On the solution of the NLO forward BFKL equation

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Introduction

The LO BFKL equation has conformally invariant kernel and can be solved in the space of conformal eigenfunctions both for the forward and nonforward cases. (Lipatov 1986)

$$\frac{\partial \hat{G}}{\partial Y} = \hat{K}\hat{G}, \quad \hat{G}|_{Y=Y_0} = \hat{1},$$

where Y is the rapidity and Y_0 is the energy scale set by the impact factors.
For the forward scattering we will work in the space of the Born kernel eigenfunctions $|n\nu\rangle$

$$\langle \vec{r}|n\nu\rangle = \frac{1}{\pi\sqrt{2}} e^{in\phi}(\vec{r}^{\,2})^{-\frac{1}{2}+i\nu}, \quad \langle \vec{r}|n\nu\rangle\langle n\nu|\vec{r}^{\,\prime}\rangle = \delta(\vec{r}-\vec{r}^{\,\prime}),$$

$$\langle n\nu|\vec{r}\rangle\langle \vec{r}|m\sigma\rangle = \delta_{nm}\delta(\nu-\sigma).$$

Introduction

$$\langle n\nu|\hat{K}_M|h\rho\rangle = \bar{\alpha}\chi(n,\rho)\delta_{nh}\delta\left(\rho-\nu\right)$$

$$\bar{\alpha} = \frac{\alpha_s N_c}{\pi}, \quad \chi(n,\nu) = 2\psi(1) - \psi(\frac{1+n}{2}+i\nu) - \psi(\frac{1+n}{2}-i\nu)\,.$$

As a result for the convolution with the impact factor $\langle n\nu|\hat{K}_M|\Phi\rangle$

$$\frac{\partial\langle n\nu|G_M|\Phi\rangle}{\partial Y} = \bar{\alpha}\chi(n,\nu)\langle n\nu|G_M|\Phi\rangle, \quad \langle n\nu|G_M|\Phi\rangle \sim e^{\bar{\alpha}\chi(n,\nu)Y}.$$

Introduction

The NLO BFKL kernel can be taken quasiconformal, i.e. the only source of the violation of the conformal symmetry is the running coupling.

$$\langle n\nu|\hat{K}_M|h\rho\rangle = \left[\bar{\alpha}\chi(n,\rho) + \frac{\bar{\alpha}^2}{4}\delta(n,\rho)\right]\delta_{nh}\delta\left(\rho-\nu\right)$$
$$-i\frac{\bar{\alpha}^2\beta}{4}\chi(n,\rho)\delta_{nh}\delta'\left(\rho-\nu\right),$$

Here

$$\bar{\alpha} = \frac{\alpha_s(\mu^2)N_c}{\pi},$$

and $\delta(n,\nu)$ contains all the NLO terms without the derivative. (Fadin, Lipatov 1998)

There are many approaches to solving the NLO BFKL or incorporating the running coupling corrections into the solution. (GLR1983, Kovchegov and Mueller 1998, Levin 1999,... Chirilli and Kovchegov 2013)

Solution

Action of the kernel on the impact factor in the conformal igenfunction space reads

$$\frac{\partial\langle n\nu|\hat{G}|\Phi\rangle}{\partial Y} - i\frac{\bar{\alpha}^2\beta}{4}\frac{\partial\left(\chi(n,\nu)\langle n\nu|\hat{G}|\Phi\rangle\right)}{\partial\nu}$$
$$= \left[\bar{\alpha}\chi(n,\nu) + \frac{\bar{\alpha}^2}{4}\delta(n,\nu)\right]\langle n\nu|\hat{G}|\Phi\rangle$$

with the initial condition

$$\langle n\nu|\hat{G}|\Phi\rangle|_{Y=Y_0} = \Phi_B(n,\nu)\,.$$

For simplicity we will write $\chi(n,\nu)\langle n\nu|\hat{G}|\Phi\rangle = \mathbf{G}$. In this notation the equation is

