Covariance analysis and an example: SLy5-min

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1 Introduction

2 Formalism

Let us consider a model characterized by a number of parameters $\mathbf{p} = (p_1, ..., p_n)$ defining the model space. Those parameters may be, for example, coupling constants of the effective Hamiltonian and effective charges characterizing operators in the assumed Hilbert space. Calculated observables are functions of these parameters. Because the number of parameters is usually much smaller than the number of observables, correlations exist between computed quantities. Moreover, because the model space has been optimized to a limited set of observables, there may also exist correlations between parameters.

2.1 χ^2 definition

Usually, most of the model space produces observables that are far from reality. Therefore, one needs to confine the model space to a "physically reasonable" domain. That can be achieved by a least-squares regression analysis. To this end, one selects a pool of fit observables $\mathcal{O}$ that are used to calibrate p. The optimum parametrization p_0 is determined by a least-squares fit with the global quality measure,

$$\chi^2(\mathbf{p}) = \sum_{\imath=1}^{m} \left(\frac{\mathcal{O}_\imath^{\text{theo.}} - \mathcal{O}_\imath^{\text{ref.}}}{\Delta\mathcal{O}_\imath^{\text{ref.}}} \right)^2 \tag{1}$$

where "theo." stands for the calculated values, "ref." for experimental and/or semi-empirical data, and $\Delta\mathcal{O}^{\text{ref.}}$ for the adopted errors. In cases where the experimenatal data is used one could adopt, in principle, the experimental error. The latter choice is not always reasonable since, *e.g.* in the nuclear case, experiments on masses are beyond the accuracy of available models and generally more precise than other measured observables which also use to enter in the fits such as charge radius, neutron skins, spin orbit splittings,etc.. Such a situation force one to have some freedom in choosing a convinient value for each $\Delta\mathcal{O}^{\text{ref.}}$. There is not a unique criterium for this and one should try to "equilibrate" the different terms in the χ^2 according to the final purpose of the fit. In this sense the adopted error can also be undesrtood as a weight in the various terms defining the function to be minimized. A post-optimiztion check to know if one did a proper selection of such values is to see if each term contribute around one to the χ^2. One can easily see why form Eq. 1,

$$\mathcal{O}_\imath^{\text{theo.}} - \mathcal{O}_\imath^{\text{ref.}} = \Delta\mathcal{O}_\imath^{\text{ref.}}, \tag{2}$$

which means that for all $\mathcal{O}_\imath$ for $\imath = 1, ..., m$, the deviation in the theoretical description of each observable is inside the adopted uncertainty or, if experimetal errors are used, within one standard deviation.

2.2 Covariance analysis of parameters and observables

Asuming that the χ^2 is a well behaved (analytical) function in the vicinity of the minimum, *i.e.* around $\chi^2(\mathbf{p}_0)$,

$$\partial_{\mathbf{p}}\chi^2(\mathbf{p}) \mid_{\mathbf{p}=\mathbf{p}_0} = 0 \quad , \tag{3}$$

and the χ^2 near the minimum can be approximated as an hyper-parabola in the parameter space $\mathbf{p}$,

$$\begin{aligned} \chi^2(\mathbf{p}) - \chi^2(\mathbf{p}_0) &\approx \frac{1}{2}\sum_{\imath,\jmath}^{n}(p_\imath - p_{0\imath})\partial_{p_\imath}\partial_{p_\jmath}\chi^2(p_\jmath - p_{0\jmath}) \\ &\equiv \sum_{\imath,\jmath}^{n}(p_\imath - p_{0\imath})\mathcal{M}_{\imath\jmath}(p_\jmath - p_{0\jmath}) \end{aligned} \tag{4}$$

where the curvature matrix, $\mathcal{M}$, provide us access to estimate the errors ($\mathbf{e}$) of the fitted parameters —according to our definition of the χ^2— since its inverse is the so called error matrix ($\mathcal{E}$),

