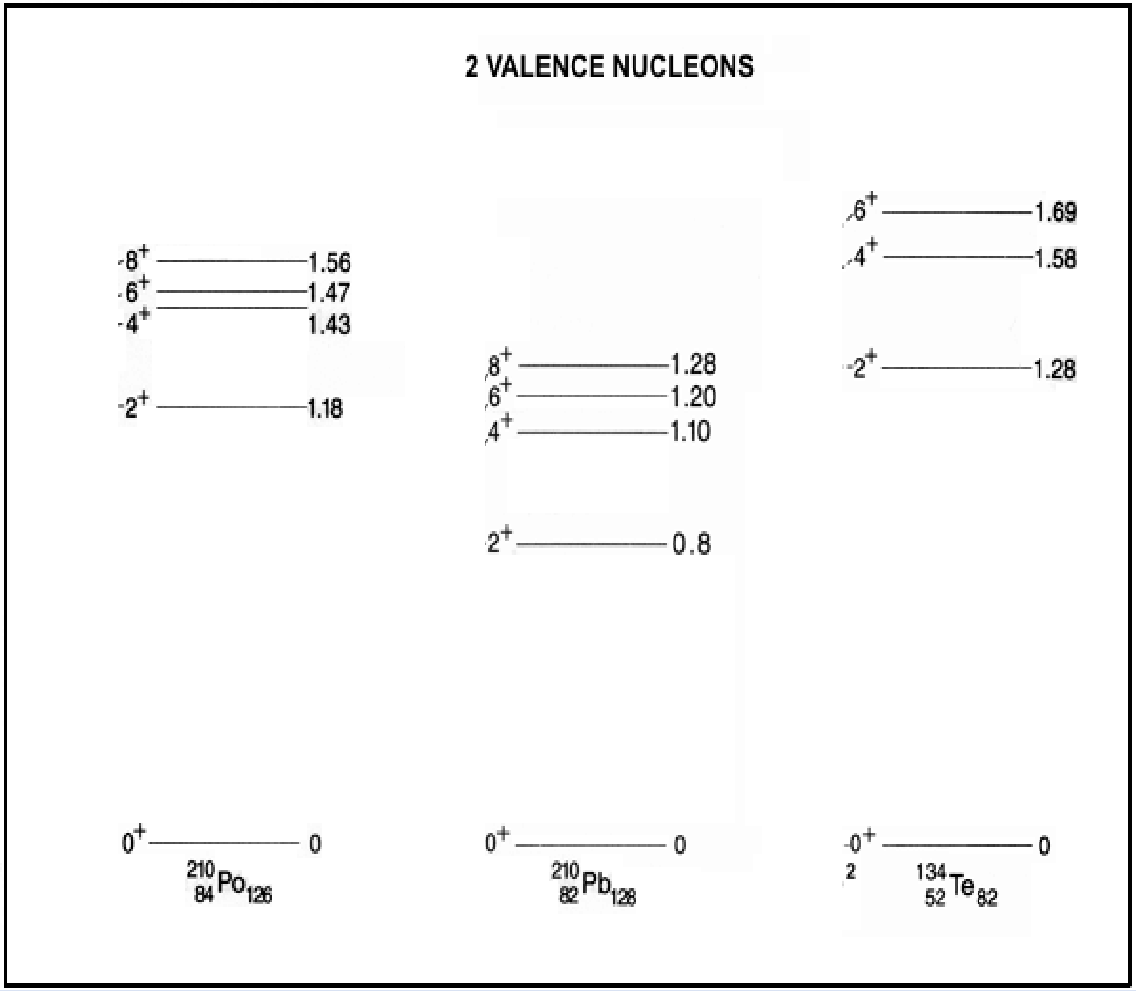
Nuclei with more than one valence nucleon

Multi-particle systems

The simplest case: nuclei with 2 "valence" particles outside doubly magic core. Universal result: J = 0,2,4,6…(2j-1). Ground state always 0^+, large energy to 2^+ first excited state



Why? Can we understand such simple results by extending the IPM to multi-valence-nucleon cases?

Independent Particle Model:

1 particle (or hole)

outside closed shell

(very few nuclei)

→ Multi-particle systems

Recall mean field approximation

$H = H_{IPM} + H_{Residual}$

↓

Residual Interactions

Effects <u>not</u> included in independent particle model <u>potential</u>

Residual interactions among valence nucleons

Dominate the evolution of Structure

- **Pairing – coupling of two identical nucleons to angular momentum zero. No preferred direction in space, therefore drives nucleus towards spherical shapes. We will see the basis of this in a few minutes.**

- **p-n interactions – generate configuration mixing, unequal magnetic state occupations, therefore drive towards collective structures and deformation. See later lecture.**

- **Monopole component of p-n interactions generates changes in single particle energies and shell structure. See discussion of exotic nuclei and the fragility of magicity.**

Residual Interactions

Need to consider a more complete Hamiltonian:

$$H = H_0 + H_{residual}$$

$H_{residual}$ reflects interactions not in the single particle potential.

NOT a minor perturbation. In fact, these residual interactions determine almost everything we know about most nuclei.

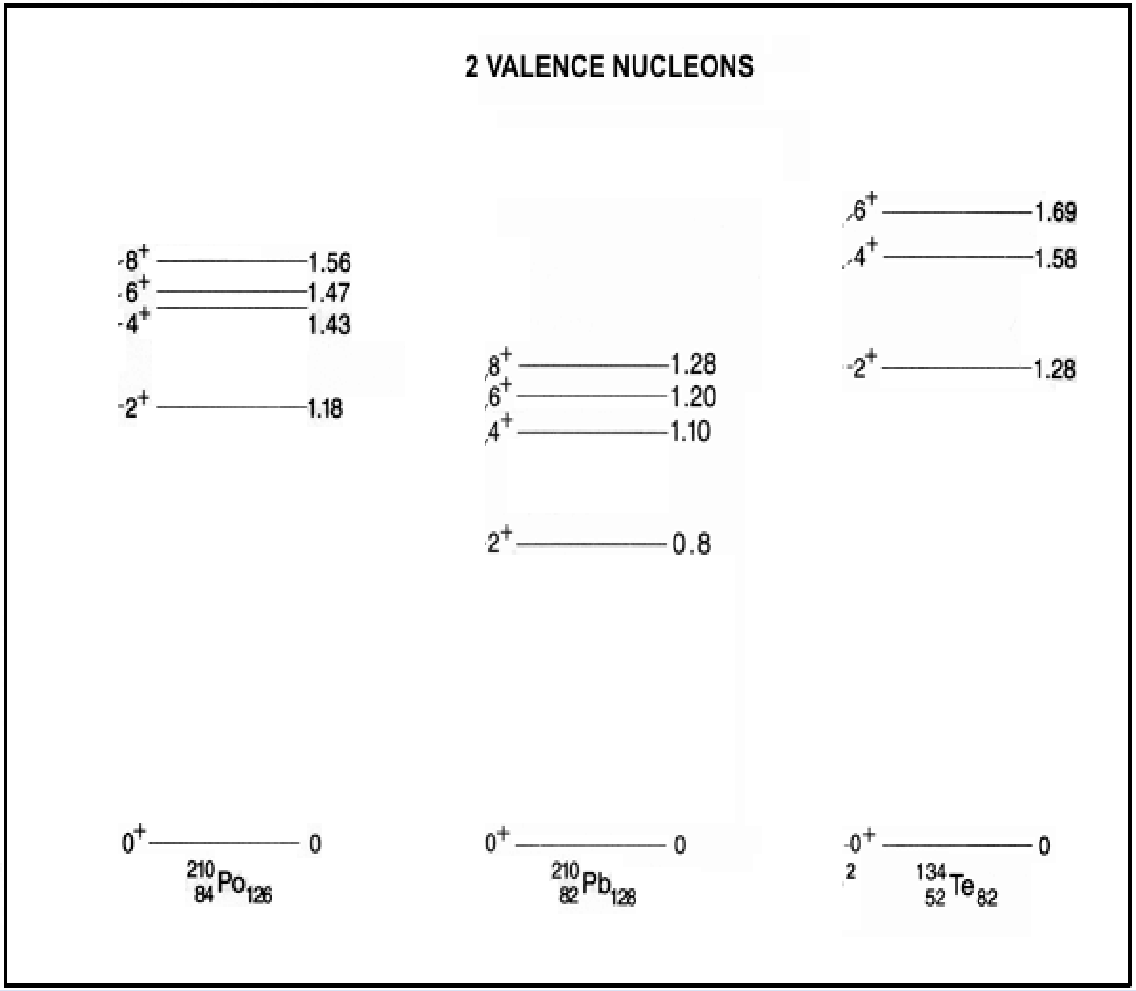
Start with 2- particle system, that is, a nucleus "doubly magic + 2".

$$H_{residual} \quad \text{is} \quad H_{12}(r_{12})$$

Consider two identical valence nucleons with j_1 and j_2.

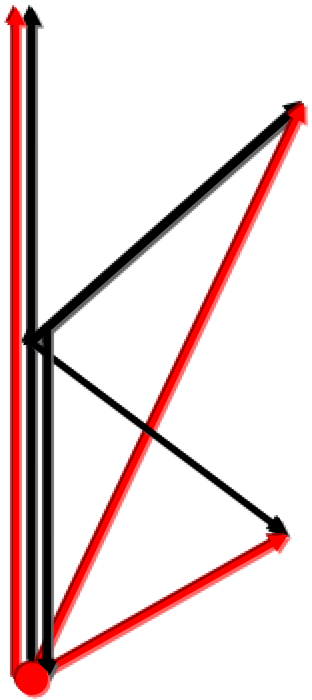
Two questions: What total angular momenta $j_1 + j_2 = J$ can be formed?
What are the energies of states with these J values?

First problem – what angular momenta for multi-particle systems?



WHY these?

Coupling of two angular momenta



$\mathbf{j_1 + j_2}$ **All values from:** $j_1 - j_2$ to $j_1 + j_2$ $(j_1 \neq j_2)$

Example: $j_1 = 3$, $j_2 = 5$: $J = 2, 3, 4, 5, 6, 7, 8$

BUT: For $j_1 = j_2$: $J = 0, 2, 4, 6, \ldots (2j - 1)$ (Why these?)

How can we know which total J values are obtained for the coupling of two identical nucleons in the same orbit with total angular momentum j? Several methods: easiest is the "m-scheme".

