Generalized Loop Space and TMDs

High Energy Physics in the LHC Era.

Valparaìso - CHILE

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December 17, 2013

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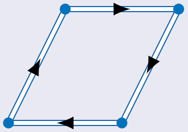
Wilson loops ⇔ $\mathcal{N} = 4$ Super Yang-Mills



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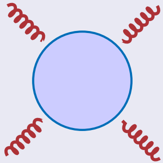

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- Duality between a planar Wilson loop made up from N light-like segments and the N-gluon planar scattering amplitude in $\mathcal{N} = 4$ SYM.
- Momenta p_i of external gluons in SYM are equal to light-like segment lengths $p_i \equiv x_i - x_{i+1}$ of the loop, motivating our quadrilateral parametrization.

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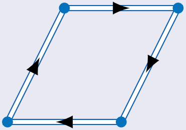

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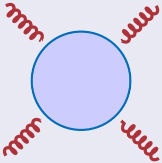

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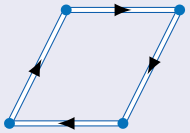

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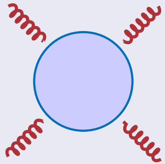

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- UV singularities of the Wilson loop $\leftrightarrow$ IR singularities of the $\mathcal{N} = 4$ SYM scattering amplitude.
- The evolution equation for the amplitude as function of IR-cutoff is governed by the cusp anomalous dimension of the Wilson loop.

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Motivation

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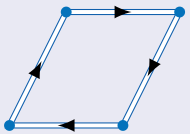

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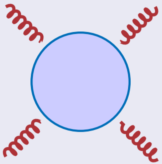

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- Investigate geometric evolution of Wilson loops.
- These might lead to evolution equations for TMDs, as the singular parts of the latter are related to those of Wilson loops.

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Definition of Wilson Loops

Definition

$$\mathcal{W}[C] = \frac{1}{N_c} \text{Tr} \left\langle 0 \left| \mathcal{P} e^{ig \oint_C dz^\mu A^a_\mu(z) t_a} \right| 0 \right\rangle$$

$$\mathcal{C} : \quad z^\mu(s) \quad s = 0 \ldots 1 \quad \text{where} \quad z^\mu(0) \equiv z^\mu(1)$$

Wilson Loops...

- are characterised by their geometry
- can be used to construct a loop space representation of the gauge theory under investigation (Ambrose-Singer Theorem)
- can exhibit dualities to objects in standard QCD, depending on the structure of the loop

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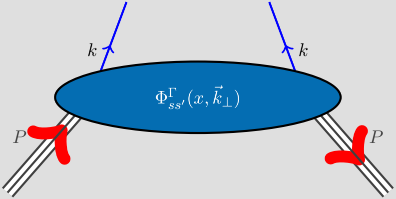
Transverse Momentum Distributions (TMDs) and Wilson Lines

Recap definition TMDs



$$x = \frac{k^+}{P^+},\ \vec{k}_\perp$$

$$\Phi^{\Gamma}_{ss'}(x,\vec{k}_\perp) = \frac{1}{2}\int \frac{dz^- d^2 z_\perp}{(2\pi)(2\pi^2)} e^{ik\cdot z} \left\langle P, s \left| \overline{\psi_i}(0)\, \Gamma_{ij}\, \mathcal{W}_{\text{TMD}}\, \psi_j(z) \right| P, s' \right\rangle_{z^+=0}$$

- TMD Evolution $\leftrightarrow$ Singularity Evolution
- Singularity Evolution $\leftrightarrow$ Evolution Wilson Loop (light cone quadrilateral)
- Area Evolution $\sim \frac{1}{\text{Rapidity}}$ Evolution (hence the minus sign)
- Surprise!!! We get Collins-Soper kernel for the double Π contour (which is for off light cone), but we have different boundary conditions! Can we improve Collins-Soper?

