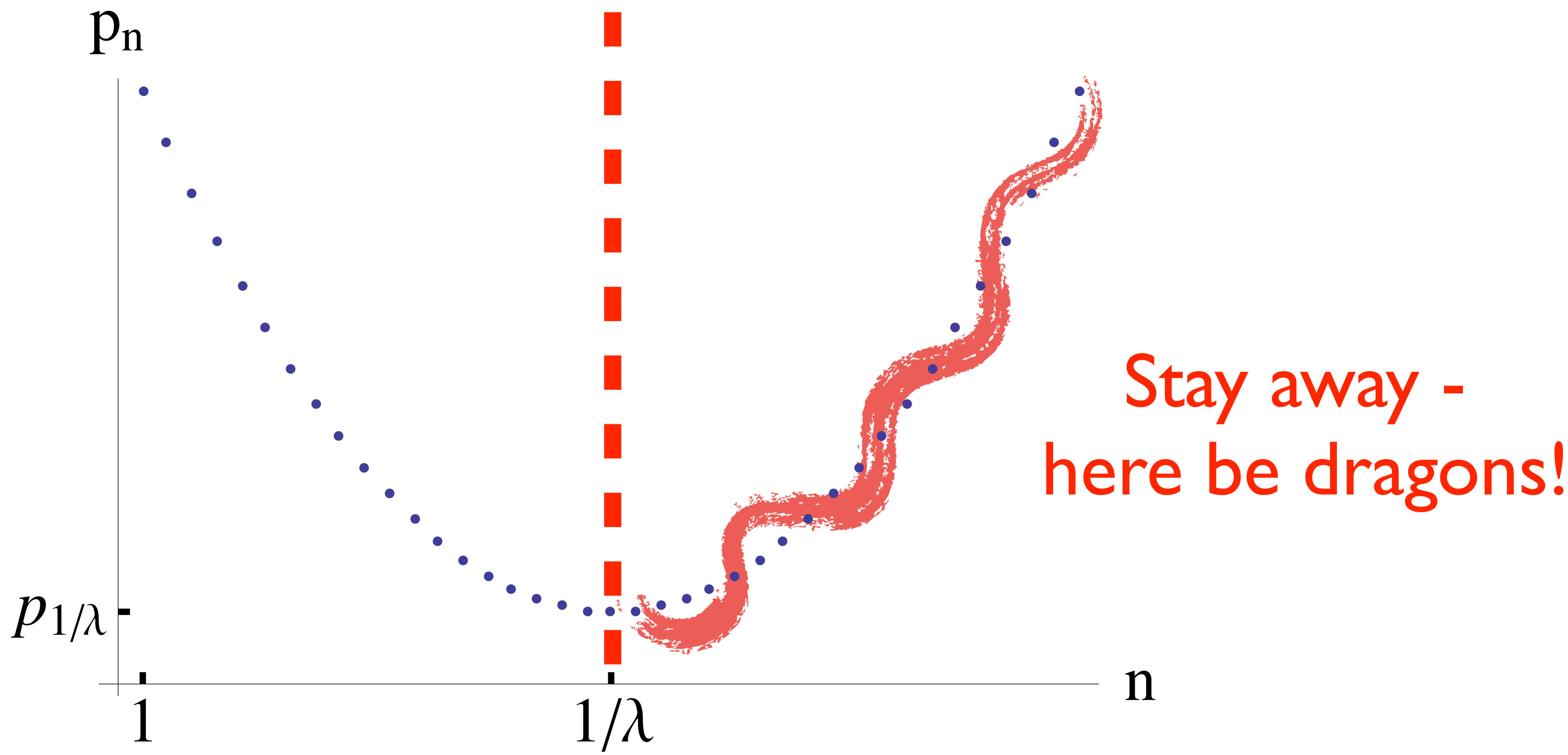


This is hardly systematic!

A more systematic approach would be nice - and with recent developments, it is finally possible…

Result is some deep insight into non-perturbative structure of path integrals in QFTs

Perturbation theory as a semiclassical expansion

$$\langle \mathcal{O}[\lambda] \rangle = Z[\lambda]^{-1} \int d[U]\, e^{-S(U;\lambda)} \mathcal{O}$$ regularized path integral

Use saddle-point approximation for small λ :

$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n} \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n} \lambda^n$$

Usually *all* of these series are sick, suffer from divergences!

Smooth well-defined <O(λ)> requires intricate relations connecting $p_{c,n}$ for different saddles

$\Rightarrow$ relations between perturbative and non-perturbative physics

Vainshtein, 1964; Bender+Wu 1969…

Resurgence theory is the detailed implementation of this idea

Dingle, Berry 1960+...

