On the loop-tree Duality

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LHCPhenonet annual meeting, CERN, 5/12/2013

The Duality Hall of Fame

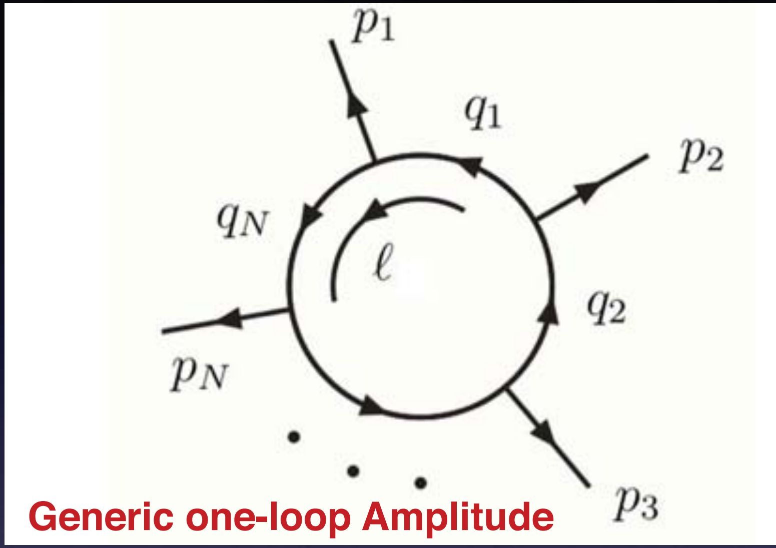
- Bierenbaum,I.
- Buchta, S
- Catani, S
- Chachamis, G
- Draggiotis, P
- Gleisberg, T
- Krauss, F
- M.I.
- Rodrigo, G
- Winter, J-C

- [Catani,Gleisberg,Krauss,Rodrigo,Winter, JHEP0809(2008)065]
- [Bierenbaum,Catani,Draggiotis,Rodrigo, JHEP1010(2010)073]
- [Bierenbaum,Buchta,Draggiotis,M.I.,Rodrigo, JHEP 1303(2013)025]
- [Buchta, Chachamis,Draggiotis, M.I.,Rodrigo, in preparation]

Outline of the talk

- Feynman Tree theorem and Duality theorem
- Duality theorem for higher loops
- Singularities of the loop integrands
- Example of cancellation of singularities
- Conclusions

Notation

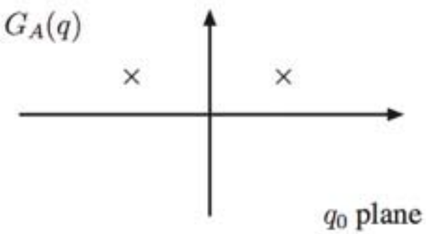


$$q_i = \ell + k_i \quad \text{with} \quad k_i = p_1 + \cdots + p_i$$

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \quad \text{and} \quad \int_\ell = -i \int \frac{d^d\ell}{(2\pi)^d}$$

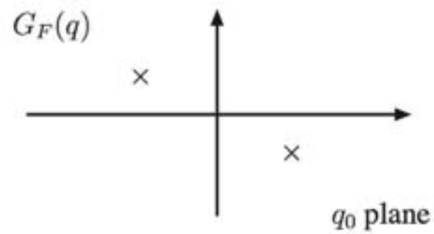
- All momenta outgoing

Feynman´s tree theorem and a Duality theorem



$$G_A(q) \equiv \frac{1}{q^2 - i0\, q_0}$$

Both poles are placed above the real axis, independently of the sign of the energy



$$G_F(q) \equiv \frac{1}{q^2 + i0}$$

$+i0$: positive frequencies are propagated forward in time, negatives backward

$$G_A(q) \quad \equiv \quad G_F(q) + \tilde{\delta}(q) \ , \qquad \tilde{\delta}(q) \equiv 2\pi\, i\, \theta(q_0)\, \delta(q^2) = 2\pi\, i\, \delta_+(q^2)$$

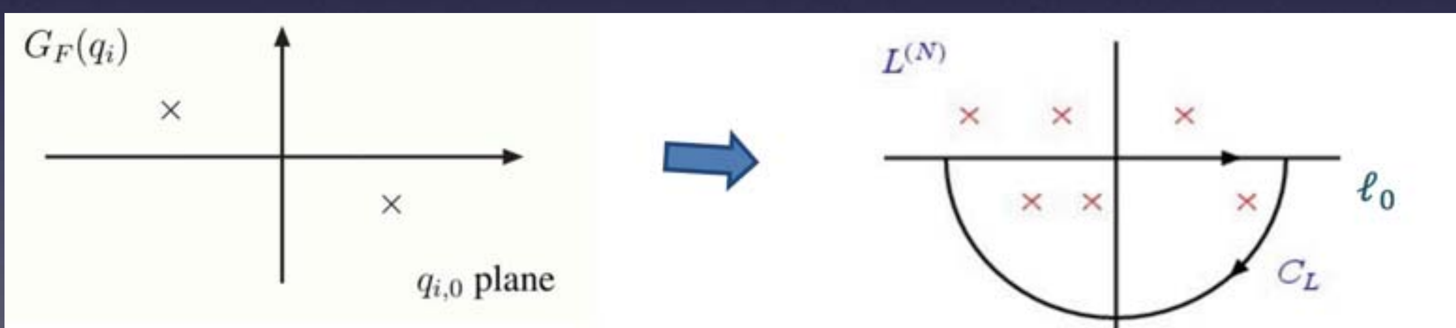
- Advanced one-loop integral vanishes
- Amplitude is given as an integral of Feynman propagators