$$e_\imath \equiv e(p_\imath) = \sqrt{(\mathcal{M}^{-1})_{\imath\imath}} \equiv \sqrt{\mathcal{E}_{\imath\imath}} \quad , \tag{5}$$

and also the correlations ($\mathcal{C}$) between parameters,

$$\mathcal{C}_{\imath\jmath} \equiv \frac{\mathcal{E}_{\imath\jmath}}{\sqrt{\mathcal{E}_{\imath\imath}\mathcal{E}_{\jmath\jmath}}} \tag{6}$$

where $\mathcal{C}_{\imath\jmath}$ takes values form -1 to 1. $\mid \mathcal{C}_{\imath\jmath} \mid \approx 1$ indicates a large correlation between parameters $p_\imath$ and $p_\jmath$ which means that one of them is redundant and can be fixed during the fit setting its value according to the other and $\mathcal{C}_{\imath\jmath}$ around zero which means that no correlation holds at all between parameters $p_\imath$ and $p_\jmath$.

Moreover, once determined the set of parameters minimizing the χ^2, an expectation value (and deviation) of an observable A, not included in the fit, can be computed at $A(\mathbf{p}_0)$. The uncertainties in the prediction of such observable are originated by the adopted errors in the fitted observables and from the reliability of the model —which cannot be estimated quantitatively if it is not compared with other models. To estimate such an error —its "adopted-standard deviation" in a sense— one can expand the observable under study, $A(\mathbf{p})$, around the minimum $\mathbf{p}_0$ assuming a smooth behavior of the former as a function of the latter and, then, neglecting the second order derivatives,

$$\begin{aligned} A(\mathbf{p}) &= A(\mathbf{p}) + (\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}} A(\mathbf{p}) \mid_{\mathbf{p}=\mathbf{p}_0} \\ &\equiv A_0 + (\mathbf{p}-\mathbf{p}_0)\mathbf{A}_0 \end{aligned} \tag{7}$$

within this approximation the statistical expectation value of the observable $\overline{A}$ would coincide exactly with A_0. This can be demonstrated if we assume a Gaussian distribution of the different parametrizations around the minimum, *i.e.* if we assume that the set of reasonable models fulfilling the condition $\chi^2(\mathbf{p}) - \chi^2(\mathbf{p}_0) \leq 1$ is distributed following a Gaussian probability law. This would mean that the probablity distribution can be written as

$$\mathcal{P}(\mathbf{p}) = \mathcal{N} \exp\left(-\frac{1}{2}(\mathbf{p}-\mathbf{p}_0)\mathcal{M}(\mathbf{p}-\mathbf{p}_0)\right) \tag{8}$$

where $\mathcal{N}$ is the normalization constant and, therefore, the expected value, $\overline{A}$, can be calculated as

$$\begin{aligned} \overline{A} &= \int A(\mathbf{p})\mathcal{P}(\mathbf{p})d\mathbf{p} \\ &\approx A_0 \int \mathcal{P}(\mathbf{p})d\mathbf{p} + \int (\mathbf{p}-\mathbf{p}_0)\mathbf{A}_0 \mathcal{P}(\mathbf{p})d\mathbf{p} \\ &\approx A_0 \end{aligned} \tag{9}$$

since —remember— we have neglected second derivatives with respect the parameters, $\mathbf{p}$, of the observables, $A(\mathbf{p})$. The second integral does not contribute since $\mathcal{P}(\mathbf{p})$ is symmetric with respect to all parameters, $\mathcal{P}(\mathbf{p}) = \mathcal{P}(-\mathbf{p})$, and the factor $(\mathbf{p}-\mathbf{p}_0)$ is not. Note that $\mathbf{A_0} \equiv \partial_{\mathbf{p}} A(\mathbf{p}) \mid_{\mathbf{p}=\mathbf{p}_0}$ is just a constant.