Ecalle: 1980s

Argyres, Unsal: YM
Dunne, Unsal: CP^N

Borel sum technology

Given
$$\mathcal{O} = \sum_{n=1}^{\infty} p_n \lambda^n, \quad p_n \sim n!$$

Define
`Borel transform'
$$B[\mathcal{O}](t) \equiv \sum_{n=1}^{\infty} \frac{p_n}{(n-1)!} t^{n-1}$$

BO(t) has finite radius of convergence, analytic near t=0

Borel
representation:
$$\tilde{\mathcal{O}}(\lambda) = \frac{1}{\lambda} \int_0^{\infty} dt\, e^{-t/\lambda} B[\mathcal{O}](t)$$

Doing integral term by term just gives O(λ)

Integrating all at once gives unique
smooth function, assuming integral exists

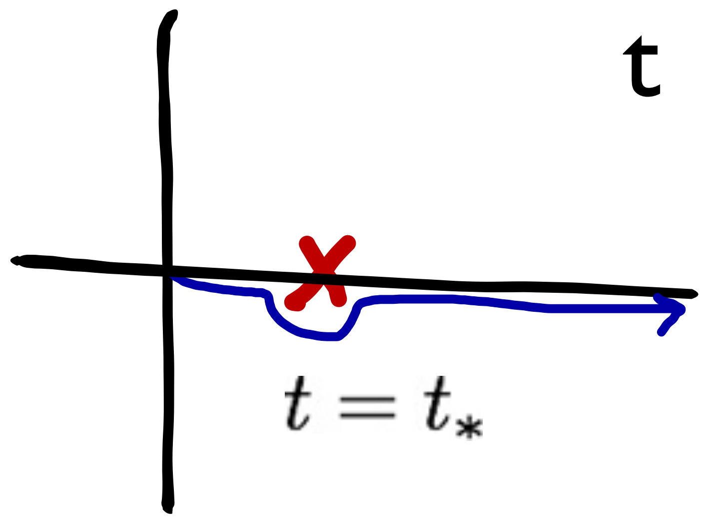
Singularities in BO(t) <~> factorial divergences in original series

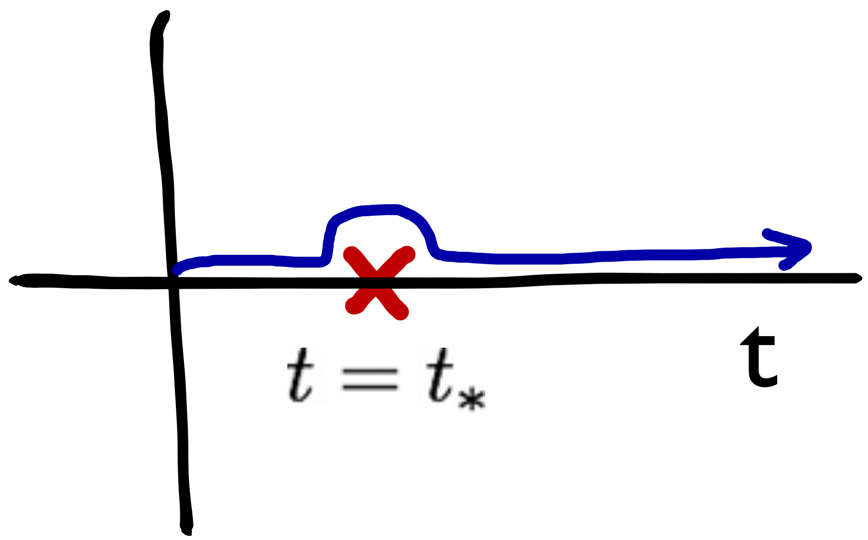
Singularity at t = t* means $p_n \sim n!\ (1/t^*)^n$

Borel sum technology

What if integrand has a pole on the integration contour C = R$^+$?

Can try to deform contour to avoid it - but two choices on how to do it...





Equivalently, can avoid singularity by shifting $\lambda \to \lambda(1 \pm i\epsilon)$