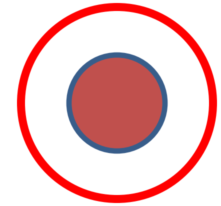


Table 5.1 *m scheme for the configuration* $|(7/2)^2 J\rangle$*

$j_1 = 7/2$ m_1	$j_2 = 7/2$ m_2	M	J
7/2	5/2	6	
7/2	3/2	5	
7/2	1/2	4	
7/2	–1/2	3	6
7/2	–3/2	2	
7/2	–5/2	1	
7/2	–7/2	0	
5/2	3/2	4	
5/2	1/2	3	
5/2	–1/2	2	4
5/2	–3/2	1	
5/2	–5/2	0	
3/2	1/2	2	
3/2	–1/2	1	2
3/2	–3/2	0	
1/2	–1/2	0	0

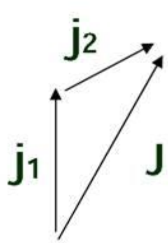
* Only positive total M values are shown. The table is symmetric for $M < 0$.

Residual Interactions—Diagonal Effects

Consider 2 particles, in orbits j_1, j_2 coupled to spin J_i, and interacting with a residual interaction, V_{12}.

2 Identical Nucleons

$| j_1, j_2, J >$



NO RESIDUAL INTERACTION

What are Energies of 2-particle configurations

$$\Delta E(j_1 j_2 J) = \langle j_1 j_2 J M | H_{12} | j_1 j_2 J M \rangle$$

$$= \frac{1}{\sqrt{2J+1}} \langle j_1 j_2 J || H_{12} || j_1 j_2 J \rangle$$

Separate radial and angular coordinates

$$\Psi = \frac{1}{r} R_{nl}(r) Y_{lm}(\theta, \phi)$$

where $$\frac{d^2 R_{nl}}{dr^2} - \frac{l(l+1)}{r^2} R_{nl} + \frac{2m}{\hbar^2}(E_{nl} - V) R_{nl} = 0$$

R_{nl} depends on potential – but generally not very much.

Now, what is H_{resid}?

Many choices possible. Let's start with simplest.
Nuclear force is short range and attractive. So, take
δ-force

$$V_\delta = \frac{-V_0}{r_1 r_2}\,\delta(r_1 r_2)\,\delta(\cos\Theta_1, \cos\Theta_2)\,\delta(\Phi_1, \Phi_2)$$

in spherical coordinates

Need to evaluate the matrix element (ME) of the form

$$\left\langle \Psi \middle| ``\delta" \middle| \Psi \right\rangle = \left\langle \tfrac{1}{r} R_{nl} \middle| V_{\delta_r} \middle| \tfrac{1}{r} R_{nl} \right\rangle \times \left\langle Y_{lm}(\Phi, \Phi) \middle| V_{\delta_{\Theta,\Phi}} \middle| Y_{lm}(\Phi, \Phi) \right\rangle$$

First factor is just a constant independent of J,

i.e., does not depend on J in $|j_1 j_2 J\rangle$.

<u>So energy shifts for different J's are independent of the form of the radial wave functions and hence of the radial form of the potential</u> ! !

⇒ Great simplification

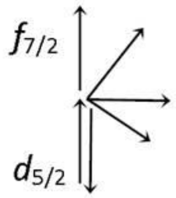
⇒ Typical of many results – radial effects disappear

How can we understand the energy patterns that we have seen for two – particle spectra with residual interactions? Easy – involves a very beautiful application of the Pauli Principle.

Need 2 ideas <u>only</u>

- Nuclear force (including residual interactions) is
  - Short range and attractive
  - Pauli Principle

Physical Interpretation



J depends on angle between the two orbital planes

Interaction strongest when the 2 particles are closest to each other

i.e., when the orbits are co-planar

$\Rightarrow$ strongest interaction either for

J_{min} ***or*** J_{max}

Which one?

Consider *L*, *S* composition of state *J*

$$\bar{L} = \bar{l}_1 + \bar{l}_2 \qquad S = \bar{\frac{1}{2}} \pm \bar{\frac{1}{2}} = 1 \text{ or } 0$$

Pauli Principle

Fermions:

No two fermions can occupy the same state/place

Wave functions must be totally antisymmetric

$$\Psi(\bar{r}) = -\Psi(-\bar{r}) \qquad \bar{r} = \bar{r}_2 - \bar{r}_1$$

$\therefore$ If particles are at same place ----- $\bar{r}$ = 0-----

then Ψ (0) = -Ψ (0)

$\Rightarrow \Psi$ (0) = 0

so PP is satisfied

We split wave functions into 2 parts - spatial part (L), and spin part (S). PP $\Rightarrow$

$$\Psi_{Tot} = \Psi_{spat} \times \Psi_{spin} = \text{Anti-sym}$$

This is the most important slide: understand this and all the key ideas about residual interactions will be clear !!!!!

PP:

Key Physics Ideas

$\Psi_{spatial}$	Ψ_{spin}
A	S
S	A

$S = ½ + ½ = 1 = \text{Sym}$

$S = ½ - ½ = 0 = A\text{-Sym}$

$\Psi_{spat}\,(A) \times \Psi_{spin}\,(S = 1)$ $\qquad$ $\Psi_{spat}\,(S) \times \Psi_{spin}\,(S = 0)$

$S = 1$ case $\qquad$ $\Psi_{spat} = A$ $\qquad$ $\Psi\,(r_{12}) = -\Psi\,(-\,r_{12})$

For δ force, which only acts at $r_{12} = 0$

$$\Psi\,(r_{12} = 0) = 0\ !\ !$$

So, at the ONLY place where a δ-int acts, the wave fct. vanishes—*i.e.*, No effect of δ fct int on $S = 1$ states *! ! !*

$S = 0$ case $\qquad$ $\Psi_{spat} = S$

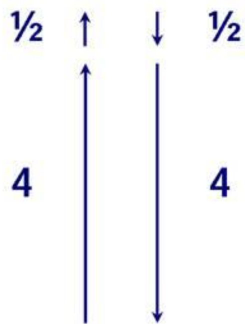
No restriction on $\Psi\,(r_{12} = 0)$, hence δ-int can have big effect *! ! !*

Equivalent Orbits

$j_1 = j_2 \qquad |j^2\, J\rangle$

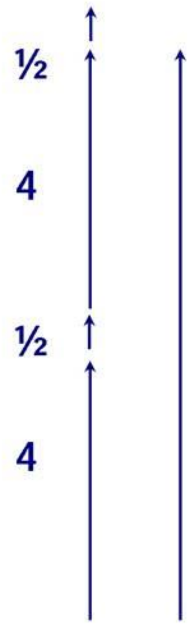
$J = 0, 2, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 2j - 1$

e.g. $|g^2_{9/2}\, J\rangle \quad J = 0, 2, 4, 6, 8$



J = 0

S = 0

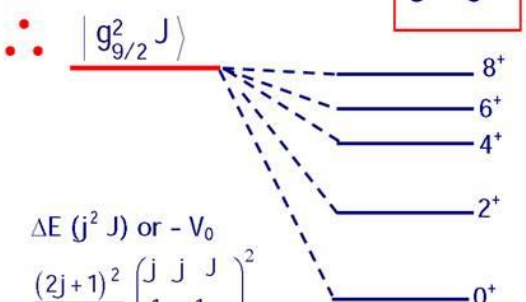


J = 8

S = 1

S = 0 lowered ⇒ J = 0 lowest

∴



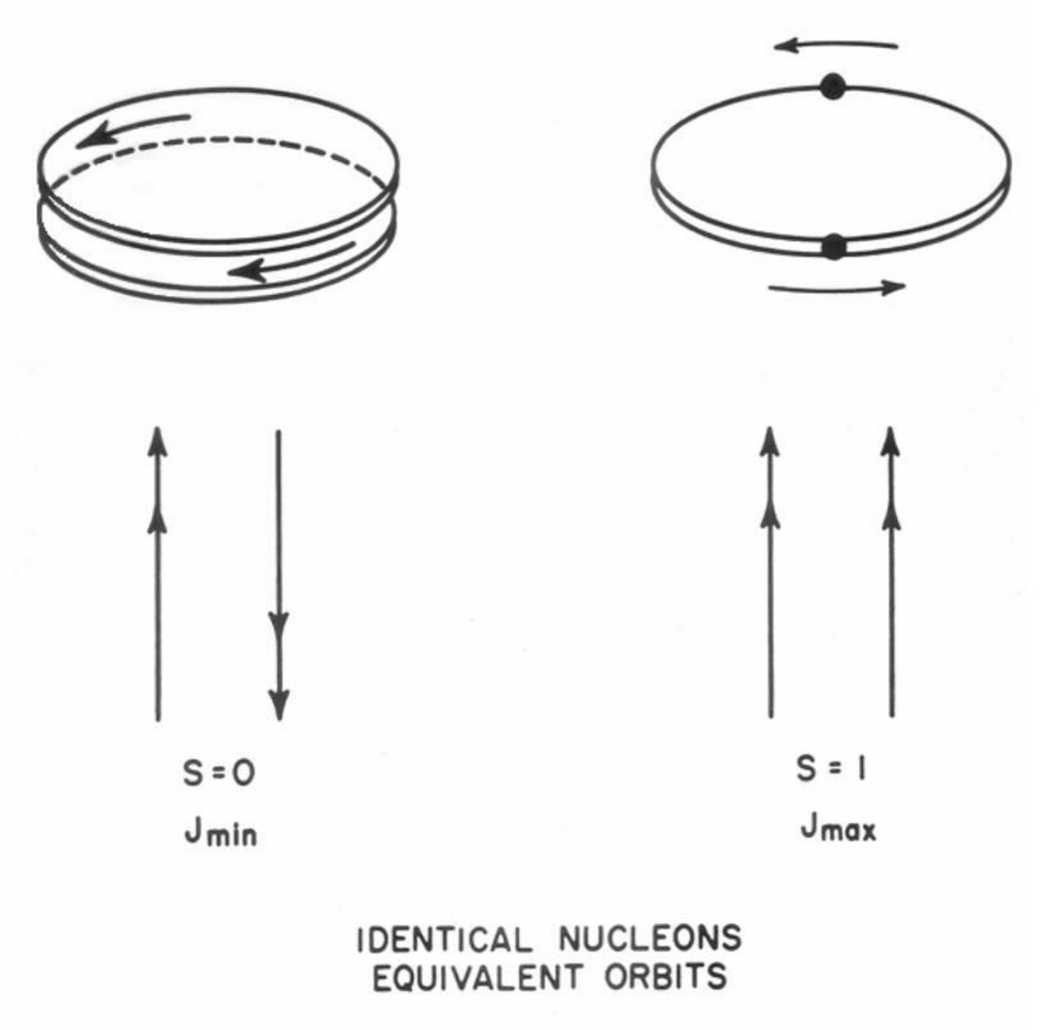
0+ lowest
all e - e
nuclei have
0+ g.s. ***!!!***

