Two-loop amplitudes from generalized unitarity

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Theoretical uncertainty $\leq$ experimental uncertainty $\Rightarrow$
NNLO (two-loop) or higher.

Most relevant calculations completed (and automated) at one-loop

This talk will present an application of similar methods at two-loop.

The methods has been developed over a couple of papers:

arXiv:1202.2019	arXiv:1207.2976
arXiv:1205.5707	arXiv:1310.1051

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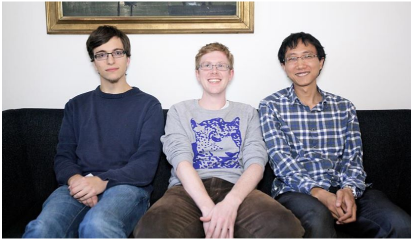

In theory, everything can be calculated using traditional Feynman diagram methods.

In practice this will become (and is becoming) computationally infeasible.
The number of diagrams and the number of terms in each diagram scales very badly with the number of loops and legs -
even though the final result is usually simple.

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even though the final result is usually simple.

To find another way is desirable, from both a practical and a theoretical point of view.

Outline:

- Introduction
- An one-loop example
- A two-loop application
- Perspectives

$$I = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \cdots \int \frac{\mathrm{d}^d k_L}{(2\pi)^d} \frac{N}{D_1 D_2 \cdots D_P}$$

N is a function of k_i through a number of scalar products:

$$k_i \cdot k_j\,,\ k_i \cdot p_j\,,\ k_i \cdot \omega_j\,,\ \ldots$$

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$$N = \Delta + \sum_j \kappa_j D_j$$

Δ is the Irreducible Numerator, and it is a function of the remaining, Irreducible Scalar Products (ISPs).
This expansion is step one of the OPP ([hep-ph/0609007]) method.

$$I = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \cdots \int \frac{\mathrm{d}^d k_L}{(2\pi)^d} \sum_i \frac{\Delta_i}{D_1^{a_{1i}} D_2^{a_{2i}} \cdots D_P^{a_{Pi}}}$$

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$$\Downarrow$$

$$\mathcal{A} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \cdots \int \frac{\mathrm{d}^d k_L}{(2\pi)^d} \sum_j \frac{\Delta_j}{D_1^{a_{1j}} D_2^{a_{2j}} \cdots D_{P'}^{a_{P'j}}}$$

An expression for $\mathcal{A}$ can be found by finding all Δ_j.

In general we will want to work in d dimensions.

We will use the FDH (Four Dimensional Helicity) scheme, which keeps external particles and internal polarizations in four dimensions, but describe the momenta of the internal particles in $d = 4 - 2\epsilon$ dimensions.

All information about the -2ϵ extra components of the loop momenta are contained in the parameters

$$\mu_{ij} = -k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]} \qquad \text{with} \qquad k_i = \bar{k}_i + k_i^{[-2\epsilon]}$$

Example: $gg \to gg$ in Yang-Mills at one loop (first by BCF in [hep-th/0412103]).

$$\mathcal{A} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{\Delta_\Box}{D_1 D_2 D_3 D_4} + \sum \frac{\Delta_\triangle}{D^3} + \sum \frac{\Delta_\bigcirc}{D^2} \right)$$

$$\Delta_\Box^{[4]} = c_1 + c_2\, k \cdot \omega$$

$$\omega^\nu \propto \varepsilon^{123\nu}$$

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$$\Delta_\Box^{[d]} = c_1 + c_2\, k\cdot\omega + c_3\, \mu_{11} + c_4\, \mu_{11}\, k\cdot\omega + c_5\, \mu_{11}^2$$

$$\mu_{11} = -\left(k^{[-2\epsilon]}\right)^2 \qquad \text{and} \qquad \omega^\nu \propto \varepsilon^{123\nu}$$

How to find the c_i-coefficients in Δ:
Unitarity cuts. Schematically

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$$\frac{1}{D_i} \to \delta(D_i)$$

$$\mathcal{A} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{\Delta_\square}{D_1 D_2 D_3 D_4} + \sum \frac{\Delta_\triangle}{D^3} + \sum \frac{\Delta_\bigcirc}{D^2} \right) \Rightarrow$$

$$\mathcal{A}|_{4\times\mathrm{cut}} = \Delta_\square|_{4\times\mathrm{cut}}$$

$$\mathcal{A}|_{4\times\mathrm{cut}} = \sum_{\substack{\mathrm{particles} \\ \mathrm{helicities}}} A(-l_1^{-h_1}, p_1, l_2^{h_2}) A(-l_2^{-h_2}, p_2, l_3^{h_3}) A(-l_3^{-h_3}, p_3, l_4^{h_4}) A(-l_4^{-h_4}, p_4, l_1^{h_1})$$

Tree-amplitudes only!