Leads to imaginary non-perturbative ambiguity in resummation

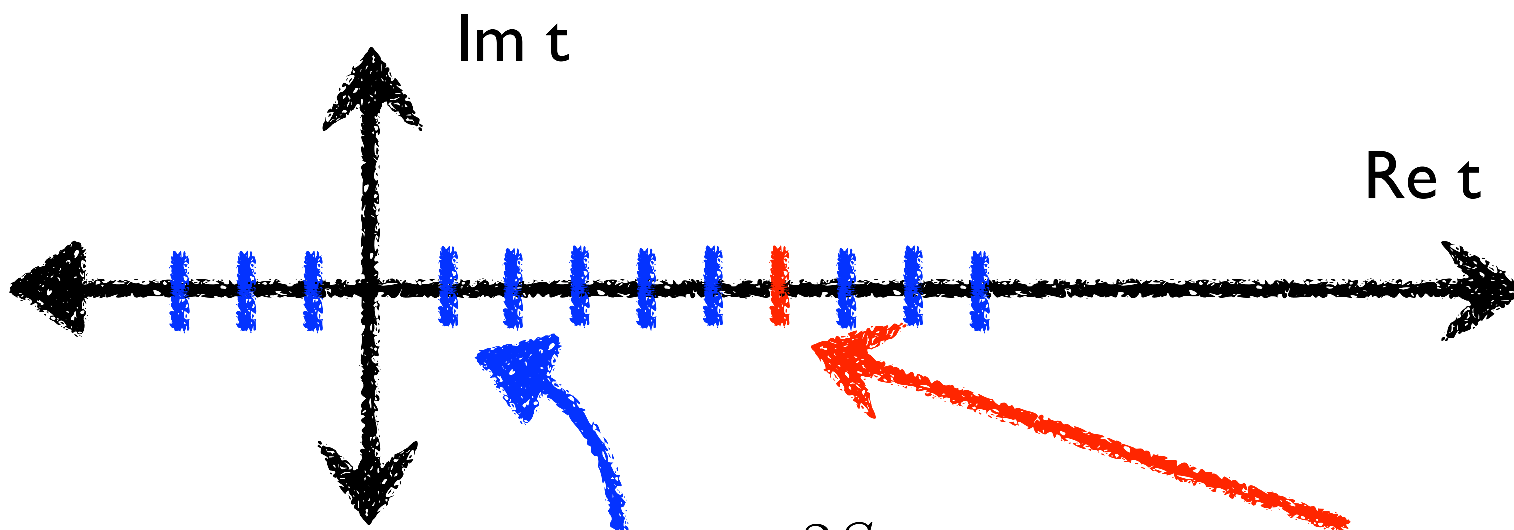
$$\tilde{\mathcal{O}}_{\pm}(\lambda) = \mathrm{Re}\left[\tilde{\mathcal{O}}(\lambda)\right] \pm 2\pi i\, e^{-t_*/\lambda}$$

Looks like contribution from 'instanton' with action t_*

Smallest t_* associated to largest ambiguity

Borel plane singularities for QCD

't Hooft, 1979



instanton contribution?

't Hooft's IR renormalons

$$t = 2S_I/|\beta_0| \Rightarrow e^{-\frac{2S_I}{|\beta_0|g^2}} \qquad t = 2S_I \Rightarrow e^{-2S_I/g^2}$$

$\beta_0 \sim N$, so fractional instanton contribution?

Renormalon ambiguity >> instanton ambiguity

Renormalons cause issues in pQCD calculations for e.g. collider physics

effect parametrized by introducing phenomenological 'power corrections'

Borel singularities for QCD and its relatives

't Hooft's dream: renormalons associated to some kind of fractional instantons, related to confinement

No such configurations known in QCD on R^4

Related issue: expansion in coupling constant generally does not make sense in QCD

Strong coupling in IR makes semiclassical approach problematic

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What is to be done to make semiclassics sensible?

(1) Put theory in finite spatial box

(2) For small volumes, weak coupling guaranteed

(3) Devise adiabatic small volume limit

Argyres, AC, Dorigoni, Dunne, **Unsal**, Yaffe 2008-now

$\boxed{\text{Precisely the needed fractional instantons emerge when this is done!}}$

Example: 2D Principal Chiral Model

$$S = \frac{1}{2g^2}\int_M d^2x \,\mathrm{Tr}\ \partial_\mu U \partial^\mu U^\dagger, \qquad U \in SU(N)$$

Set M = R x S^1(L)

Why is it interesting?

Asymptotically free field theory with a dynamically generated mass gap, with a matrix-like large N limit!

N^2 microscopic degrees of freedom, but for large L, known that free energy F ~ N^0

To have adiabatic small-volume limit, better have F~N^0 even when L is small

Turns out there is a unique set of boundary conditions on the circle which give such a limit

Z_N twisted boundary conditions

$$U(x_1, x_2 + L) = \Omega\, U(x_1, x_2)\, \Omega^\dagger$$

$$\Omega = \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix}$$

Free energy scales as F ~ N^0 at large N for any L

Maximal possible insensitivity of theory to changes in BCs

$$\frac{\partial\left[\mathcal{V}^{-1}\log Z(L)\right]}{\partial H_V} = \langle J_x^V \rangle_{H_V, H_A} = 0$$

$$\frac{\partial\left[\mathcal{V}^{-1}\log Z(L)\right]}{\partial H_A} = \langle J_x^A \rangle_{H_V, H_A} = 0$$

Large order perturbation theory at small L

Can show:

$$p_n \sim -\frac{2}{\pi}\left(\frac{1}{8\xi}\right)^n n! \left[1 - \frac{5}{2n} + \mathcal{O}(n^{-2})\right]$$

Factorially growing and non-alternating series…

Not Borel summable!