$\Delta E\,(j^2 J)$ or $-V_0 \dfrac{(2j+1)^2}{2} \begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2$

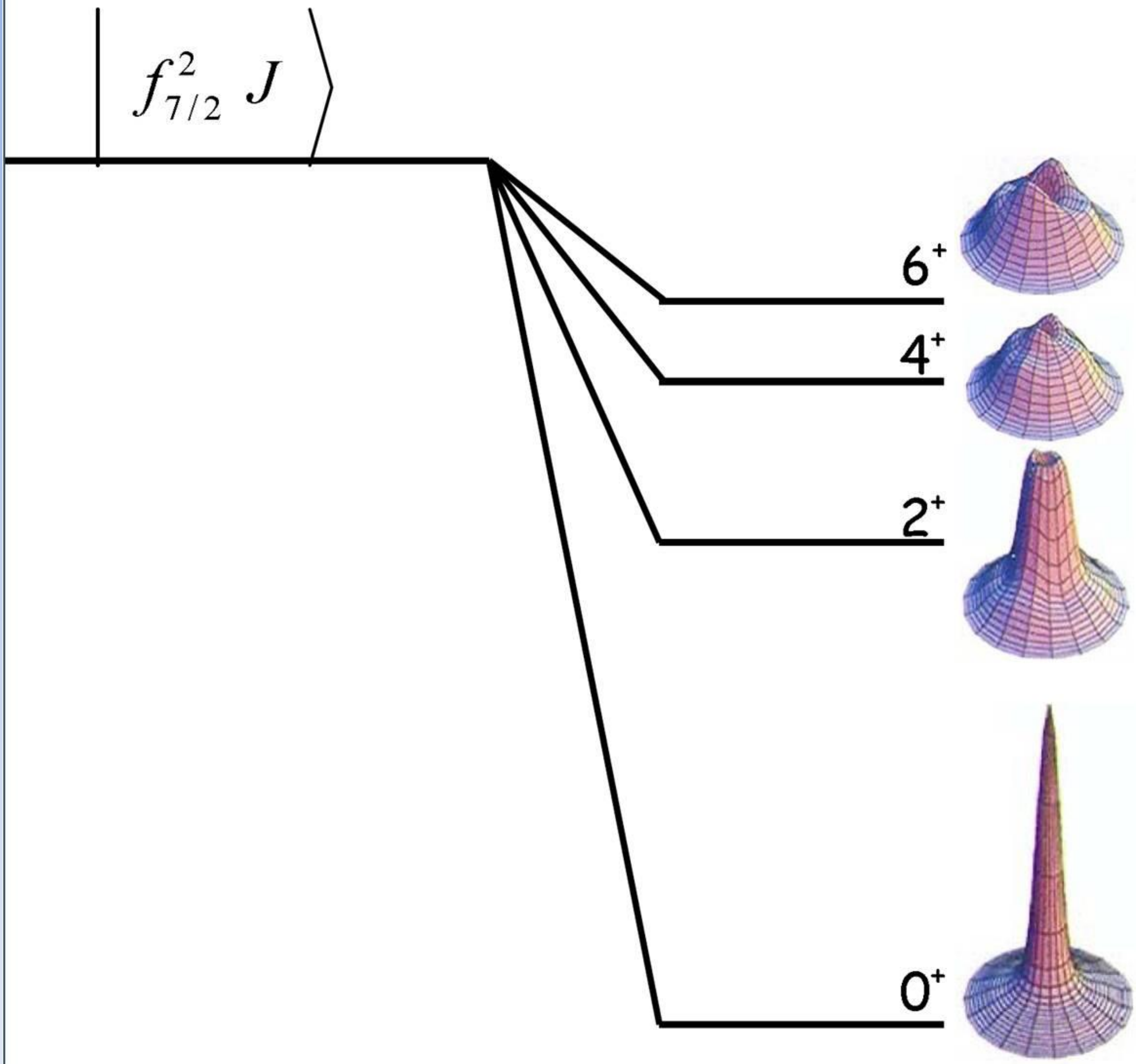
For $J = 0 \quad \Delta E \propto -V_0 \dfrac{2j+1}{2}$ ⇒ ΔE larger for larger j: Why???

Geometrical Interpretation

for $\left| j^2 J=0 \right\rangle$ being lowest



Pauli Principle is ~ repulsive interaction !


$|f_{7/2}^2\ J\rangle$
6+
4+
2+
0+

δ Interaction **Analytic formulas**

$$V_{12}(\delta) = -V_0\,\delta(r_1 - r_2) = \frac{-V_0}{r_1 - r_2}\,\delta(r_1 - r_2)\,\delta(\cos\theta_1 - \cos\theta_2)\,\delta(\Phi_1 - \Phi_2)$$

$$\Delta E(j_1 j_2 J) = -V_0 F_R(n_1 l_1 n_2 l_2) A(j_1 j_2 J)$$

where

$$F_R(n_1 l_1 n_2 l_2) = \frac{1}{4\pi}\int \frac{1}{r^2} R^2_{n_1 l_1}(r) R^2_{n_2 l_2}(r)\,dr$$

and

$$A(j_1 j_2 J) = (2j_1+1)(2j_2+1)\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \quad (\text{if } l_1 + l_2 - J \text{ is even})$$

$$= 0 \quad (\text{if } l_1 + l_2 - J \text{ is odd})$$

(Non-equivalent orbits)

$$\Delta E(j^2 J) = -V_0 F_R(nl) A(j^2 J) \quad (J \text{ even})$$

where

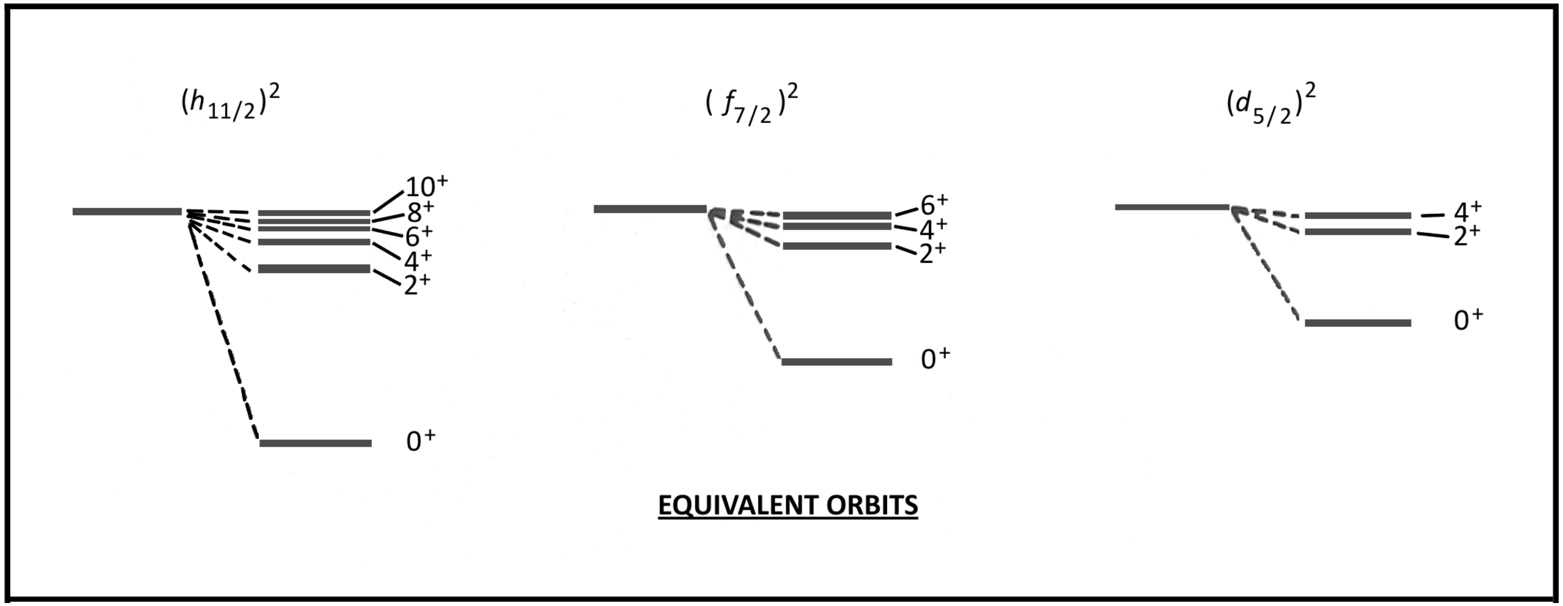
$$F_R(nl) = \frac{1}{4\pi}\int \frac{1}{r^2} R^4_{nl}(r)\,dr$$

and

$$A(j^2 J) = \frac{(2j+1)^2}{2}\begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \quad (J \text{ even})$$

(Equivalent orbits)

MULTIPLET SPLITTINGS; δ INTERACTION (Identical Particles)



NOTE: $R_{4/2} < 2.0$

Simple treatment of residual interactions accounts for universal fact that even-even nuclei have 0^+ ground states.

Note that the 0^+ level is lowered more for higher j orbits

Lowering of $0^{\pm}$ States

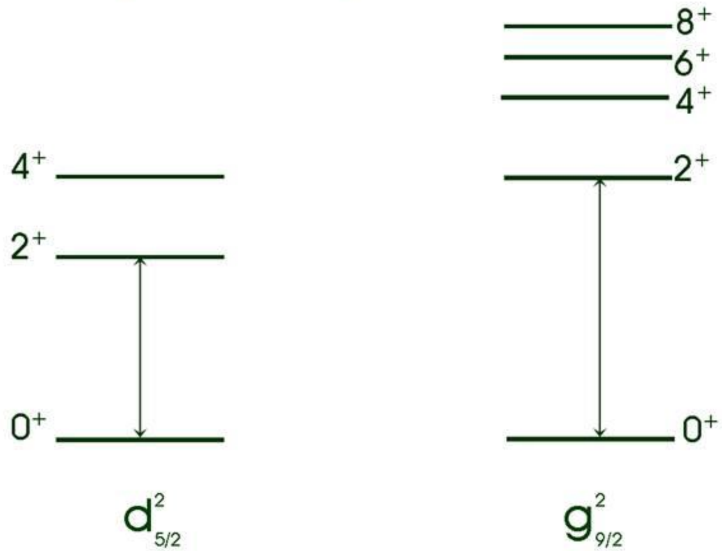
$$\Delta E\,(j^2\,J) \propto -V_0\,\frac{(2j+1)^2}{2}\begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

For $J = 0$

$$\Delta E\,(j^2\,J = 0) \propto -V_0\,\frac{(2j+1)}{2}$$

$\Longrightarrow$ $\Delta E \propto 2j + 1$

Energy lowering of 0^+
is larger for larger j



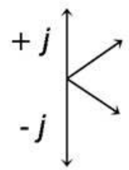
Why ?