Generalized loop Space

Problems with naive Loop space

- Reparametrization
- Over-complete
- Algebraic constraints
- Mandelstam constraints (Infinitely many)

Generalized loop space

- Loops are generalized using Chen's algebraic paths (i.e. they become like distributions ⇒ Chen Iterated integrals). Using the notation $\omega_k(t) \equiv \omega_k(\gamma(t)) \cdot \dot{\gamma}(t)$:

$$X^{\omega_1 \cdots \omega_r}(\gamma) = \int_\gamma \omega_1 \cdots \omega_r = \int_0^1 \left(\int_{\gamma^t} \omega_1 \cdots \omega_{r-1} \right) \omega_r(t) dt,$$

- Reparametrization invariance follows from Chen integrals
- Wilson Loops from Chen integrals: $\text{Tr}[U_\Gamma] = \text{Tr}[1] + \text{Tr}\left[\int_\Gamma \omega\right] + \text{Tr}\left[\int_\Gamma \omega_1\omega_2\right] + \cdots$
- Equivalence relation introduced by Wilson loop solves over-completeness and takes care of the Mandelstam constraints.
- Generalized Loops space can be turned into a topological group, solving the algebraic constraints and introducing a Banach, Hopf, Hausdorff, Tychonov, commutative, nuclear and multiplicative-convex structure by use of the Gel'Fand spectrum that has an associated Lie Algebra (Infinite dim) allowing for different differential operators.

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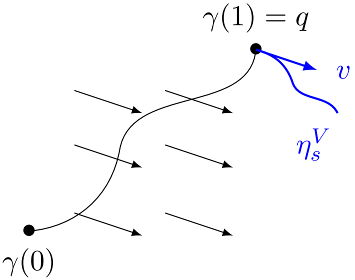
Differential Operators: Path derivative



Terminal Endpoint Derivative

Let Ψ be a path functional on $\mathcal{PM}$, with values in $\mathbb{R}$ (resp., $\mathbb{C}$; $gl(m)$).We define the *Terminal Endpoint Derivative* of Ψ, at γ, in the direction of $v \in T_{\gamma(1)}\mathcal{M}$, as the limit:

$$\partial_v^T \Psi(\gamma) = \lim_{s\to 0} \frac{\Psi(\gamma_s) - \Psi(\gamma)}{s} \quad (1)$$

provided this limit exists independently of the choice of the vector field $V \in \mathcal{XM}$, such that $V(\gamma(1)) = v$.

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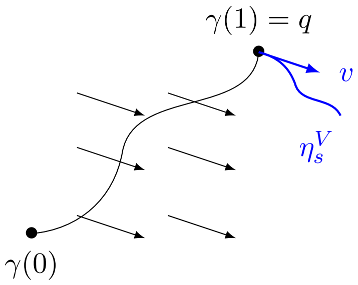

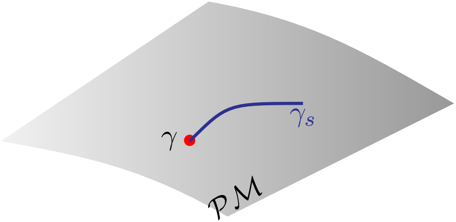

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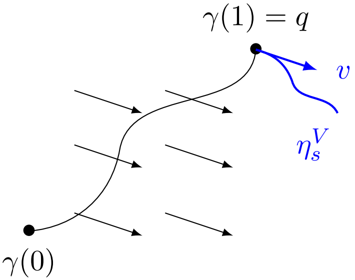

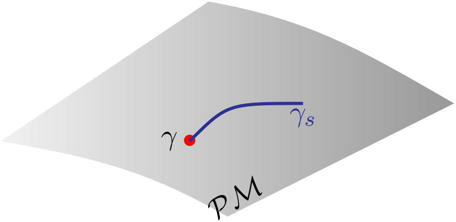

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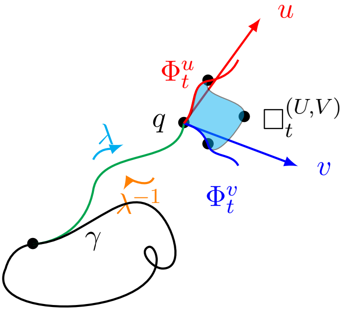
Area Derivative