Four cut constraints

$$l_1^2 = l_2^2 = l_3^2 = l_4^2 = 0$$

$$(k + p_1)^2 = k^2 = (k - p_2)^2 = (k - p_2 - p_3)^2 = 0$$

Five degrees of freedom, four constraints, one free parameter τ

$$k = \tau \frac{\langle 23 \rangle}{\langle 13 \rangle} \frac{\langle 1|\gamma^\mu|2]}{2} + (1 - \tau) \frac{[32]}{[31]} \frac{\langle 2|\gamma^\mu|1]}{2} + k^{[-2\epsilon]}$$

$$\Delta_{\square}^{[d]} = c_1 + c_2\, k\cdot\omega + c_3\, \mu_{11} + c_4\, \mu_{11}\, k\cdot\omega + c_5\, \mu_{11}^2$$

$$\mu_{11}|_{\mathrm{cut}} = \frac{st}{u}\tau(1-\tau) \qquad\qquad k\cdot\omega|_{\mathrm{cut}} = t(2\tau-1)$$

$$\Delta_{\square}^{[d]}|_{\mathrm{cut}} = \sum_{i=0}^{4} d_i \tau^i$$

$$\mathbf{d} = M\mathbf{c} \qquad\qquad (5\times 5)$$

$$\mathcal{A} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{\Delta_\Box}{D_1 D_2 D_3 D_4} + \sum \frac{\Delta_\triangle}{D^3} + \sum \frac{\Delta_\bigcirc}{D^2} \right)$$

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$$\mathcal{A}|_{3\times\mathrm{cut}} = \left(\frac{\Delta_\square}{D_4} + \Delta_{\triangle 123} \right)\Big|_{3\times\mathrm{cut}}$$

$$\mathcal{A}|_{3\times\mathrm{cut}} = \sum_{\substack{\text{particles} \\ \text{helicities}}} A(-l_1^{-h_1}, p_1, l_2^{h_2}) A(-l_2^{-h_2}, p_2, l_3^{h_3}) A(-l_3^{-h_3}, p_3, p_4, l_1^{h_1})$$

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$$\Delta_{\triangle 123} = \sum_i c_i (k\cdot\omega_1)^{a_{1i}} (k\cdot\omega_2)^{a_{2i}} \mu^{a_{3i}} \qquad (10 \text{ terms})$$

Three constraints $\Rightarrow$ two free parameters

$$\Delta_{\triangle 123}|_{\mathrm{cut}} = \sum_j d_j \tau_1^{a_{1j}} \tau_2^{a_{2j}} \qquad \mathbf{d} = M\mathbf{c}$$

$$\mathcal{A} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{\Delta_\Box}{D_1 D_2 D_3 D_4} + \sum \frac{\Delta_\triangle}{D^3} + \sum \frac{\Delta_\bigcirc}{D^2} \right)$$

$$\Delta_\Box^{[d]} = c_1 + c_2\, k\cdot\omega + c_3\, \mu_{11} + c_4\, \mu_{11}\, k\cdot\omega + c_5\, \mu_{11}^2$$

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$$\Delta_\Box^{[d]} = c_1 + c_2\, k\cdot\omega + c_3\, \mu_{11} + c_4\, \mu_{11}\, k\cdot\omega + c_5\, \mu_{11}^2$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{D_1 D_2 D_3 D_4} = \frac{1}{st} \left(\frac{2}{\epsilon^2} \left((-s)^{-\epsilon} + (-t)^{-\epsilon} \right) - \log^2(s/t) - \pi^2 \right) + \mathcal{O}(\epsilon)$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k\cdot\omega}{D_1 D_2 D_3 D_4} = 0$$