From here, one can calculate the covariance between two observables which, easily, becomes the calculation of the variance of an observable when both of them are the same. Specifically, the covariance is defined as the statistical product (*i.e.* expectation value) of the predicted values for the two observables respect their expectatcion values. That is within our approximation,

$$\begin{aligned} C_{AB} &= \overline{(A(\mathbf{p})-\overline{A})(B(\mathbf{p})-\overline{B})} \\ &\approx \overline{(A(\mathbf{p})-A_0)(B(\mathbf{p})-B_0)} \end{aligned} \tag{10}$$

and using Eqs. 7 and 9,

$$\begin{aligned} C_{AB} &\approx \overline{(\mathbf{p}-\mathbf{p}_0)\mathbf{A}_0\mathbf{B}_0(\mathbf{p}-\mathbf{p}_0)} \\ &\approx \mathbf{A}_0\overline{(p_\imath - p_{0\imath})(p_\jmath - p_{0\jmath})}\mathbf{B}_0 \\ &\approx \sum_{\imath\jmath}^{n} \partial_{p_\imath} A(\mathbf{p}) \mid_{\mathbf{p}=\mathbf{p}_0} (\mathcal{M}^{-1})_{\imath\jmath} \partial_{p_\imath} B(\mathbf{p}) \mid_{\mathbf{p}=\mathbf{p}_0} \end{aligned} \tag{11}$$

since $\overline{(p_\imath - p_{0\imath})(p_\jmath - p_{0\jmath})} = (\mathcal{M}^{-1})_{\imath\jmath} = \mathcal{E}_{\imath\jmath}$. Hence,

$$C_{AB} \approx \sum_{\imath\jmath}^{n} \partial_{p_\imath} A \mathcal{E}_{\imath\jmath} \partial_{p_\jmath} B \tag{12}$$

The variance of A which estimates the uncertainty (squared error) in the observable is, then, easily calculated from the last expression as C_{AA}. Furthermore, it can be very useful for the study of the predicted observables of a model, if one analyze the Pearson-product moment correlation coefficient between those observables. It is defined as,

$$c_{AB} \equiv \frac{C_{AB}}{\sqrt{C_{AA} C_{BB}}} \tag{13}$$

2.3 Numerical details

In order to cope with the covariance analysis described in the previous subsection, one should calculate the matrix $\mathcal{M}$, its inverse $\mathcal{E}$ and the derivatives of the observables respect to the parameters of the model in order to find their covariances, variances and Pearson-product correlation coefficient.

2.3.1 The curvature matrix $\mathcal{M}$

The calculation of the curvature matrix, proportional to the Hessian matrix, can be done by using different numerical approximations. The first one, can be the following. Assuming Eq. 3 is fulfilled, one can define the $\chi^2(\mathbf{p})$ around the minimum, $\mathbf{p}_0$, by an hyper-parabola without linear terms in the parameters,

$$\chi^2(\mathbf{p}; \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \sum_{\imath\jmath}^{n} a_{\imath\jmath} + b_{\imath\jmath} p_\imath^2 + c_{\imath\jmath} p_\jmath^2 + d_{\imath\jmath} p_\imath p_\jmath \tag{14}$$

where the parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ should be determined from a least-squares fitting after the evaluation of the $\chi^2(\mathbf{p})$ around the minimum (how to choose a good step size for evaluating this function and avoiding numerical inacuracies will be discussed at the end of the present subsection). Note that for $\imath = \jmath$ one should set parameters $\mathbf{b}$ and $\mathbf{c}$ equal to zero and then perform the least-squares fitting. The curvature matrix using this numerical approach becomes,

$$\frac{1}{2} \partial_{p_\imath} \partial_{p_\jmath} \chi^2(\mathbf{p}; \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \frac{d_{\imath\jmath}}{2} \tag{15}$$