Lowering of 0^+ States in $|j^2 J\rangle$

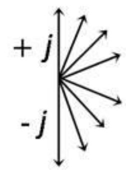
$\Delta E \propto 2j+1$. Why?

Note: $2j+1 = \#$ magnetic substates

low j



high j



Semi-classical picture-localized

$\Psi(J, m, \Theta)$ is localized to an angular range* centered about normal to ang. mom. vector:

spread of Ψ roughly given by angular "distance" to next substate

*quantum fluctuations

$\therefore$ Larger j $\Rightarrow$ more magnetic substates

$\Rightarrow$ greater localization

$\Rightarrow$ greater spatial overlap in $|j, m\rangle$ and $|j, -m\rangle$

$\Rightarrow$ lower energy

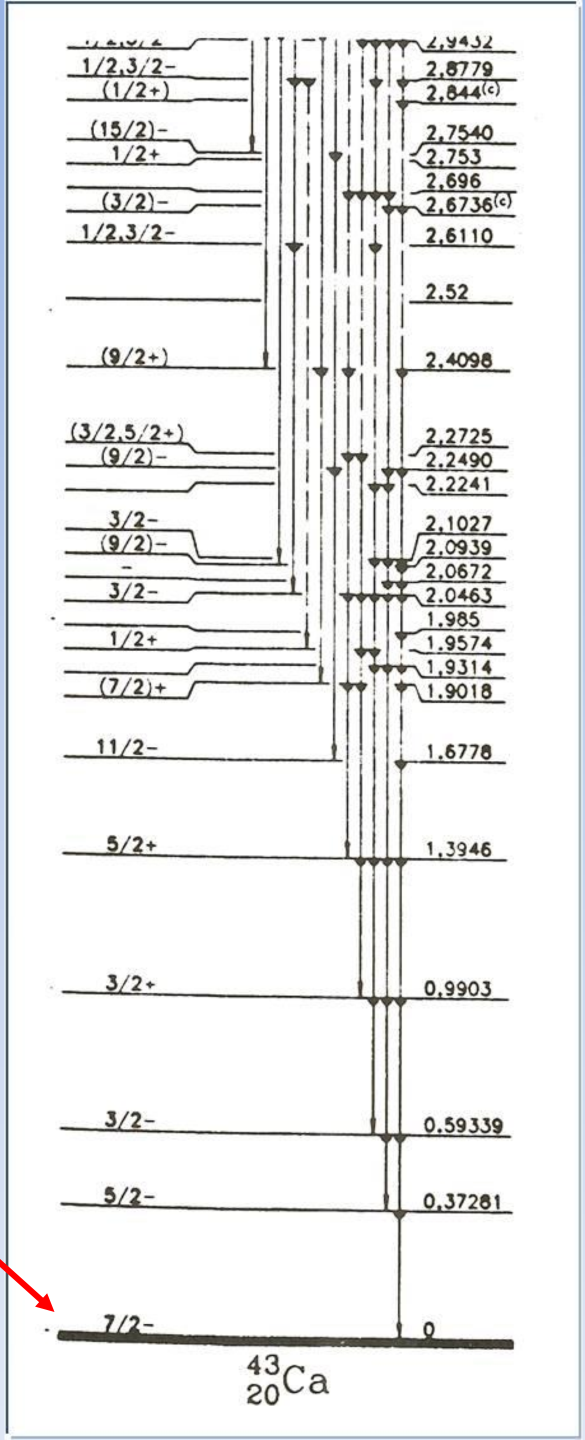
Extending the IPM with residual interactions

- Consider now an extension of, say, the Ca nuclei to ^{43}Ca, with three particles in a j= 7/2 orbit outside a closed shell?
- How do the three particle angular momenta, j, couple to give final total J values?
- If we use the m-scheme for three particles in a 7/2 orbit the allowed J values are 15/2, 11/2, 9/2, 7/2, 5/2, 3/2.
- For the case of J = 7/2, two of the particles must have their angular momenta coupled to J = 0, giving a total J = 7/2 for all three particles.
- For the J = 15/2, 11/2, 9/2, 5/2, and 3/2, there are no pairs of particles coupled to J = 0.
- Since a J = 0 pair is the lowest configuration for two particles in the same orbit, that case, namely total J = 7/2, must lie lowest !!

^{43}Ca

Treat as 20 protons and 20 neutrons forming a doubly magic core with angular momentum J = 0. The lowest energy for the 3-particle configuration is therefore J = 7/2.

Note that the key to this is the results we have discussed for the 2-particle system !!



GROUND STATE J^{π} VALUES OF SOME ODD MASS NUCLEI

Z = 20	$\frac{3/2^+}{^{37}\mathrm{Ca}}$	$\frac{3/2^+}{^{39}\mathrm{Ca}}$	$\frac{7/2^-}{^{41}\mathrm{Ca}}$	$\frac{7/2^-}{^{43}\mathrm{Ca}}$	$\frac{7/2^-}{^{45}\mathrm{Ca}}$	$\frac{7/2^-}{^{47}\mathrm{Ca}}$	$\frac{3/2^-}{^{49}\mathrm{Ca}}$
Z = 40		$\frac{9/2^+}{^{87}\mathrm{Zr}}$	$\frac{9/2^+}{^{89}\mathrm{Zr}}$	$\frac{5/2^+}{^{91}\mathrm{Zr}}$	$\frac{5/2^+}{^{93}\mathrm{Zr}}$	$\frac{5/2^+}{^{95}\mathrm{Zr}}$	
Z = 39	$\frac{1/2^-}{^{85}\mathrm{Y}}$	$\frac{1/2^-}{^{87}\mathrm{Y}}$	$\frac{1/2^-}{^{89}\mathrm{Y}}$	$\frac{1/2^-}{^{91}\mathrm{Y}}$	$\frac{1/2^-}{^{93}\mathrm{Y}}$	$\frac{1/2^-}{^{95}\mathrm{Y}}$	$\frac{1/2^-}{^{97}\mathrm{Y}}$
Z = 41		$\frac{9/2^+}{^{91}\mathrm{Nb}}$	$\frac{9/2^+}{^{93}\mathrm{Nb}}$	$\frac{9/2^+}{^{95}\mathrm{Nb}}$	$\frac{9/2^+}{^{97}\mathrm{Nb}}$	$\frac{9/2^+}{^{99}\mathrm{Nb}}$	
N = 50	$\frac{3/2^-}{^{85}_{35}\mathrm{Br}}$	$\frac{3/2^-}{^{87}_{37}\mathrm{Rb}}$	$\frac{1/2^-}{^{89}_{39}\mathrm{Y}}$	$\frac{9/2^+}{^{91}_{41}\mathrm{Nb}}$	$\frac{9/2^+}{^{93}_{43}\mathrm{Tc}}$	$\frac{9/2^+}{^{95}_{45}\mathrm{Rh}}$	

Now, lets move beyond near-magic regions

What about nuclei with more valence nucleons, with valence nucleons of both types, and with nucleons able to occupy more than one single particle state?

Can form many states of a given angular momentum.

What happens? Emergence of collectivity due to configuration mixing. Lets look at the problem first and then possible "solutions".

THE PROBLEM

As the number of valence nucleons grows, the number of ways of making states of a given J grows hugely.

Those "basis states" will mix. How many states do we need to mix? What are the resulting structures? How difficult a calculation is this? Consider a couple of simple cases and a more typical one.

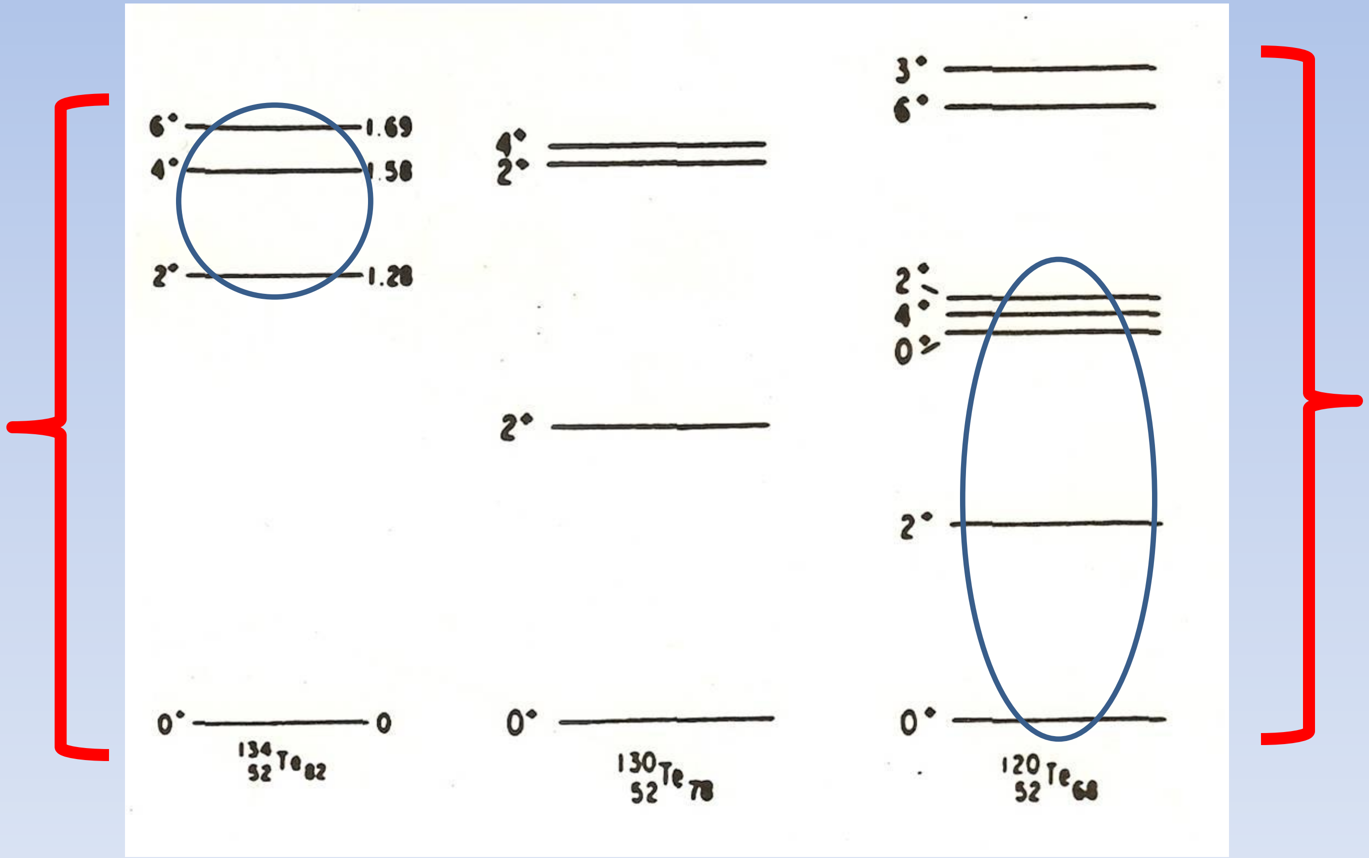
The Need for Simplification in Multiparticle Spectra

Example: How many 2^+ states?