What is the scale of the ambiguity?

$$\mathcal{S}_{\pm}\mathcal{E}(\lambda) = \int_{C_\pm} dt e^{-t/g^2} B\mathcal{E}(t)$$

$$= \mathrm{Re}\mathcal{S}\mathcal{E}(\lambda) \mp i\frac{32\pi}{\lambda}e^{-16\pi/\lambda}$$

$\lambda = g^2N$

This is a semiclassical realization of renormalon ambiguity!

Small L 'fractons'

Leading term of small L effective field theory for SU(2) PCM

$$S = \frac{L}{g^2} \int dt \left[\dot{\theta}^2 + \cos^2\theta \dot{\phi}_1^2 + \sin^2\theta \dot{\phi}_2^2 + \xi^2 \sin^2\theta \right]$$

Explicit finite-action `fracton' solutions

$$\theta(t; t_0) = 2 \, \mathrm{arcCot} \left[e^{-\xi(t-t_0)} \right]$$

$$\bar{\theta}(t; t_0) = \pi - 2 \, \mathrm{arcCot} \left[e^{-\xi(t-t_0)} \right]$$

ϕ_1=const

ϕ_2=const

$$S_{\mathrm{fracton}} = \frac{8\pi}{g^2 N}$$

N types of minimal-action fractons in SU(N)

Mass gap calculable and due to fracton events

The sum over finite-action configurations

$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n} \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n} \lambda^n$$

Small-L theory is weakly coupled, so a dilute fracton gas approximation can be used

Must do sum over fluctuations around multi-fracton configurations

Correlated fracton-anti-fracton events are the interesting ones

The issue is that the $\mathcal{F}\bar{\mathcal{F}}$ interaction is attractive!

Dilute gas approximation means all fractons must be widely separated

Integral over the pairs becomes dominated by nearby configurations, which do not make sense!

Making sense of fracton-anti-fracton events

Bogomolny-Zinn-Justin recipe: analytically continue $g^2 \to g^2(1 \pm i\epsilon)$

Gives repulsive component to interaction

Away from Im[g^2]=0, integral dominated by well-separated fractons

$$r_F \sim L \ll r_{F\bar{F}-\mathrm{correlated}} \sim L \log\left(1/g^2\right) \ll r_{F\bar{F}-\mathrm{uncorrelated}} \sim L e^{+S_F}$$

Analytic continuation back to real g^2 is **ambiguous!**

$$[\mathcal{F}\bar{\mathcal{F}}]_\pm \sim e^{-2S_{\mathrm{fracton}}} \frac{1}{\xi} \left(\log\left[\frac{g^2}{8L\xi}\right] - \gamma \pm i\pi \right)$$

But remember, perturbation theory also
gave result with an imaginary ambiguity!

Cancellation of ambiguities

Need to check whether resurgence relation works...

$$\mathrm{Im}\left[\mathcal{S}_{\pm}\mathcal{E}(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0, \text{ up to } \mathcal{O}\left(e^{-4S_F}\right)$$

Requires calculation of prefactor of fracton-anti-fracton amplitude

Result is that the imaginary ambiguous parts cancel precisely!

Systematic demonstration that leading renormalon ambiguities of perturbation theory cancel against ambiguities in saddle-point sum

Illustrates that exact information about non-perturbative physics is present in perturbation theory, albeit in coded form!

Only showed leading resurgence effect - at higher order resurgence implies intricate cancellations and relations ...