Another approximation is to forget about condition 3 and perform the numerical derivatives of the $\chi^2(\mathbf{p})$ evaluated at $\mathbf{p}_0$ with numerical formulas available, for example, in Ref. [1]. Finally, following the approximation used along the previous section in which we consider all nuclear observables to behave smoothly with a change in the parameters $\mathbf{p}$ and, therefore, we have neglected its second order derivatives, one can calculate the curvature matrix starting form Eq. 1 in a simplified and numerically convinient way (see [2]) as follows,

$$\begin{aligned}
\partial_{p_\imath}\partial_{p_\jmath}\chi^2(\mathbf{p}) &= \partial_{p_\imath}\partial_{p_\jmath}\left[\sum_{k=1}^{m}\left(\frac{\mathcal{O}_k^{\text{theo.}}-\mathcal{O}_k^{\text{ref.}}}{\Delta\mathcal{O}_k^{\text{ref.}}}\right)^2\right] \\
&= 2\sum_{k=1}^{m}\partial_{p_\imath}\left[\left(\frac{\mathcal{O}_k^{\text{theo.}}-\mathcal{O}_k^{\text{ref.}}}{\Delta\mathcal{O}_k^{\text{ref.}}}\right)\frac{\partial_{p_\jmath}\mathcal{O}_k^{\text{theo.}}}{\Delta\mathcal{O}_k^{\text{ref.}}}\right] \\
&\approx 2\sum_{k=1}^{m}\frac{\partial_{p_\imath}\mathcal{O}_k^{\text{theo.}}}{\Delta\mathcal{O}_k^{\text{ref.}}}\frac{\partial_{p_\jmath}\mathcal{O}_k^{\text{theo.}}}{\Delta\mathcal{O}_k^{\text{ref.}}} \qquad (16)
\end{aligned}$$

and then, only first derivatives should be calculated. In the results the latter aproximation have been used.

2.3.2 The error matrix $\mathcal{E}$

The error matrix $\mathcal{E}$ which is the inverse matrix of the curvature matrix $\mathcal{M}$ can be calulated also using different approaches. Our choice have been to use the subroutine *gaussj* available in the numerical recipes libraries. It is based on the Gauss-Jordan elimination with full pivoting for solving a set of linear algebraic equations. We test numerically for each inversion that $\mathcal{M}\mathcal{E} = 1$ is accurate (precision found is 10^{-8} or better).

2.3.3 The covariance and correlation between two observables

The covariance between two observables and their correlation or its variance and error are calculated within the same numerical approximations than the ones already described. In this respect, the only important detail is to be consitent regarding the adopted approximation along the calcultions. The question here is to decide, after a proper step size for changing the parameters is known, the number of points needed for an accurate calculation of the derivatives of the different observables: a problem present all along this subsection. There is no general solution for doing this. One needs to check the different recipes and decide after that the more convinient way to proceed. However, if our approximation of a smooth variation between the observables and the parameters is good enough (almost all expressions in these notes are based on that), a two point symmetric formula for the derivation is accurate for our purposes. Therefore, the first derivative of the observables entering in the fit are calculated as,

$$\partial_{p_\imath}\mathcal{O} \approx \frac{\mathcal{O}(p_{01},...,p_{0\imath}+\Delta p_\imath,...p_{0m}) - \mathcal{O}(p_{01},...,p_{0\imath}-\Delta p_\imath,...p_{0m})}{2\Delta p_\imath} \qquad (17)$$

The same fromula have been used for the first derivative of the observables in which we are interested for the calculation of their correlations ($\mathcal{O} \rightarrow A$). The error of using this formula is,

$$\delta\left[\partial_{p_\imath}\mathcal{O}\right] \approx \frac{(\Delta p_\imath)^2}{6}\partial^3_{p_\imath}\mathcal{O} \tag{18}$$