# nucl.		
2	$d^2_{5/2}$	1

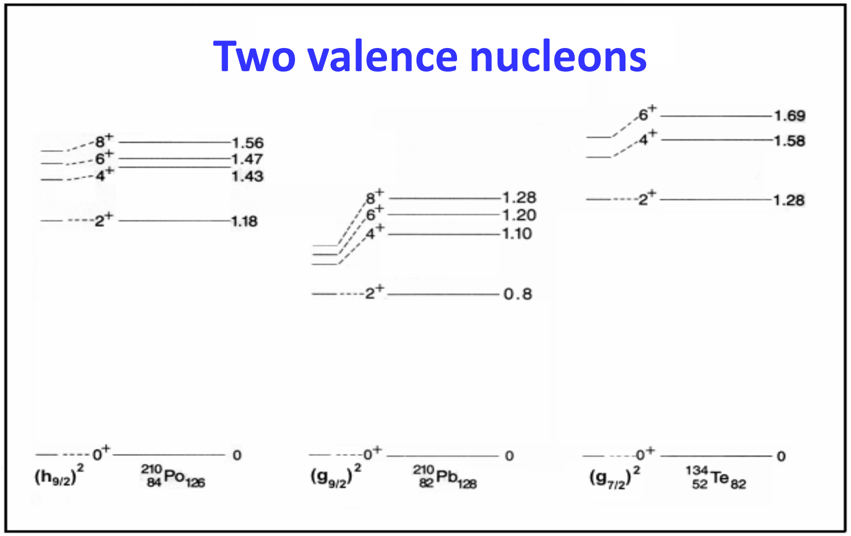
These states mix !!

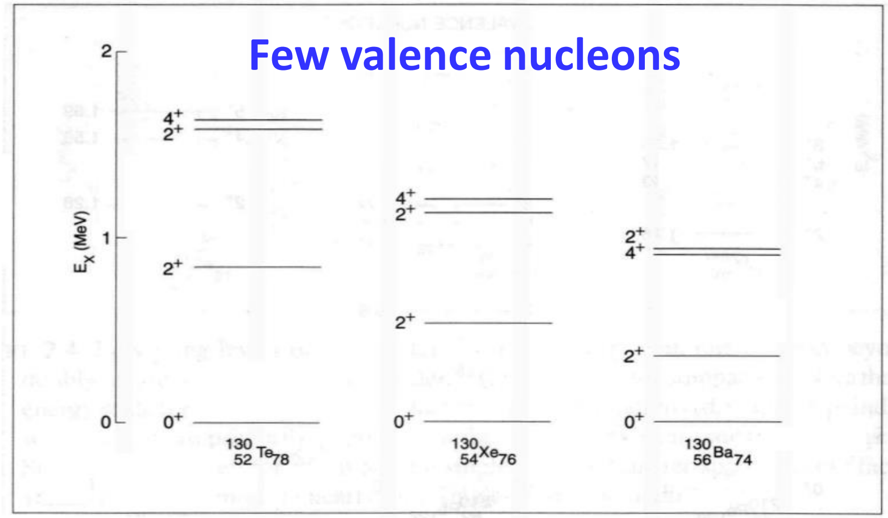
So, with even just a few valence nucleons, such calculations become intractable by simple diagonalization. But yet, nuclei show very simple patterns despite the complexity and chaotic behavior one might expect. Emergence of collective behavior.

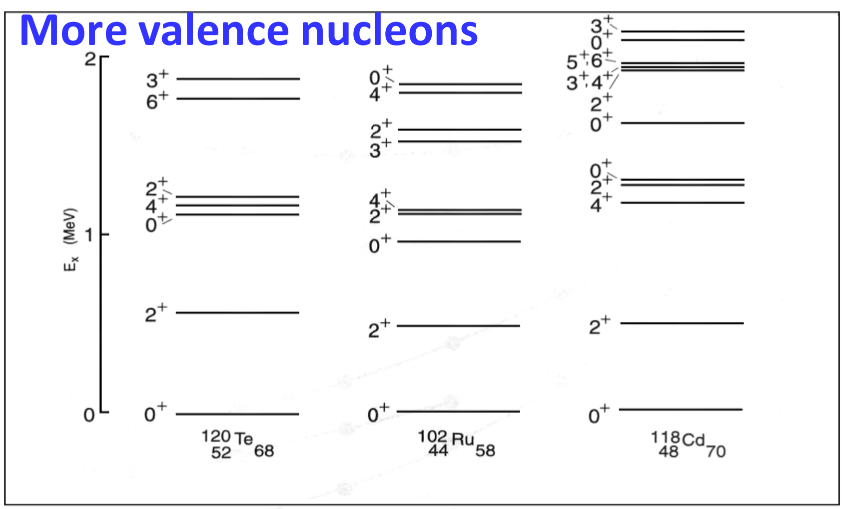


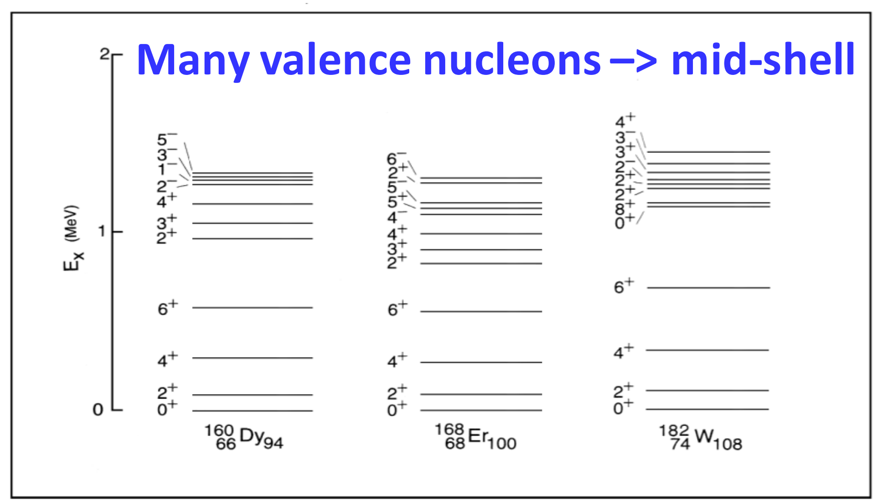
How can we understand emergent collectivity?

Reminder: several types of spectra and where they occur









Two approaches

a) **Advanced methods at the level of nucleons and their interactions**

b) **Collective models that look at the many-body system as a whole, with its shapes, oscillations, quantum numbers, selection rules, etc.**

We will follow this second route but then return to ask what the microscopic drivers of structural evolution and emergent collectivity are.

Development of collective behavior in nuclei

- Results primarily from correlations among valence nucleons.
- Instead of pure "Independent Particle model" or 2-particle configurations configurations, the wave functions are mixed – linear combinations of many components.
- Leads to a lowering of the collective states and to enhanced transition rates as characteristic signatures.
- How does this happen? Consider mixing of states.

In any many-body system it will generally be possible to make states of a given angular momentum in more than one way.

In the simplest model those ways are independent and separate states.

In realistic situations those states form a "basis" for the construction of real physical states that are mixtures of the basis states.

Conceptually:

Start from a simple model, construct basis states.

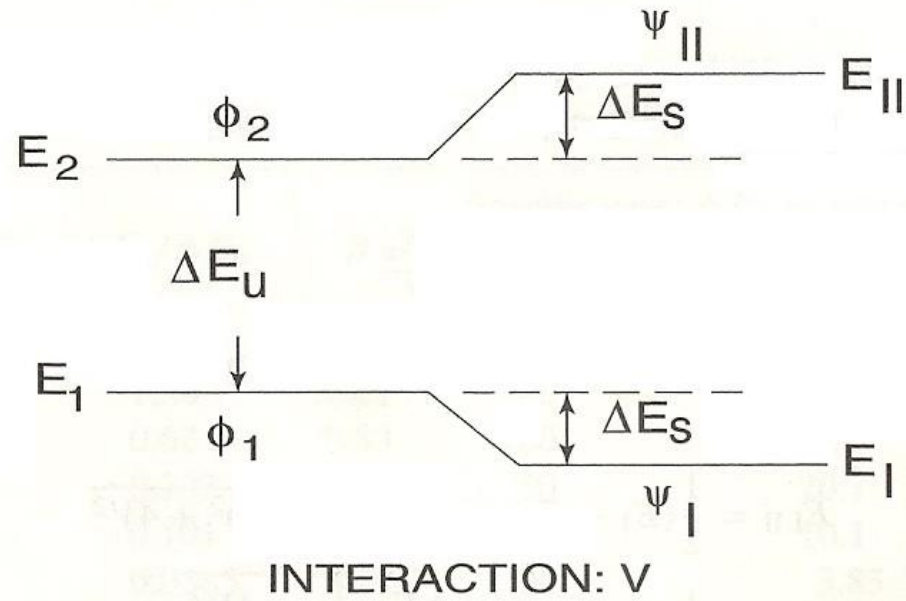
Compare with data. Usually find reasonable agreement but significant discrepancies.

Improve the model by introducing interactions ("residual interactions") that mix the basis states.