$$\frac{\partial\mathbf{G}}{\partial Y} - i\frac{\bar{\alpha}^2\beta}{4}\chi(n,\nu)\frac{\partial\mathbf{G}}{\partial\nu} = \left[\bar{\alpha}\chi(n,\nu) + \frac{\bar{\alpha}^2}{4}\delta(n,\nu)\right]\mathbf{G},$$

$$\mathbf{G}_0(n,\nu) = \mathbf{G}|_{Y=Y_0} = \chi(n,\nu)\Phi_B(n,\nu)\,.$$

Solution

Its two first integrals are

$$c_1 = Y - Y_0 - \frac{4i}{\bar{\alpha}^2\beta}\int_{\nu_0}^{\nu}\frac{dl}{\chi(n,l)}, \quad c_2 = \mathbf{G}e^{-i\left[\frac{4}{\bar{\alpha}\beta}(\nu-\nu_0)+\int_{\nu_0}^{\nu}\frac{\delta(n,l)}{\chi(n,l)\beta}dl\right]}.$$

The general solution to this equation depends on an arbitrary function f

$$\mathbf{G} = e^{i\left[\frac{4}{\bar{\alpha}\beta}(\nu-\nu_0)+\int_{\nu_0}^{\nu}\frac{\delta(n,l)}{\chi(n,l)\beta}dl\right]} f\left(\frac{\bar{\alpha}^2\beta}{4}i(Y - Y_0) + \int_{\nu_0}^{\nu}\frac{dl}{\chi(n,l)}\right).$$

Using initial condition, we can find f from the equation

$$\mathbf{G}_0(n,\nu) = e^{i\left[\frac{4}{\bar{\alpha}\beta}(\nu-\nu_0)+\int_{\nu_0}^{\nu}\frac{\delta(n,l)}{\chi(n,l)\beta}dl\right]} f\left(\int_{\nu_0}^{\nu}\frac{dl}{\chi(n,l)}\right).$$

Solution

$$\mathbf{G}_0(n,\nu) = e^{i\left[\frac{4}{\bar{\alpha}\beta}(\nu-\nu_0)+\int_{\nu_0}^{\nu}\frac{\delta(n,l)}{\chi(n,l)\beta}dl\right]} f\left(\int_{\nu_0}^{\nu}\frac{dl}{\chi(n,l)}\right).$$

To solve it we introduce

$$F(\nu) = \int_{\nu_0}^{\nu}\frac{dl}{\chi(n,l)}.$$

Hence

$$f(t) = \mathbf{G}_0(n, F^{-1}(t)) \exp\left[\frac{-4i}{\bar{\alpha}\beta}(F^{-1}(t)-\nu_0) - \frac{i}{\beta}\int_{\nu_0}^{F^{-1}(t)}\frac{\delta(n,l)}{\chi(n,l)}dl\right].$$

Solution

Then the formal general solution reads

$$\mathbf{G} = \mathbf{G}_0(n, F^{-1}(F(v) + \Delta F))$$

$$\times e^{-\frac{4i}{\bar{\alpha}\beta}(F^{-1}(F(\nu)+\Delta F)-\nu)} e^{-\frac{i}{\beta}\int_{\nu}^{F^{-1}(F(\nu)+\Delta F)} \frac{\delta(n,l)}{\chi(n,l)} dl}.$$

$$\Delta F = \frac{\bar{\alpha}^2 \beta i (Y - Y_0)}{4}.$$

The convolution of the impact factor and the Green function in the space of conformal eigenfunctions

$$\langle n\nu | \hat{G} | \Phi \rangle = \Phi_B(n, F^{-1}(F(\nu) + \Delta F)) \frac{\chi(n, F^{-1}(F(\nu) + \Delta F))}{\chi(n, \nu)}$$

$$\times e^{-\frac{4i}{\bar{\alpha}\beta}(F^{-1}(F(\nu)+\Delta F)-\nu)} e^{-\frac{i}{\beta}\int_{\nu}^{F^{-1}(F(\nu)+\Delta F)} \frac{\delta(n,l)}{\chi(n,l)} dl}.$$