$$0 \quad = \quad L_A^{(1)}(p_1, p_2, \ldots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) \quad = \int_q \prod_{i=1}^{N} \left[G_F(q_i) + \tilde{\delta}(q_i) \right]$$

$$L^{(1)}(p_1, p_2, \dots, p_N) = -\left[L^{(1)}_{1-\mathrm{cut}}(p_1, p_2, \dots, p_N) + \dots + L^{(1)}_{\mathrm{N-cut}}(p_1, p_2, \dots, p_N) \right]$$

Feynman´s tree theorem

- ¨N-cut¨ is the term with N delta functions (For N>d terms vanish)
- The Duality produces the one-loop Amplitude with only one cut
- Apply the Cauchy residue theorem and select residues with positive energy and negative imaginary part



- In each contribution every uncut propagator becomes ¨dual¨

$$G_D(q_i; q_j) := \frac{1}{q_j^2 - i0\, \eta(q_j - q_i)}$$

- The first argument in the parenthesis stands for the cut propagator

- The i0 prescription changes but notice that at one-loop the modification does not depend on the loop momentum

- *n is a future-like momentum,* its dependence should (and does) cancel when summing all contributions

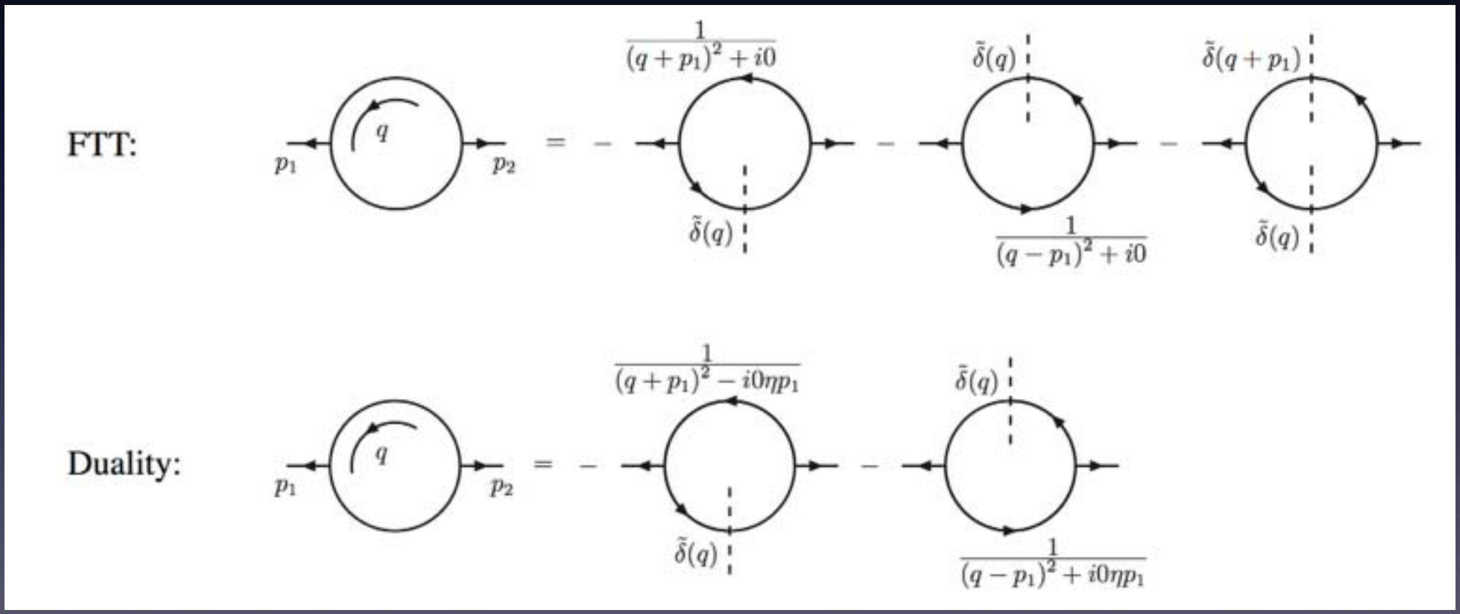
$$\eta_0 \geq 0,\ \eta^2 = \eta_\mu \eta^\mu \geq 0$$

$$L^{(1)}(p_1, \ldots, p_N) \;=\; -\sum_{i=1}^{N} \int_q \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j)$$

Loop-tree Duality theorem

- Virtual contributions take similar form to the real corrections (return to that later)