Compare with data

Mixing of quantum mechanical states

The essential key to understanding any many-body system



The mixed wave functions are

$$\psi_{\mathrm{I}} = \alpha\phi_1 + \beta\phi_2$$

$$\alpha^2 + \beta^2 = 1$$

$$\psi_{\mathrm{II}} = -\beta\phi_1 + \alpha\phi_2$$

where the smaller amplitude β is given by

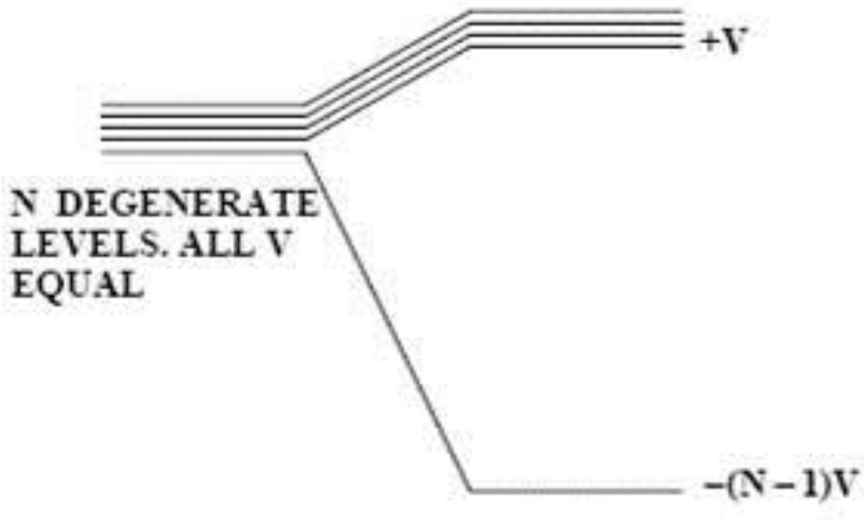
$$\beta = \frac{1}{\left\{1 + \left[R/2 + \sqrt{1 + R^2/4}\right]^2\right\}^{\frac{1}{2}}} \qquad R = \frac{\Delta E_u}{V}$$

Table 1.2 *Examples of two-state mixing energy shifts and mixing amplitudes (from Eqs. 1.6 and 1.8).* $R = \Delta E_u / V$

R^*	$\Delta E_s / \Delta E_u$	β	Specific case: $\Delta E_u = 100$ keV	
			V (keV)	ΔE_s (keV)
0.2	4.52	0.67	500	452
0.5	1.56	0.61	200	156
1	0.62	0.53	100	62
2	0.207	0.38	50	20.7
3	0.101	0.29	33.3	10.1
5	0.0385	0.19	20	3.85
10	0.0099	0.099	10	0.99
20	0.0025	0.050	5	0.25

*For $R = 0$, $\beta = 0.707$, and $\Delta E_s = V$.

Extend to many-level case – Mixing of N configurations. Collective states at low energy



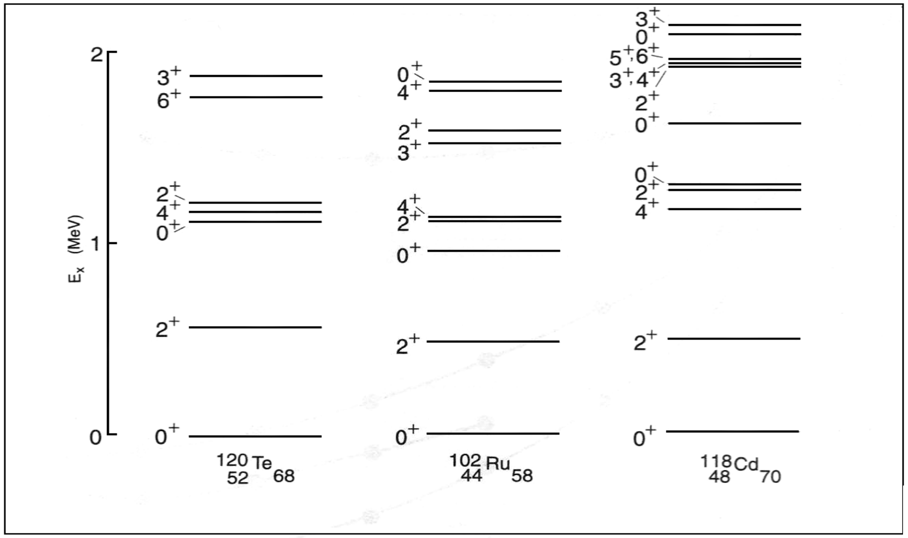
$$\Psi_{\text{LOWEST}} = \frac{1}{\sqrt{N}}\left[\phi_1 + \phi_2 + \ldots + \phi_N\right]$$

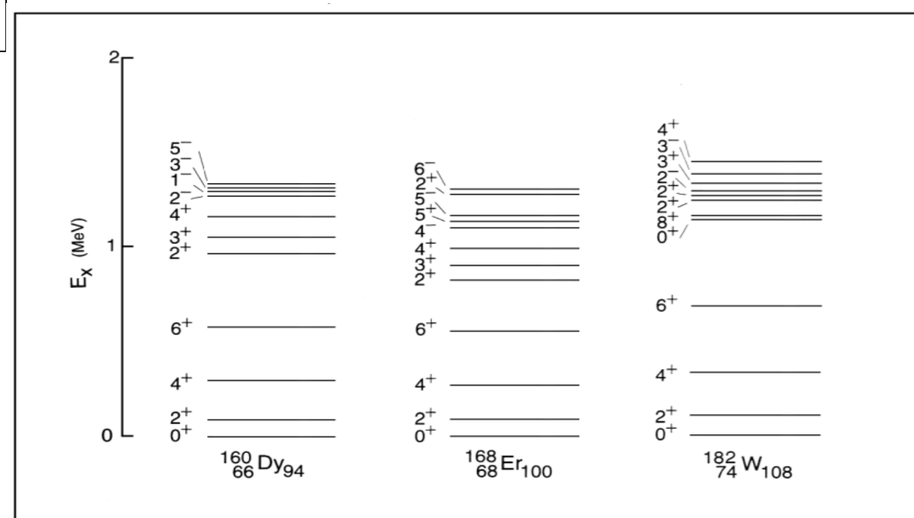
Lowering of one state. Note that the components of its wave function are all equal and in phase

Please think about this carefully – it is one of the most important and most general concepts in all of many-body physics

Consequences of this: Lower energies for collective states, and enhanced transition rates.

Now lets go back to some data we saw earlier and try to understand it in terms of collective structures





First consider nuclei with a moderate number of valence nucleons (~ 6-16).

These nuclei retain the spherical shapes of nuclei near closed shells but are “soft” -- they can take on oscillatory vibrational motion. The lowest lying such excitation is a small amplitude surface quadrupole oscillation with angular momentum 2

2^+ ———— **J = 2 one “phonon” vibrational excitation**

0^+ ————

More than one phonon? What angular momenta? M-scheme for phonons

Table 6.1 *m scheme for two-quadrupole phonon states**

$J_1 = 2$ m_1	$J_2 = 2$ m_2	$M = \sum m_i$	J
2	2	4	4
2	1	3	
2	0	2	
2	−1	1	
2	−2	0	
1	1	2	2
1	0	1	
1	−1	0	
0	0	0	0

*Only positive total M values are shown: the table is symmetric for $M < 0$. The full set of allowable m_i values giving $M \geq 0$ is obtained by the conditions $m_1 \geq 0$, $m_2 \leq m_1$.

Homework: What angular momenta are allowed for three quadrupole bosons?

Types of collective structures

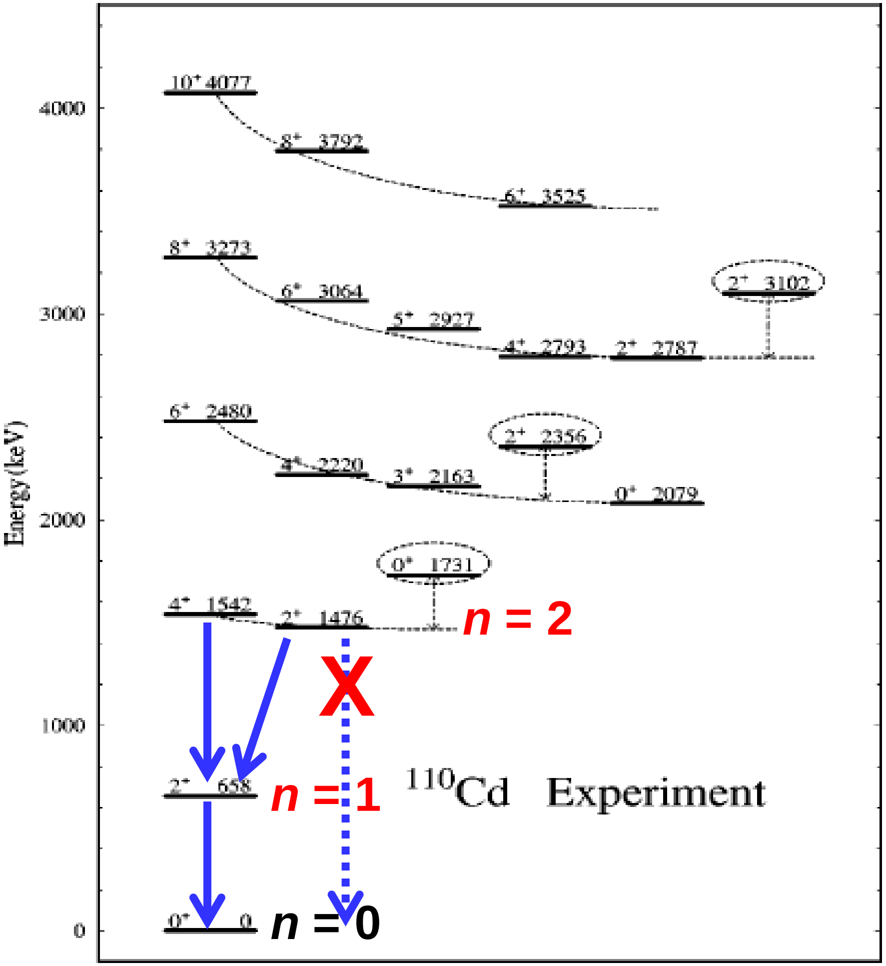
Few valence nucleons of each type: Remember this? Now we see it as a spherical vibrator

Vibrator (H.O.)

$$E(J) = n\,(\hbar\,\omega_0)$$

$$R_{4/2} = 2.0$$

Gamma-ray transitions:
Selection rule: Can destroy only one phonon



Deformed Nuclei

- What is different about non-spherical nuclei?