Summary

★ Found adiabatic semiclassical limits of asymptotically-free QFTs

Mass gap still there, notorious renormalon divergences present

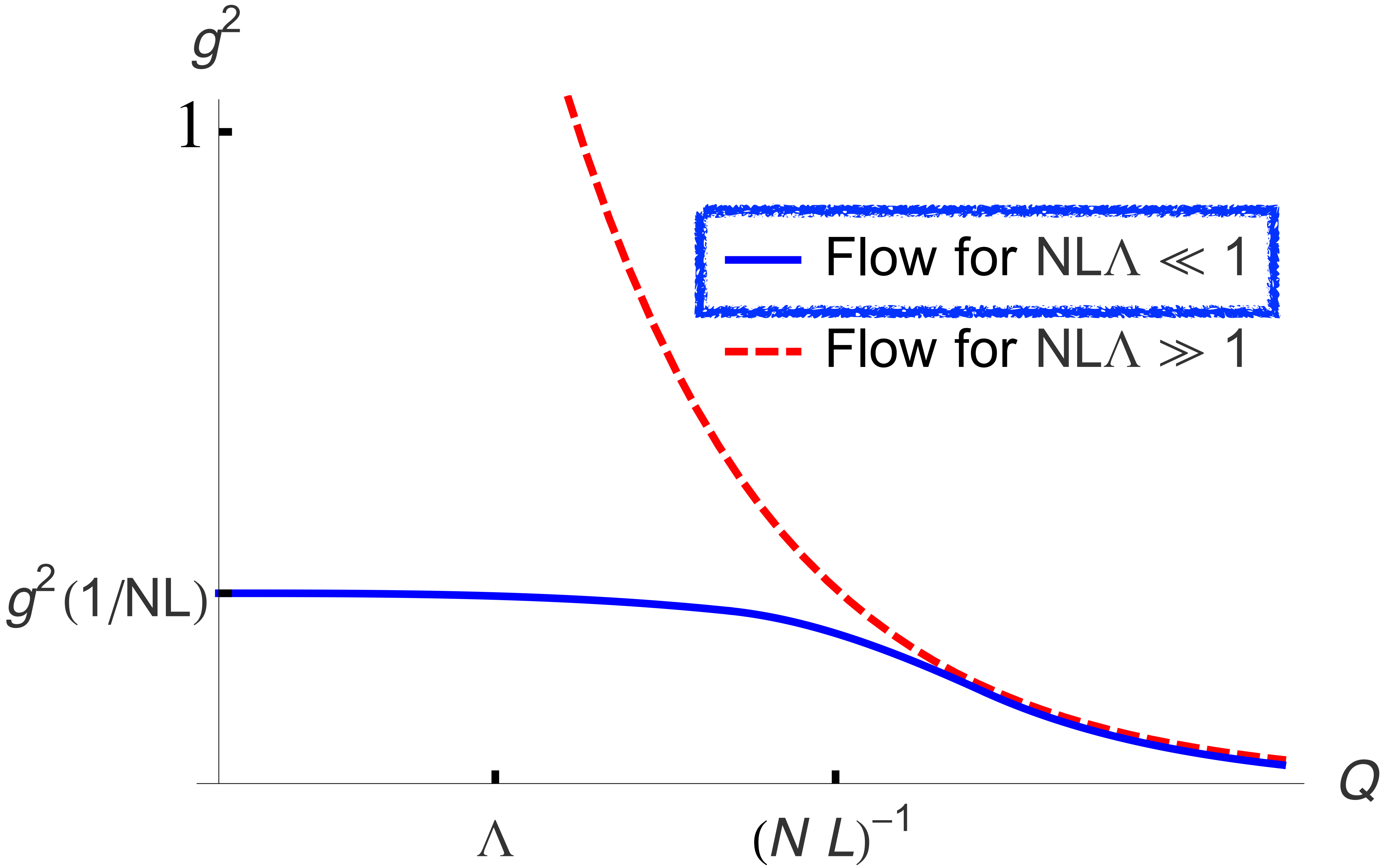
★ At small L, found novel finite-action objects, the fractons, which are related to the renormalons

★ Lack of ambiguities in physical observables due to conspiracies between perturbative and non-perturbative contributions

★ Gives new insight into non-perturbative structure of path integrals, which may have practical implications.

’t Hooft’s dream that renormalons related to mass gap realized

Flow of coupling constant in Z_N-twisted PCM



Scale NL appears due to Z_N-symmetric form of H_V

We focus on NLΛ << 1 to get a weakly-coupled theory

Physics remains very rich - mass gap, renormalons still remain!

Mass gap vs renormalons

Small L theory weakly coupled, mass gap is calculable!

The mass gap ~ one-fracton amplitude

$$\text{renormalon} \sim e^{-\frac{2\times 8\pi}{g^2 N}} = e^{-\frac{2\times 8\pi}{\lambda}}$$

Gap between ground state and first excited state in

$$\Delta_{\mathrm{SU(N)\ PCM}} = \frac{1}{NL}\frac{8\pi}{\sqrt{\lambda}}e^{-\frac{8\pi}{\lambda}}$$

Same relation in all small-L cases checked so far:

$$\Delta \sim \text{renormalon}^{1/2}$$

PCM, CP^N, YM