Terminal Endpoint Derivative

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$$\Delta_{\lambda;(u,v)}(q)\Psi(\gamma) = \lim_{t\to 0} \frac{\Psi(\lambda_t \cdot \gamma) - \Psi(\gamma)}{t^2}$$

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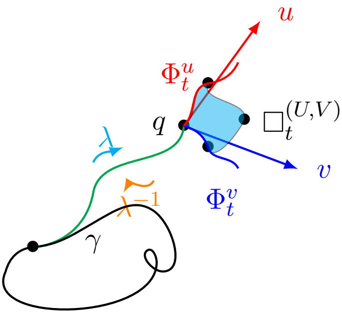

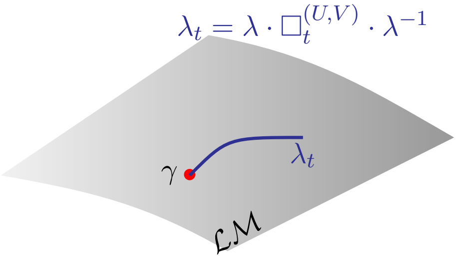

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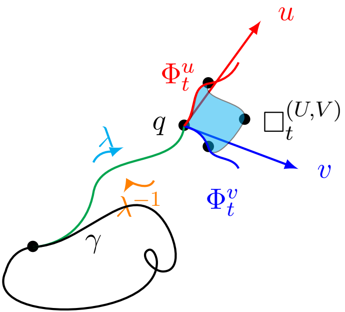

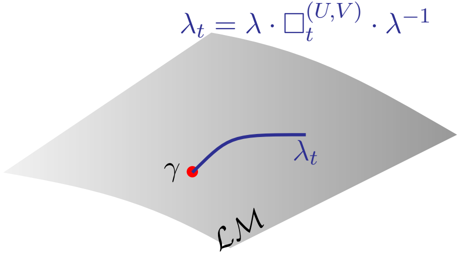

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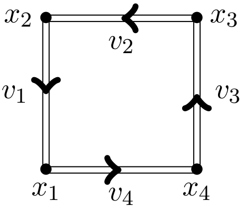
provided this limit exists independently of the choice of the vector fields $U, V \in \mathcal{XU}$, considered above.

Wilson Quadrilateral on the light cone

One-loop Result

$$W_{\text{L.O.}}(\Gamma_\square) = 1 - \frac{\alpha_s C_F}{\pi}\left(2\pi\mu^2\right)^\epsilon \Gamma(1-\epsilon) \left[\frac{1}{\epsilon^2}\left(-\frac{s}{2}\right)^\epsilon + \frac{1}{\epsilon^2}\left(-\frac{t}{2}\right)^\epsilon - \frac{1}{2}\left(\ln^2\frac{s}{-t} + \pi^2\right)\right] + \mathcal{O}(\alpha_s^2)$$



Mandelstam variables:

$$s = (v_1 + v_2)^2$$
$$t = (v_2 + v_3)^2$$

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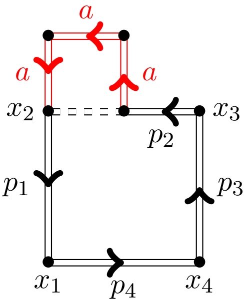
Problem with area derivative on the LC



Preliminary Result

$$\frac{\Psi(\lambda_a \cdot \gamma) - \Psi(\gamma)}{a^2} \sim \frac{1}{a^{2-2\epsilon}\epsilon^2}$$

(To be submitted for publication together with gravitational case)

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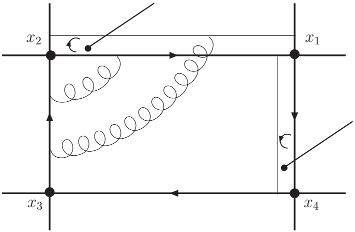
Defining a new derivative



$$\delta\sigma^{+-} = N^{+}\delta N^{-} \to v_1 \delta v_2 = \frac{1}{2}\delta s \, ,$$

$$\delta\sigma^{-+} = -N^{-}\delta N^{+} \to -v_2 \delta v_1 = \frac{1}{2}\delta t \, .$$

$$\frac{\delta}{\delta \ln \sigma} \equiv \sigma_{+-}\frac{\delta}{\delta\sigma_{+-}} + \sigma_{-+}\frac{\delta}{\delta\sigma_{-+}}$$