- They can **ROTATE** !!!

- They can also **VIBRATE**
  - For axially symmetric deformed nuclei there are two low lying vibrational modes called β and γ

- So, levels of deformed nuclei consist of the ground state, and vibrational states, with rotational sequences of states (rotational bands) built on top of them.

A subtle concept about deformation

Ground states of all e-e nuclei, including deformed ones, are J = 0^+

Such wave functions are spherically symmetric (Y_{00})

So how can the nucleus be deformed?

The angular momentum 0^+ is in the laboratory frame of reference. The ellipsoidal shape is in the nuclear (body-fixed frame). In going from the body-fixed frame to the lab, the nucleus can have any orientation so all are equally probable, hence the density distribution in the lab is spherically symmetric.

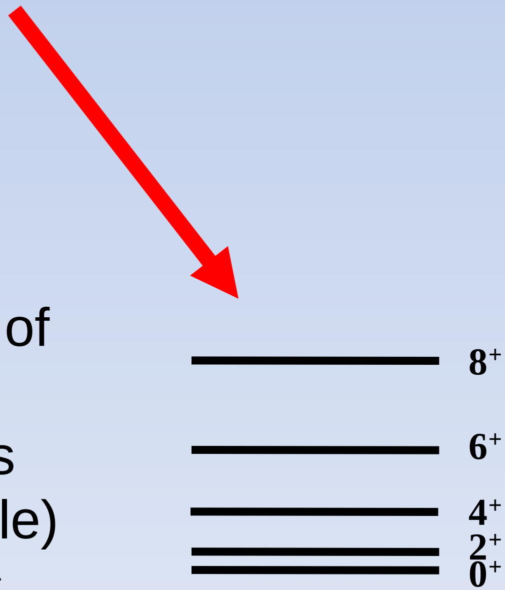
A related point: How can circular nucleon orbits give a deformed shape? "Circular" is not the same as "spherical". Deformation is equivalent to a non-uniform occupation of magnetic substates resulting from mixing.

Deformed nuclei – rotational spectra

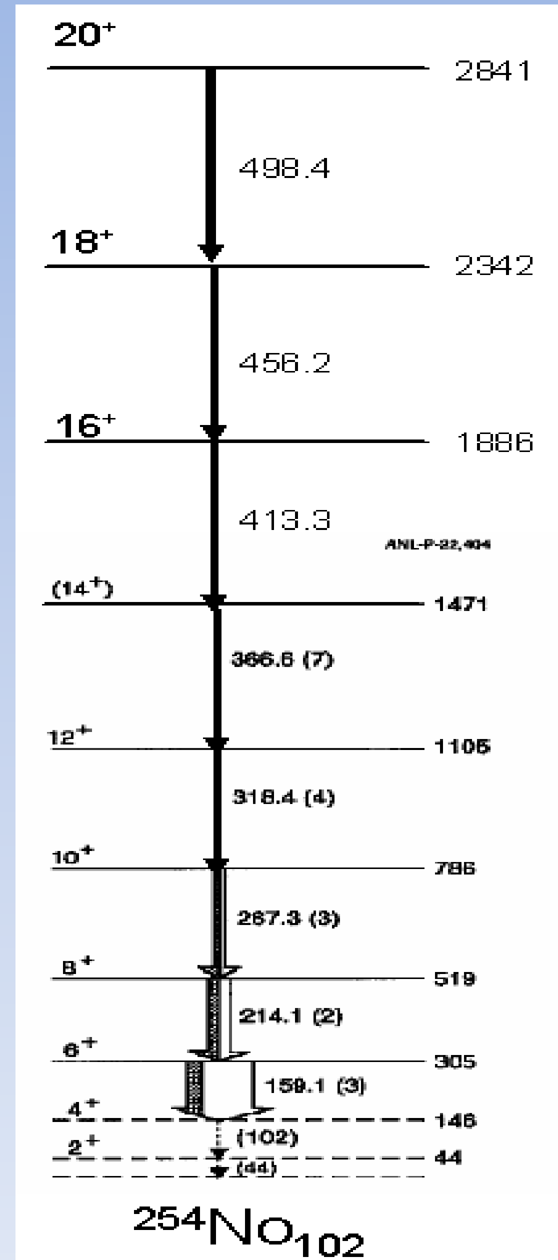
Rotor

$E(I) \propto (\hbar^2/2I)I(I+1)$

$R_{4/2} = 3.33$



BTW, note value of paradigm in spotting physics (otherwise invisible) from deviations



Value of paradigms

J	E (keV)	Paradigm Benchmark: Rotor J(J + 1)
6^+	690	700
4^+	330	333
2^+	100	100
0^+	0	0

?

Without rotor paradigm

Paradigm Benchmark

Rotor J(J + 1)

Amplifies structural differences

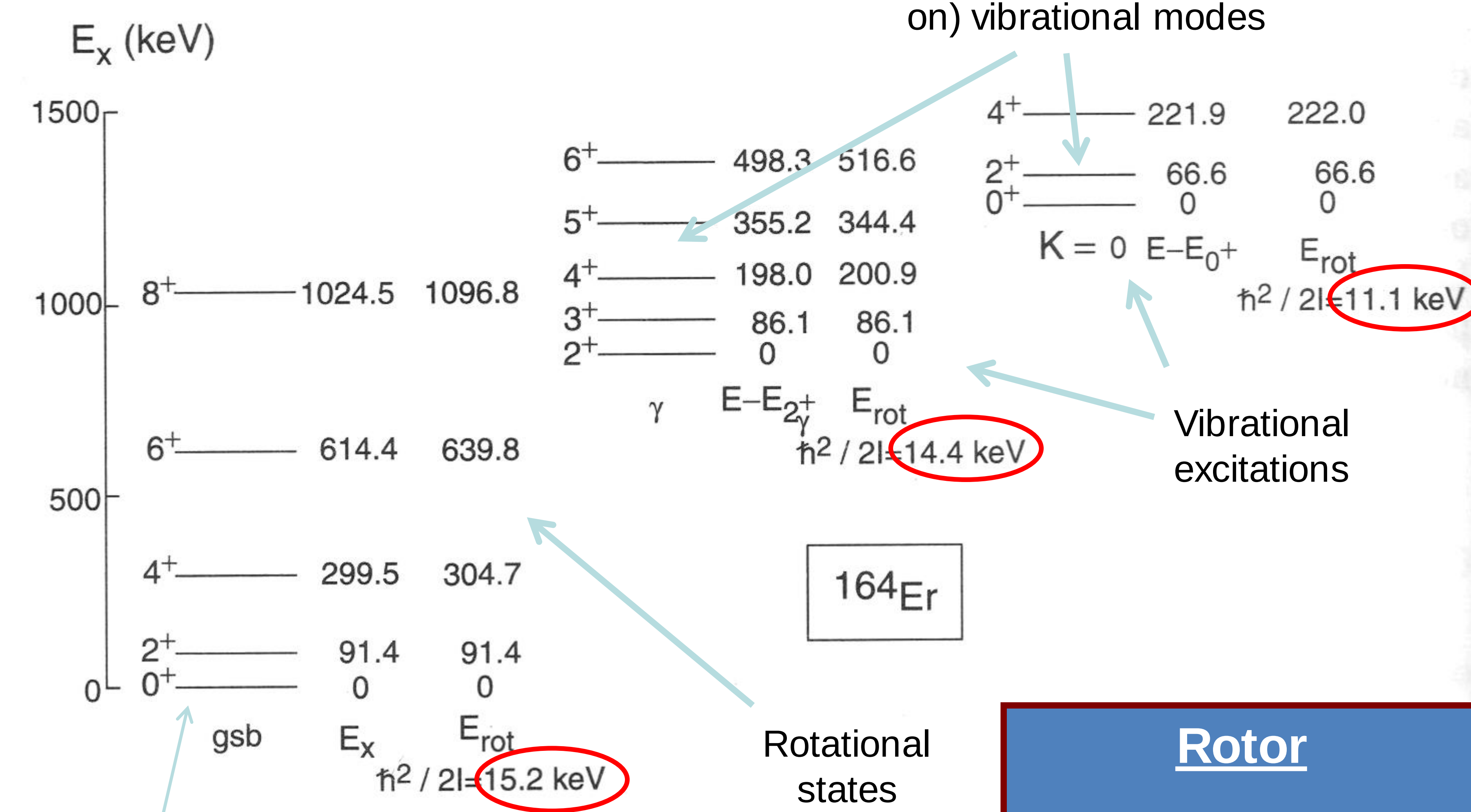
Centrifugal stretching

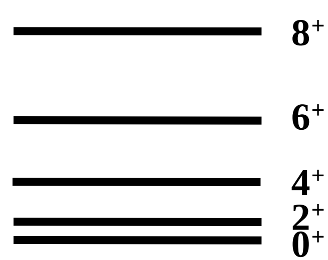
Deviations

↓

Identify additional degrees of freedom

Typical deformed nucleus





Rotor

$$E(I) \propto (\hbar^2/2I)I(I+1)$$

$$R_{4/2} = 3.33$$

Doubly magic plus 2 nucleons

$R_{4/2} < 2.0$

Vibrator (H.O.)