Result

We can keep only the terms up to $\bar{\alpha}^{m+1}Y^m$ order since we work in NLO. With this accuracy

$$F^{-1}\left(F(\nu)+\Delta F\right)\simeq \nu+\chi\left(n,\nu\right)\frac{\bar{\alpha}^2\beta i}{4}(Y-Y_0)+$$

$$+\frac{\chi\left(n,\nu\right)\chi'\left(n,\nu\right)}{2}\left(\frac{\bar{\alpha}^2\beta i}{4}Y\right)^2$$

Therefore with NLO accuracy

$$\langle n\nu|\hat{G}|\Phi\rangle=\left(\Phi_B(n,\nu)+\frac{\bar{\alpha}^2\beta i}{4}[\Phi_B(n,\nu)\chi\left(n,\nu\right)]'Y\right)$$

$$\times e^{\bar{\alpha}\chi(n,\nu)(Y-Y_0)+\frac{\bar{\alpha}^2}{4}\delta(n,\nu)Y+\frac{\bar{\alpha}^3\beta i}{8}\chi(n,\nu)\chi'(n,\nu)Y^2}.$$

The solution satisfies the equation up to terms proportional to $\bar{\alpha}^{n+2}Y^n$ including them.

Result

It is also interesting to consider the next term in series $F^{-1}\left(F(\nu)+\Delta F\right)$, though it is beyond the NLO accuracy.

$$F^{-1}\left(F(\nu)+\Delta F\right)^{(3)} = \frac{\chi^2\chi'' + \chi\chi'^2}{3!}\left(\frac{\bar{\alpha}^2\beta i}{4}Y\right)^3$$

Therefore the solution will have the form

$$\langle\Phi_A|\hat{G}\,|\Phi_B\rangle = \sum_n \int d\nu\,\Phi_A^*\left(\Phi_B + \frac{\bar{\alpha}^2\beta i}{4}[\Phi_B\chi]'Y\right)$$

$$\times e^{\bar{\alpha}\chi(Y-Y_0)+\frac{\bar{\alpha}^2}{4}\delta Y+\frac{\bar{\alpha}^3\beta i}{8}\chi\chi' Y^2-\frac{\bar{\alpha}^5\beta^2 Y^3}{3!4^2}\left(\chi^2\chi''+\chi\chi'^2\right)}.$$

Result

If one integrates with respect to ν treating the terms $\sim \bar{\alpha}^2 Y, \bar{\alpha}^3 Y^2, \bar{\alpha}^5 Y^3$ as small and takes only the contribution of the saddle point of $\chi(0,0)$, one gets

$$\langle \Phi_A | \hat{G} | \Phi_B \rangle \simeq \Phi_A^*(0,0) \left(\Phi_B(0,0) + \frac{\bar{\alpha}^2 \beta i}{4} \Phi_B'(0,0) \chi(0,0)\, Y \right)$$

$$\times \sqrt{\frac{2\pi}{\bar{\alpha}\, |\chi''(0,0)|\, Y}}\, e^{\bar{\alpha}\chi(0,0)(Y-Y_0) + \frac{\bar{\alpha}^2}{4}\delta(0,0)\, Y - \frac{\bar{\alpha}^5 \beta^2 Y^3}{3!4^2}\chi(0,0)^2 \chi''(0,0)}.$$

Here the term $\sim \bar{\alpha}^5 \beta^2 Y^3$ is known.
The term $\sim \bar{\alpha}^3 \beta i \chi'(n,\nu)$ is zero in the saddle point of χ.

Conclusion

- The NLO forward BFKL equation can be transformed into the partial derivative equation in the space of conformal eigenfunctions.
- This equation can be solved with the NLO accuracy.
- The solution contains the exponential $e^{\frac{\bar{\alpha}^3 \beta i}{8}\chi(n,\nu)\chi'(n,\nu)Y^2}$.

Thank you!