One of the advantages of using this formula is that for the same nucmerical effort one can calculate the second derivaties of the observables as,

$$\partial^2_{p_\imath}\mathcal{O} \approx \frac{\mathcal{O}(p_{01},...,p_{0\imath}+\Delta p_\imath,...p_{0m}) - 2\mathcal{O}(\mathbf{p}_0) + \mathcal{O}(p_{01},...,p_{0\imath}-\Delta p_\imath,...p_{0m})}{(\Delta p_\imath)^2} \tag{19}$$

and find, in this way, an overestimation of the error done in Eq. 19 (assuming that the second derivative is larger than the third derivative). For completeness, if we had used the non symmetric formula for the calculation of the the first dervatives,

$$\partial_{p_\imath}\mathcal{O} \approx \frac{\mathcal{O}(p_{01},...,p_{0\imath}+\Delta p_\imath,...p_{0m}) - \mathcal{O}(\mathbf{p}_0)}{\Delta p_\imath} \tag{20}$$

the computational time would have been smaller and the error in the derivatives larger,

$$\delta\left[\partial_{p_\imath}\mathcal{O}\right] \approx \frac{\Delta p_\imath}{2}\partial^2_{p_\imath}\mathcal{O} \tag{21}$$

and, of course we would have had no way of estimating the error without more computational effort.

2.3.4 How to chose step sizes, Δp, for calculating derivatives with respect the parameters

As it has been explained, the region of reasonable parametrizations is that defined by the condition, $\chi^2(\mathbf{p}) - \chi^2(\mathbf{p}_0) \leq 1$, since it ensures that (in average), the step in the parameters do not provide very large or very small changes in the fitted observables but a change comparable to the adopted errors. For this reason, a reasonable choice for the step size is that in which the variation in each parameter produce a change $\Delta\chi^2 \approx 1$. In doing that, we evaluate three points of the χ^2. One at $\chi^2(p_{01},...,p_{0\imath}-h,...,p_{0m}) \equiv \chi^2_{p_\imath-h}$, another at $\chi^2(p_{01},...,p_{0\imath}+h,...,p_{0m}) \equiv \chi^2_{p_\imath+h}$ and finally at $\chi^2(\mathbf{p}_0) \equiv \chi^2_0$. The magnitude of h is taken so that it produce an estimation of $\Delta p_\imath$ that do not change very much with the value of h itself and that the $\chi^2(p_{01},...,p_{0\imath} \pm \Delta p_\imath,...,p_{0m})$ is comparable to one. Assuming a parabolic approximation of χ^2, valid around the minimum, and using Eq. 5 the estimation of $\Delta p_\imath$ is the following,

$$\begin{aligned} e(p_\imath)^2 \equiv (\Delta p_\imath)^2 &= \left(\mathcal{M}^{-1}\right)_{\imath\imath} \equiv 2\left(\frac{\partial^2\chi^2}{\partial p_i^2}\right)^{-1} \\ &\approx 2h^2(\chi_{p_\imath-h} - 2\chi_0 + \chi_{p_\imath+h})^{-1} \end{aligned} \tag{22}$$

where this expresions provides a $\Delta p_\imath$ which induce a change in the χ^2 of around the unity. We have also checked that $\pm 10\%$ changes in the value of $\Delta p_\imath$ lead us to the same results.

3 Results

We present the correlation analysis of the interaction SLy5-min (very close to SLy5). For details on the χ^2 definition, fitting procedure, values of the parameters and properties of SLy5 see Refs. [3] and [4]. Just one comment is important here, not all the parameters (t_0, t_1, t_3, x_0, x_1, x_2, x_3, α and W_0) of the SLy-like models have been included in the analysis since three of them have been fixed manually and no χ^2 term exist for fixing them. Those are $x_2 = -1$, $\alpha = 1/6$ and $W_0 = 126$ MeV fm^5. In Fig. 1 we show results for the Pearson product-