Evolution (in the large N_c limit)

$$\mu \frac{d}{d\mu} \frac{\delta \, \ln W(\Gamma_\square)}{\delta \ln \sigma} = -4\,\Gamma_{\rm cusp} \, , \; \Gamma_{\rm cusp} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2) \, .$$

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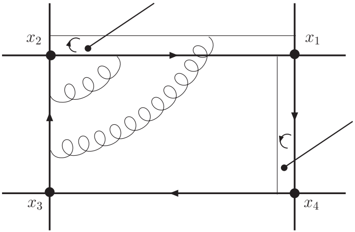

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$$\mu\frac{d}{d\mu}\frac{\delta\,\ln W(\Gamma_\square)}{\delta\ln\sigma} = -4\,\Gamma_{\text{cusp}}\,,\ \Gamma_{\text{cusp}} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2)\,.$$

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Our conjecture

Evolution Rapidity vs Π-loop

$$\mu \frac{d}{d\mu} \frac{\delta \ \ln \Phi(x, k_\perp)}{\delta \ln \theta} = 2\Gamma_{\text{cusp}} \ ,$$

for transverse momentum densities with the longitudinal gauge links on the light cone. Compare this with the area variation of the Π-loop build from two non light like parallel vectors and one finite segment on the light cone:

$$\mu \frac{d}{d\mu} \frac{\delta \ \ln W(\Gamma_\Pi)}{\delta \ln \sigma} = -2\Gamma_{\text{cusp}} \ ,$$

Evolution (in the large N_c limit)

$$\mu \frac{d}{d\mu} \frac{\delta \ \ln W(\Gamma_\Box)}{\delta \ln \sigma} = -\sum_{\text{cusps}} \Gamma_{\text{cusp}} \ ,$$

Phys.Rev. D86 (2012) 085035,
Phys.Part.Nucl. 44 (2013) 250-259,
Int.J.Mod.Phys.Conf.Ser. 20 (2012) 109-117,
AIP Conf.Proc. 1523 (2013) 272-275

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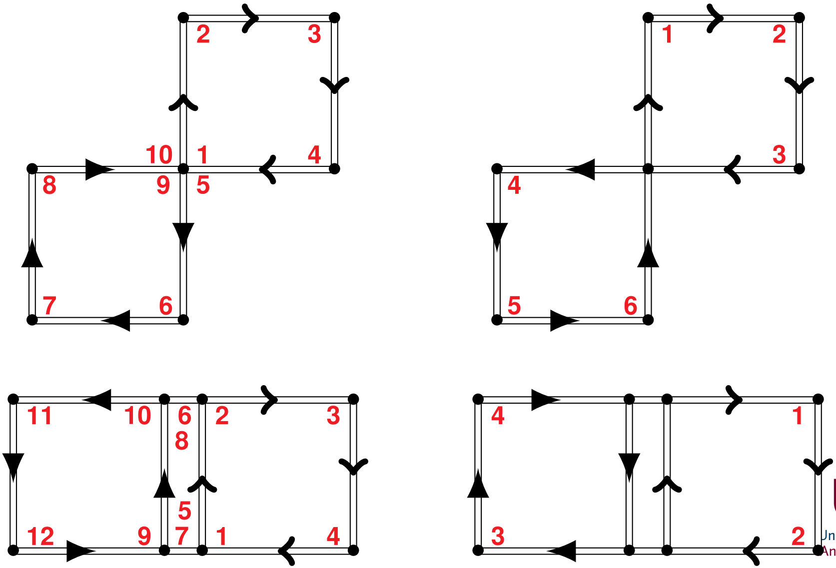
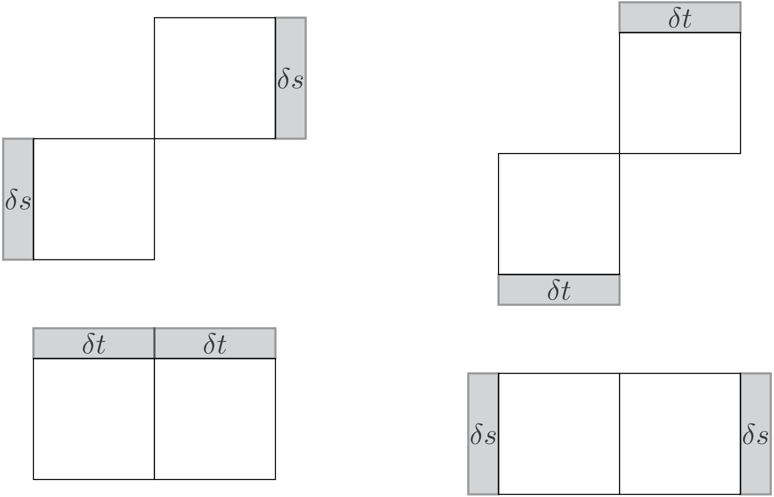
Self-intersecting WLs (Phys. Lett. B 727 (2013) 563 [arXiv:1308.5296 [hep-ph]])