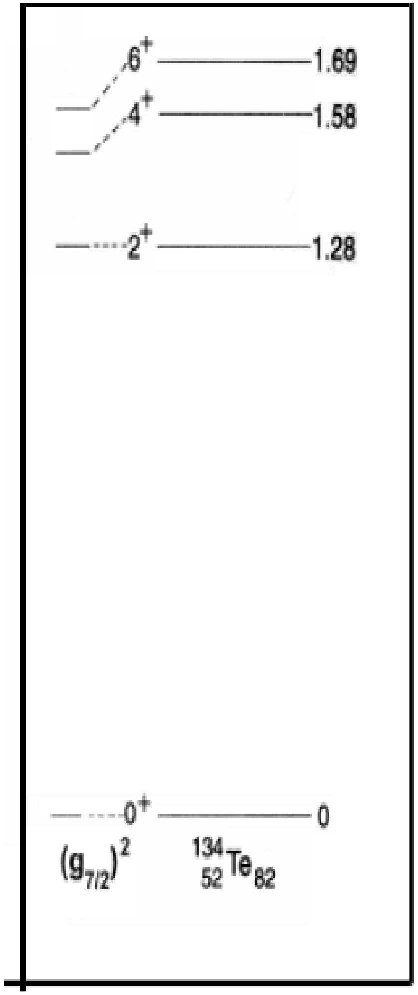
$E(J) = n\,(\hbar\,\omega_0)$

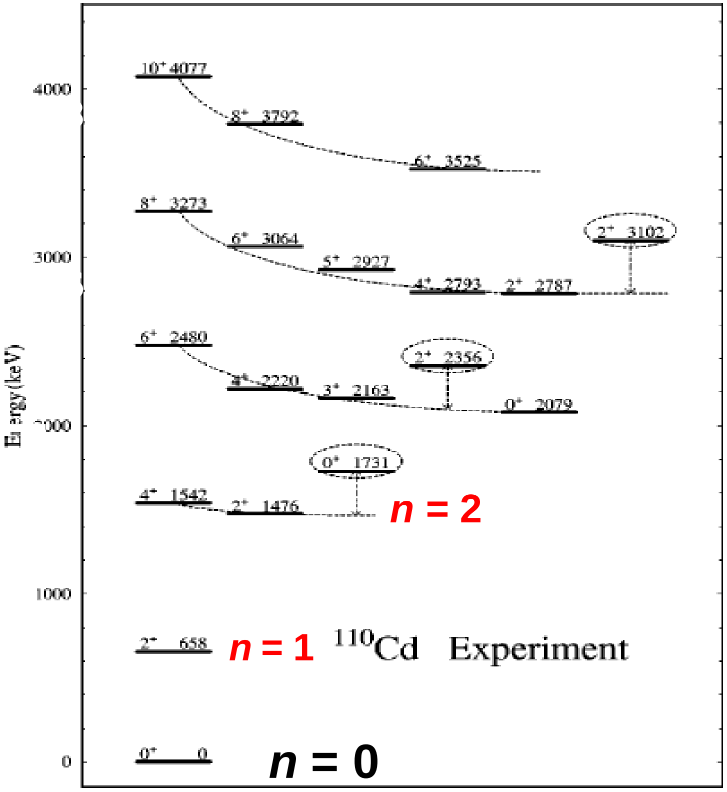
$R_{4/2} = 2.0$

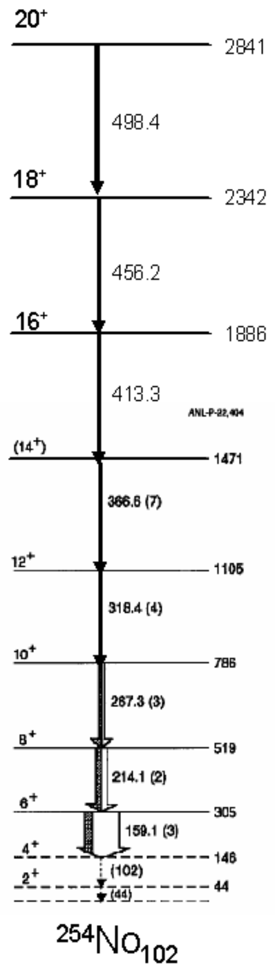
Rotor

$E(J) \propto (\hbar^2/2I)J(J+1)$

$R_{4/2} = 3.33$

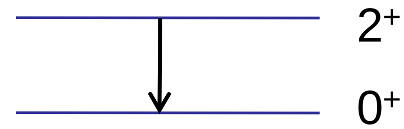


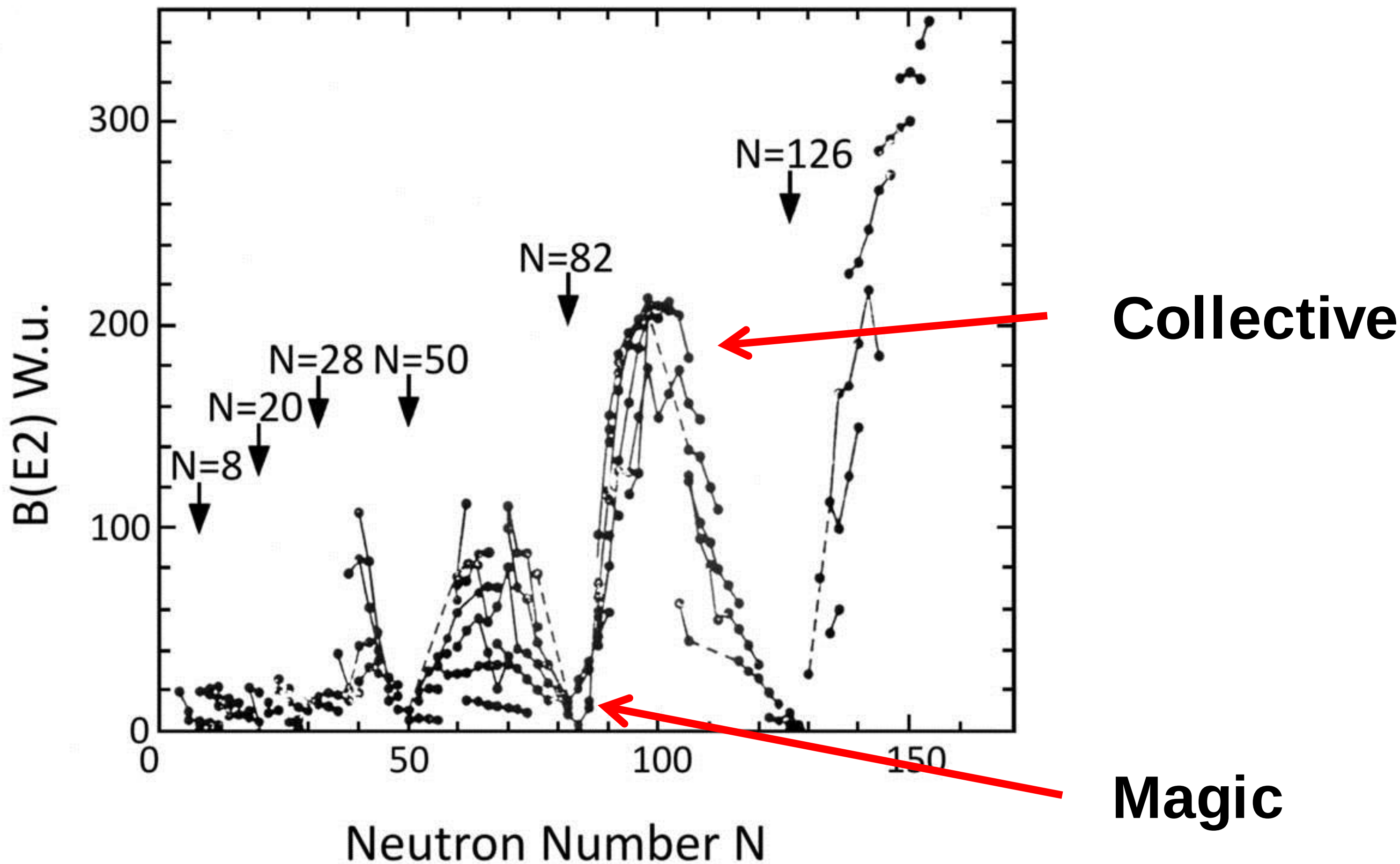




Transition rates (half lives of excited levels) also tell us a lot about structure

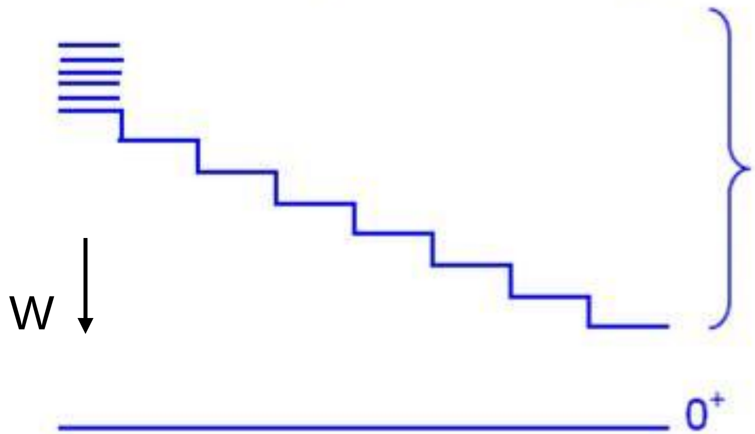
$$B(E2: 0^+_1 \rightarrow 2^+_1) \propto \langle 2^+_1 || E2 || 0^+_1 \rangle^2$$





Coherence and Transition Rates

Consider simple case of N degenerate levels: 2^+



$\Delta E = (N-1)V$

$\Psi = a\varphi_1 + a\varphi_2 + \cdots a\varphi_N$

where $a = \frac{1}{\sqrt{N}}$

$\left(\sum_{i=1}^{N} a^2 = \frac{N}{N} = 1 \right)$

Consider transition rate from $2_1^+ \rightarrow 0_1^+$

$$B(E2;\, 2_1^+ \rightarrow 0_1^+) = \frac{1}{2J_i + 1} \left\langle 0_1^+ \| E2 \| 2_1^+ \right\rangle^2$$

$$\left\langle 0_1^+ \| E2 \| 2_1^+ \right\rangle = \left\langle 0_1^+ \| E2 \| \Psi \right\rangle = a \sum_{i=1}^{N} \left\langle 0_1^+ \| E2 \| \varphi_i \right\rangle$$

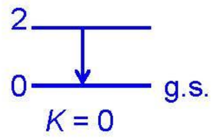
The more configurations that mix, the stronger the B(E2) value and the lower the energy of the collective state. **Fundamental property of collective states.**

Relation of B(E2) values to the nuclear shape.

Quadrupole Moment of Ellipsoid

$$Q_0 = \frac{3e}{\sqrt{5\pi}} Z R^2 \beta \left[1 + \underbrace{0.16\beta}_{} \right]$$

higher order term ($\propto \beta^2$)



$$B\left(E2; J_i \to J_f\right) = \frac{5}{16\pi} Q_0^2 \left\langle J_i K\, 20 \middle| J_f K \right\rangle^2 e^2 b^2$$

For $0_1^+ \to 2_1^+$:

$$B\left(E2;\, \mathbf{0}_1^+ \to \mathbf{2}_1^+\right) = \frac{5}{16\pi} Q_0^2$$

$$\frac{B(E2)}{e^2} = \frac{5}{16\pi} \left(\frac{3}{\sqrt{5}\,\pi} Z R^2 \beta \right)^2$$

or $$\beta = \frac{4\pi}{3 Z R^2} \sqrt{B(E2)/e^2}$$