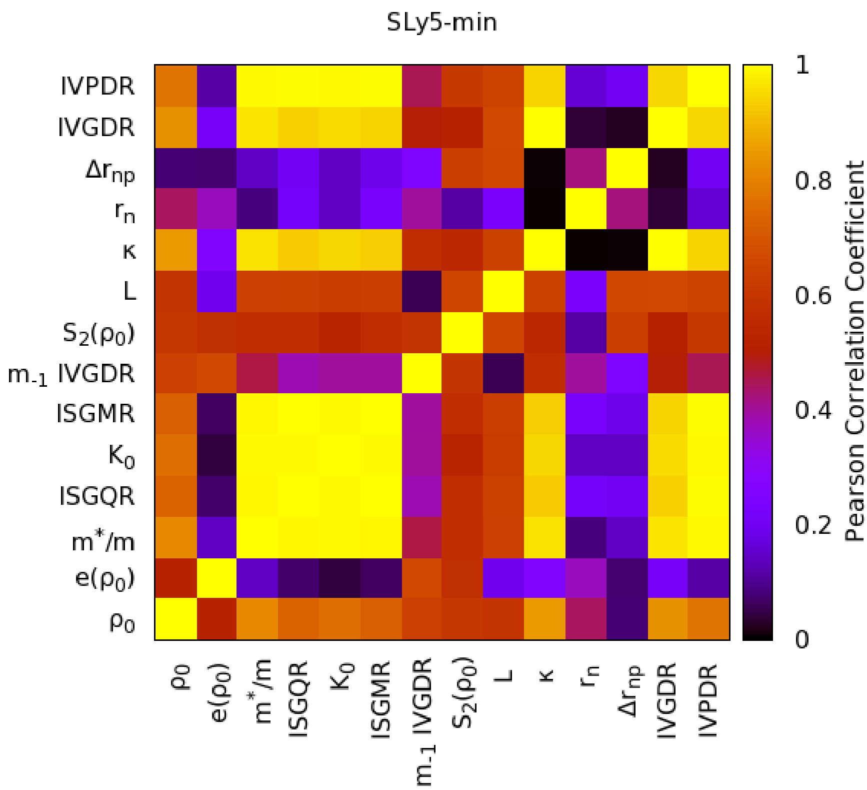

Figure 1: Pearson product-moment correlation coefficient matrix for the properties shown in the axes as predicted by the covariance analysis of SLy5-min.

moment correlation coefficient matrix for the properties shown in the axes as predicted by the covariance analysis of SLy5-min. In the three panels of Fig. 2 some details of the same magnitude are shown. In what follows we show the

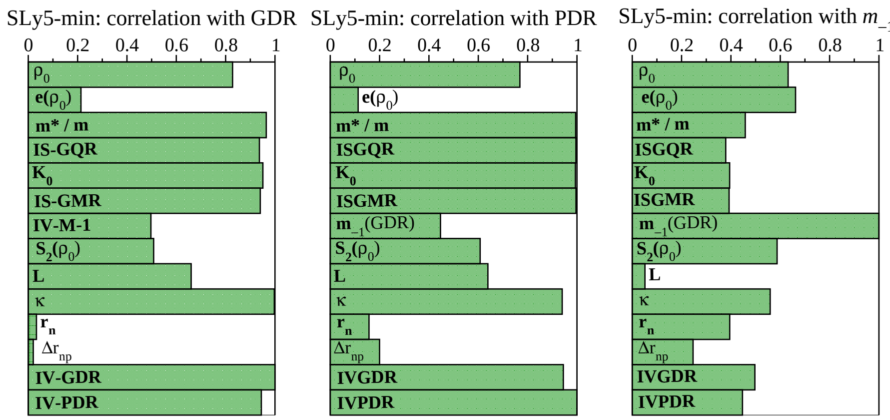

Figure 2: Pearson product-moment correlation coefficient for the IVGDR (left panel), IVPDR (middle panel) and m_{-1}(IVGDR) (right panel) with all other studied properties as predicted by the covariance analysis of SLy5.