Self-intersecting WLs (Phys. Lett. B 727 (2013) 563 [arXiv:1308.5296 [hep-ph]])



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Fréchet Derivative

Definition

Let X, Y be Banach spaces, with $U \subset X$ an open subset. Then a function $f : U \to Y$ is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A_x : X \to Y$ such that:

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|_Y}{\|h\|_X} = 0,$$

where the limit is defined as in the usual sense. If this limit exist one writes $Df(x) = A_x$, the Fréchet derivative. We call the function f, C^1 if:

$$Df : U \to B(X, Y); x \mapsto Df(x) = A_x,$$

is continuous, and where the B highlights the fact that this a map between Banach spaces.

Fréchet derivative of Wilson loop

Fréchet derivative of Chen Integral

$$D_V X^{\omega_1 \dots \omega_r}(\gamma) = \sum_{i=1}^{r} \int_\gamma \omega_1 \dots \omega_{i-1} . \iota_V(d\omega_i) \cdot \omega_{i+1} \dots \omega_r + \sum_{i=2}^{r} \int_\gamma \omega_1 \dots \omega_{i-2} . \iota_V(\omega_{i-1} \wedge \omega_i) . \omega_{i+1} \dots \omega_r$$

Fréchet derivative of Wilson loop

$$D_V[U(\gamma)] = U(\gamma) \cdot \int_0^1 dt\, U(\gamma^t) \cdot F_{\mu\nu}(t) (V^\mu(t) \wedge \dot{\gamma}^\nu(t)) \cdot U((\gamma^t)^{-1})$$

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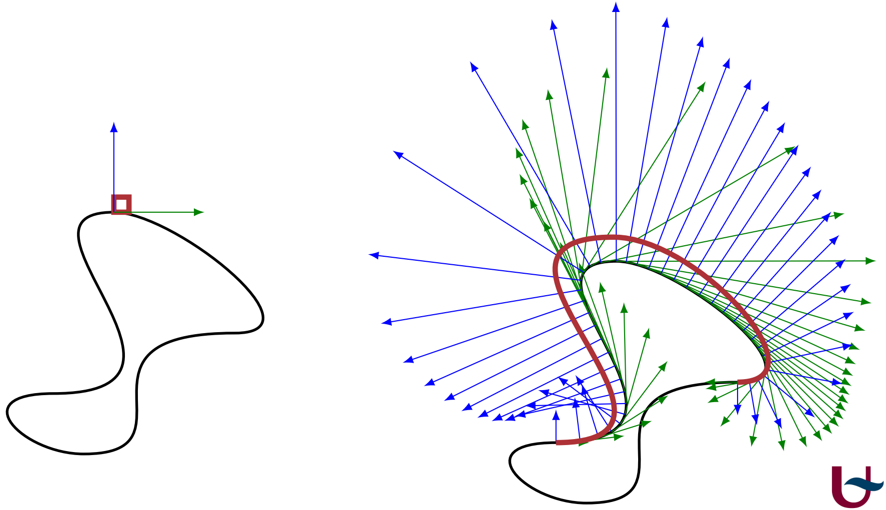
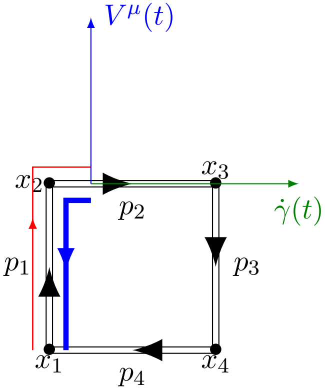
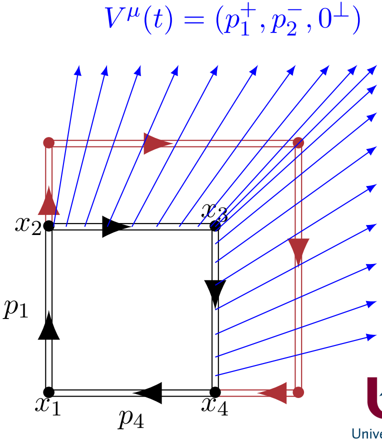
Relation between Area and Fréchet derivative