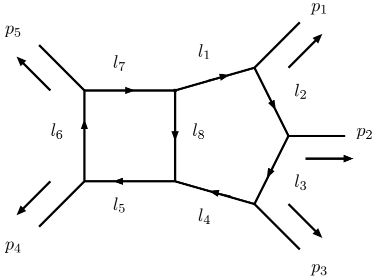
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mu_{11}}{D_1 D_2 D_3 D_4} = \epsilon \int \frac{\mathrm{d}^{(6-2\epsilon)} k}{(2\pi)^{(6-2\epsilon)}} \left(\frac{1}{D_1^2 D_2 D_3 D_4} + \frac{1}{D_1 D_2^2 D_3 D_4} + \frac{1}{D_1 D_2 D_3^2 D_4} + \frac{1}{D_1 D_2 D_3 D_4^2} \right)$$

Summary of the method:

- Find the expansion of $\mathcal{A}$ in Δ_i
- Find the expansion of Δ_i in c_j and ISPs (with Gram matrices or polynomial division)
- Find a polynomial expansion of Δ_i in the free parameters remaining after a generalized unitarity cut
- Invert the system to find the c_i-coefficients (guarantied to be possible in d-dimensions)
- (Solve the integrals corresponding to the Δ_i topologies)

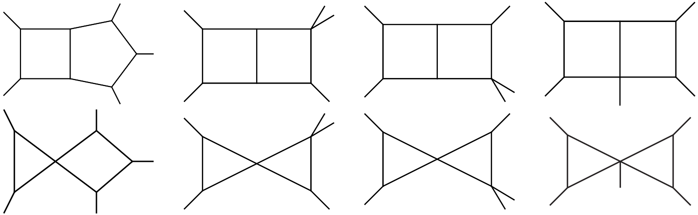
Two-loop:

Example in question: The planar contribution to $gg \to ggg$ in pure YM, with the external helicities all 'plus' [ArXiv:1310.1051].

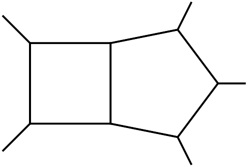


$$\mathcal{A} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left(\sum_i \frac{\Delta_i}{D_i^{a_{1i}}} \right)$$

For the all-plus case only eight of the Δ_i are non-zero:



Two of the topologies are related to others through reflection symmetry, so only six needs to be calculated.



$$\Delta_{(431)} = \sum_i c_i \, (k_1 \cdot p_5)^{a_{1i}} \, (k_2 \cdot p_1)^{a_{2i}} \, (k_2 \cdot p_2)^{a_{3i}} \, \mu_{11}^{a_{4i}} \, \mu_{22}^{a_{5i}} \, \mu_{12}^{a_{6i}}$$

11 degrees of freedom and 8 constraints leave 3 free parameters,

$$\Delta_{(431)}\big|_{8\times\text{cut}} = \sum_j d_j \tau_1^{a_{1j}} \tau_2^{a_{2j}} \tau_3^{a_{3j}}$$

$$\mathbf{d} = M\mathbf{c} \qquad 160 \times 160$$

$$\Delta_{431} = \frac{-i\, s_{12} s_{23} s_{45}\, F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5} \left(\mathrm{tr}_+(1345)(k_1 + p_5)^2 + s_{15} s_{34} s_{45}\right)$$

$$F_1 = 2\left(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}\right) + 4\left(\mu_{12}^2 - 4\mu_{11}\mu_{22}\right)$$

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$$\begin{aligned}
\Delta_{330;5L} \;&=\; -\frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \times \\
&\left(\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\left(4(\mu_{11}+\mu_{22})\mu_{12}\right.\right. \\
&\quad \left. + 4\mu_{11}\mu_{22}\frac{4(k_1\cdot p_3)(k_2\cdot p_3) + (k_1+k_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\right) \\
&\quad + 4\mu_{11}\mu_{22}\Bigg[(k_1+k_2)^2 s_{15} \\
&\quad + \mathrm{tr}_+(1235)\left(\frac{(k_1+k_2)^2}{2s_{35}} - \frac{k_1\cdot p_3}{s_{12}}\left(1 + \frac{2(k_2\cdot\omega_{453})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(k_2-p_5)^2\right)\right) \\
&\quad \left. + \mathrm{tr}_+(1345)\left(\frac{(k_1+k_2)^2}{2s_{13}} - \frac{k_2\cdot p_3}{s_{45}}\left(1 + \frac{2(k_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(k_1-p_1)^2\right)\right)\Bigg]\right)
\end{aligned}$$

Other ingredients:

- Algebraic geometry (to establish the form of $\Delta(c_i)$) [arXiv:1205.5707]
- Six-dimensional spinor-helicity formalism (to calculate the μ_{ij}-dependence) [arXiv:0902.0981].
- Schwinger parameters (for the μ_{ij}-integrals)
- Feynman diagrams (for check)
- Momentum twistors (to simplify the result) [arXiv:0905.1473]

$$A_5^{[P]}(1^+,2^+,3^+;4^+,5^+) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Big(c_{431}\, I_{431}\,[F_1]$$
$$+\, c^T_{431} I_{431}\left[F_1\ (k_1+p_5)^2\right] + c_{331;M_1} I_{331;M_1}\,[F_1] + c_{331;M_2} I_{331;M_2}\,[F_1] + c_{331;5L} I_{331;5L}\,[F_1]$$
$$+\, c_{430}\big(s_{23} I_{430}\left[F_3\ ((k_1+k_2)^2+s_{45})\right] + I_{430}\left[F_3\ ((k_1+k_2)^2+s_{45})\ 2(k_1\cdot\omega_{123})\right]\big)$$
$$+\, c_{330;M_1} I_{330;M_1}\left[F_3\ ((k_1+k_2)^2+s_{45})\right] + c_{330;M_2} I_{330;M_2}\left[F_3\ ((k_1+k_2)^2+s_{45})\right]$$
$$+\, c^a_{330;5L} I_{330;5L}\,[F_3\ N_1(k_1,k_2,1,2,3,4,5)] + c^b_{330;5L} I_{330;5L}\,[F_3\ N_2(k_1,k_2,1,2,3,4,5)]$$
$$+\, c^c_{330;5L} I_{330;5L}\,[F_3\ N_2(k_2,k_1,5,4,3,2,1)] + c^d_{330;5L} I_{330;5L}\left[F_3\ (k_1+k_2)^2\right]\Big).$$

with

$$c_{431} = -s_{12}s_{23}s_{34}s_{45}^2 s_{15}/\mathrm{tr}_5\,, \qquad c^T_{431} = -s_{12}s_{23}s_{45}\mathrm{tr}_+(1345)/\mathrm{tr}_5\,,$$
$$c_{331;M_1} = -s_{34}s_{45}^2\mathrm{tr}_+(1235)/\mathrm{tr}_5\,, \qquad c_{331;M_2} = -s_{15}s_{45}^2\mathrm{tr}_-(1234)/\mathrm{tr}_5\,,$$
$$c_{331;5L} = s_{12}s_{23}s_{34}s_{45}s_{15}/\mathrm{tr}_5\,, \qquad c_{430} = -s_{12}\mathrm{tr}_+(1345)/(2s_{13}s_{45})\,,$$
$$c_{330;M_1} = -(s_{45}-s_{12})\mathrm{tr}_+(1345)/(2s_{13}s_{45})\,, \qquad c_{330;M_2} = -(s_{45}-s_{23})\mathrm{tr}_+(1345)/(2s_{13}s_{45})\,,$$
$$c^b_{330;5L} = \mathrm{tr}_+(1235)/(2s_{35}s_{12})\,, \qquad c^c_{330;5L} = \mathrm{tr}_+(1345)/(2s_{13}s_{45})\,,$$
$$c^a_{330;5L} = -\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1235)\mathrm{tr}_+(1345)}{s_{13}s_{35}}\right)\,,$$
$$c^d_{330;5L} = c^a_{330;5L}\frac{s_{12}+s_{45}}{s_{12}s_{45}} - s_{12}c^b_{330;5L} - s_{45}c^c_{330;5L} - s_{15}\,,$$

At one-loop, it is known that [hep-th/9611127] (BDDK).

$$\Delta_n^{(1)}(1^+, \cdots, n^+) = 2\mu_{11}^2 \langle 12\rangle^{-4} \Delta_n^{(1),[\mathcal{N}=4]}(1^-, 2^-, 3^+, \cdots, n^+).$$

At two-loop, we get that

$$\Delta_n^{(2)}(1^+, \cdots, n^+) = F_1 \langle 12\rangle^{-4} \Delta^{(2),[\mathcal{N}=4]}(1^-, 2^-, 3^+, \cdots, n^+) + \text{butterfly topologies}.$$

for $n = 4, 5$.

Directions:

- Calculate the non-planar contribution (BCJ)
- Calculate the other helicity configurations (full $gg \to ggg$)
- Calculate the remaining integrals
- Calculate more amplitudes
- Automate the procedure further
- ...