output of the program we have used for the covariance analysis of SLy5-min. The parameters, observables and their associated errors as well as the curvature, error and correlation (for parameters and observables) matrices can be found.

```
 Hessian or Curvature Matrix
=============================
       1     d^2[X^2]    |
 Mij= ---   ------------|
       2    d[p_i]d[p_j]|p_0

       t_0        t_1         t_2        t_3         x_0         x_1          x_3
   31.8838    17.2803      4.5260     3.7586   1972.1206   -350.7807   -1271.3471
   17.2803     9.3900      2.4575     2.0380   1075.8922   -192.0042    -694.2176
    4.5260     2.4575      0.6453     0.5339    280.1924    -49.8234    -180.5988
    3.7586     2.0380      0.5339     0.4432    231.4662    -41.1824    -149.1819
 1972.1206  1075.8922    280.1924   231.4662 163675.5755 -28879.3109 -106873.2897
 -350.7807  -192.0042    -49.8234   -41.1824 -28879.3109   5238.2678   18977.6732
-1271.3471  -694.2176   -180.5988  -149.1819 -106873.2897  18977.6732   69945.5436

 Eigenvalues and eigenvectors of Mij
=====================================
 Eigenvalues: 0.21593E+01
              0.14722E-01
              0.16615E-02
              0.35153E-06
              0.23866E+06
              0.20708E+03
              0.36681E+02
```

Eigenvectors Vij:

t_0	t_1	t_2	t_3	x_0	x_1	x_3
0.7696E+00	-0.4643E+00	-0.1316E-01	-0.8858E-01	0.9942E-02	0.3918E-01	0.4272E+00
0.3791E+00	0.8575E+00	-0.2559E+00	-0.3425E-01	0.5426E-02	0.1756E-01	0.2323E+00
0.1058E+00	0.2208E+00	0.9639E+00	-0.8453E-01	0.1412E-02	0.5690E-02	0.6089E-01
0.9097E-01	0.6973E-02	0.7214E-01	0.9919E+00	0.1167E-02	0.4651E-02	0.5114E-01
-0.1784E+00	-0.4639E-02	0.6251E-03	0.7625E-04	0.8280E+00	0.4667E+00	0.2543E+00
0.3125E+00	0.1290E-01	-0.2683E-02	-0.2263E-03	-0.1465E+00	0.7123E+00	-0.6110E+00
-0.3392E+00	-0.9931E-02	0.1547E-02	0.1251E-03	-0.5411E+00	0.5223E+00	0.5650E+00

Error Matrix or Covariance Matrix
==================================
Eij = [M^(-1)]ij

t_0	t_1	t_2	t_3	x_0	x_1	x_3
22336.9346	8605.9939	21286.8815	-249944.5864	-19.1326	56.7466	-31.3399
8605.9939	3426.5838	8100.7505	-96652.6451	-7.8251	23.2657	-13.0625
21286.8815	8100.7505	20890.1571	-238475.4177	-18.0502	53.0719	-29.3523
-249944.5864	-96652.6451	-238475.4177	2798684.5698	215.1548	-638.6518	353.0597
-19.1326	-7.8251	-18.0502	215.1548	0.0358	-0.0826	0.0640
56.7466	23.2657	53.0719	-638.6518	-0.0826	0.2192	-0.1484
-31.3399	-13.0625	-29.3523	353.0597	0.0640	-0.1484	0.1160

Check
=======
Mij [M^(-1)]ij = 1

1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Correlation Matrix
====================
Cij = Eij/sqrt(Eii Ejj)

t_0	t_1	t_2	t_3	x_0	x_1	x_3
1.0000	0.9837	0.9854	-0.9997	-0.6766	0.8110	-0.6158
0.9837	1.0000	0.9575	-0.9870	-0.7066	0.8489	-0.6553
0.9854	0.9575	1.0000	-0.9863	-0.6601	0.7843	-0.5964
-0.9997	-0.9870	-0.9863	1.0000	0.6798	-0.8154	0.6197
-0.6766	-0.7066	-0.6601	0.6798	1.0000	-0.9327	0.9928
0.8110	0.8489	0.7843	-0.8154	-0.9327	1.0000	-0.9311
-0.6158	-0.6553	-0.5964	0.6197	0.9928	-0.9311	1.0000

```
 Errors in free parameters
===========================
 e_i = sqrt( Eii )

 t_0 =       -2475.408000 +/-        149.455460
 t_1 =         482.842000 +/-         58.537029
 t_2 =        -559.374000 +/-        144.534277
 t_3 =       13697.070000 +/-       1672.926947
 x_0 =           0.741185 +/-           0.189191
 x_1 =          -0.146374 +/-           0.468173
 x_3 =           1.162688 +/-           0.340537

 Uncertainty of an observable A
=================================
 DA = { d[A]/d[p_i] E_ij d[A]/d[p_j] }^1/2

 ro_sat =           0.161559 +/-           0.002110
 es_sat =         -16.017422 +/-           0.065051
 m_ef_m =           0.697844 +/-           0.072864
 sy_sat =          32.604439 +/-           0.706988
 K__sat =         230.461833 +/-           8.980535
 L_symm =          47.456310 +/-           4.487230
 trkivd =           0.326883 +/-           0.439172
 r_neut =           5.598887 +/-           0.012992
 rn__rp =           0.165517 +/-           0.006892
 IV-GDR =          13.936772 +/-           1.825181
 IV-M-1 =          19.428438 +/-           0.433911
 IV-PDR =           7.757677 +/-           0.344406
 IS-GMR =          14.003449 +/-           0.365507
 IS-GQR =           9.897559 +/-           0.415486
```