Fréchet derivative on Quadrilateral on the light cone





Equivalence at LO of generalized and Fréchet derivative

Equivalence

$$\left(p_1 \frac{\delta}{\delta p_1} + p_2 \frac{\delta}{\delta p_2}\right) [\mathrm{Tr}\, U(\gamma)]_{\mathrm{LO}} = \frac{1}{2}\left(\frac{\delta}{\delta s} + \frac{\delta}{\delta t}\right) [\mathrm{Tr}\, U(\gamma)]_{\mathrm{LO}} = D_V\, [\mathrm{Tr}\, U(\gamma)]_{\mathrm{LO}}$$

with $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$

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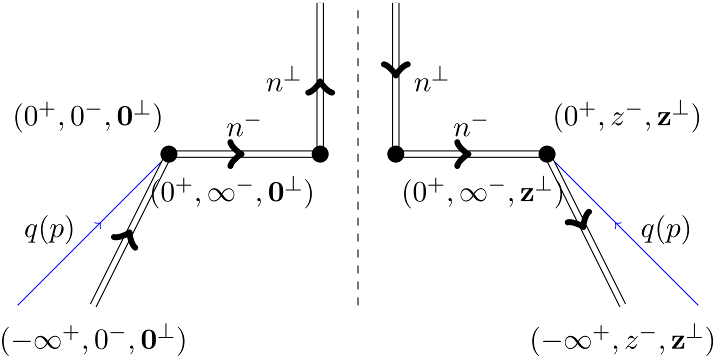
Quark TMD (under construction)



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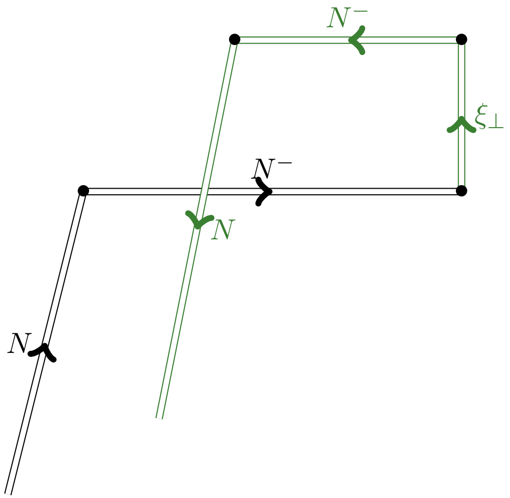
TMD dual : the double Π (under construction)



Outlook

- Further exploration of geometrical operators.
- Calculation of specific diagrams to test conjecture
- Two-loop calculation as test of conjecture
- Number of cusps and diffeomorphisms (in preparation)
- Cohomology and Maxwell
- Apply to phenomenological relevant situation $\longrightarrow$ Quark and Gluon TMD (Jlab - Large-x physics)
- Exploit Amplutihedron properties (Wilson Loops in Twistor Theory)