Pearson product-moment correlation coefficient

ro_sat	es_sat	m_ef_m	IS-GQR	K__sat	IS-GMR	IV-M-1	sy_sat	L_symm	trkivd	r_neut	rn__rp	IV-GDR	IV-PDR
1.0000	-0.5152	0.8115	-0.7290	-0.7571	-0.7210	-0.6318	0.5984	0.5888	-0.8435	-0.4394	0.0742	-0.8281	-0.7687
-0.5152	1.0000	-0.1386	0.0692	0.0397	0.0604	0.6620	-0.5692	-0.1915	0.2577	0.3668	-0.0718	0.2135	0.1132
0.8115	-0.1386	1.0000	-0.9897	-0.9948	-0.9890	-0.4581	0.5665	0.6314	-0.9601	0.0815	0.1392	-0.9647	-0.9947
-0.7290	0.0692	-0.9897	1.0000	0.9939	0.9992	0.3790	-0.5625	-0.6346	0.9269	-0.2150	-0.2024	0.9369	0.9957
-0.7571	0.0397	-0.9948	0.9939	1.0000	0.9943	0.3949	-0.5179	-0.6159	0.9419	-0.1385	-0.1366	0.9510	0.9936
-0.7210	0.0604	-0.9890	0.9992	0.9943	1.0000	0.3926	-0.5609	-0.6280	0.9315	-0.2222	-0.1864	0.9404	0.9960
-0.6318	0.6620	-0.4581	0.3790	0.3949	0.3926	1.0000	-0.5872	-0.0516	0.5592	0.3952	0.2466	0.4972	0.4468
0.5984	-0.5692	0.5665	-0.5625	-0.5179	-0.5609	-0.5872	1.0000	0.6466	-0.5340	0.1078	0.6288	-0.5083	-0.6075
0.5888	-0.1915	0.6314	-0.6346	-0.6159	-0.6280	-0.0516	0.6466	1.0000	-0.6347	0.2319	0.6513	-0.6603	-0.6390
-0.8435	0.2577	-0.9601	0.9269	0.9419	0.9315	0.5592	-0.5340	-0.6347	1.0000	0.0013	0.0026	0.9973	0.9401
-0.4394	0.3668	0.0815	-0.2150	-0.1385	-0.2222	0.3952	0.1078	0.2319	0.0013	1.0000	0.4184	-0.0331	-0.1571
0.0742	-0.0718	0.1392	-0.2024	-0.1366	-0.1864	0.2466	0.6288	0.6513	0.0026	0.4184	1.0000	-0.0203	-0.1998
-0.8281	0.2135	-0.9647	0.9369	0.9510	0.9404	0.4972	-0.5083	-0.6603	0.9973	-0.0331	-0.0203	1.0000	0.9447
-0.7687	0.1132	-0.9947	0.9957	0.9936	0.9960	0.4468	-0.6075	-0.6390	0.9401	-0.1571	-0.1998	0.9447	1.0000

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