- <u>Example-The two point function</u>



- Extension to Amplitudes

- Introducing numerators nothing changes for the method

Duality theorem at higher orders

-Bierenbaum, Catani,Draggiotis, Rodrigo, JHEP 10(2010)073

-Bierenbaum, Buchta ,Draggiotis,M.I. Rodrigo, JHEP 03(2013)025

- Duality can be extended to higher loops

- Two options
  - Number of cuts= Number of loops- i0 prescription depends on loop momenta
  - Cut more to disconnect graphs to keep the i0 prescription as in the one loop case

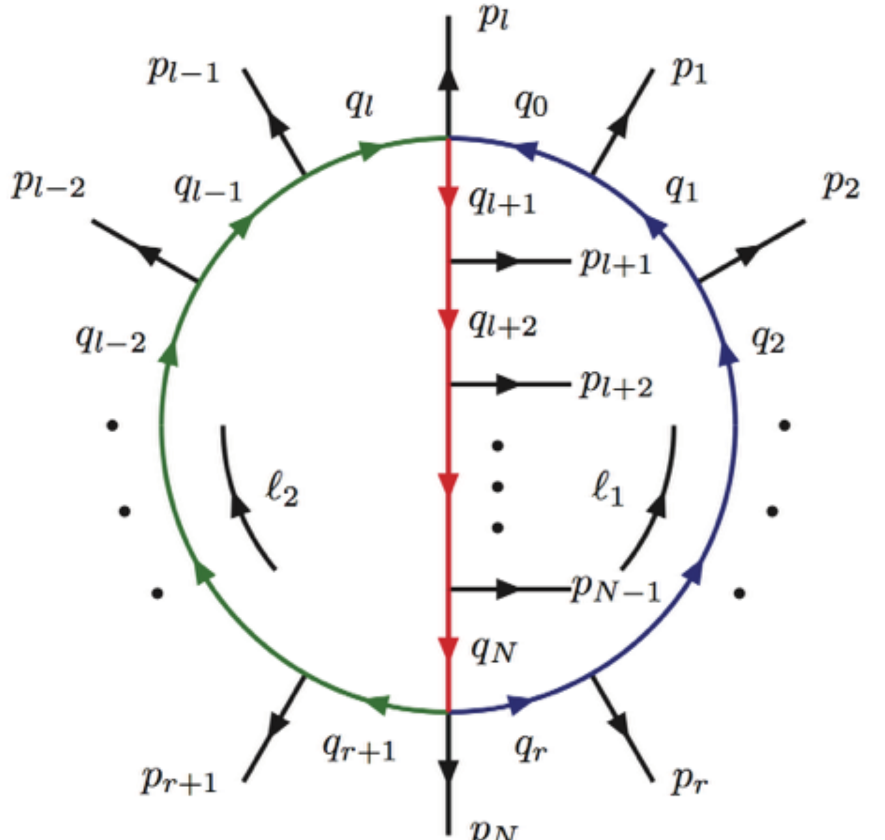
- Define sets of propagators with the same loop momentum

The "Loop Lines"

$$\alpha_1 \equiv \alpha_1(\ell_1) \equiv \{0, 1, ..., r\} \, ,$$

$$\alpha_2 \equiv \alpha_2(\ell_2) \equiv \{r+1, r+2, ..., l\} \, ,$$

$$\alpha_3 \equiv \alpha_3(\ell_1 + \ell_2) \equiv \{l+1, l+2, ..., N\}$$



Generic two loop Amplitude

- For these sets of momenta notice:

$$G_{F(A,R)}(\alpha_k) = \prod_{i\in\alpha_k} G_{F(A,R)}(q_i)$$

$$G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i; q_j)$$

- From the following equation

$$G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)$$

we can find the expression for the dual propagators over a union of sets

$$G_D(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) = \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2}) \,.$$

The sum runs over all partitions of β_N into exactly two blocks $\beta_N^{(1)}$ and $\beta_N^{(2)}$ with elements $\alpha_i, i \in \{1, \ldots, N\}$, where we include the case: $\beta_N^{(1)} \equiv \beta_N, \beta_N^{(2)} \equiv \emptyset$.

We can derive the formula for the two-loop duality theorem

$$L^{(2)}(p_1,\dots,p_N) = \int_{\ell_1}\int_{\ell_2} [-G_D(\alpha_1)G_F(\alpha_2)G_D(\alpha_3) + G_D(\alpha_1)\,G_D(\alpha_2\cup\alpha_3) + G_D(-\alpha_1\cup\alpha_2)G_D(\alpha_3)]$$

- Each term includes two Dual propagators (= two cuts)
- However, using

$$G_D(\alpha_1\cup\alpha_2) = \underbrace{G_D(\alpha_1)\,G_F(\alpha_2) + G_F(\alpha_1)\,G_D(\alpha_2)}_{\text{single cut}} + \underbrace{G_D(\alpha_1)\,G_D(\alpha_2)}_{\text{double cut}}\,.$$

we can cut more up to disconnected diagrams,

keeping the i0 prescription independent of any loop momentum

- The extension of the duality theorem to even higher loops is also known
- In the case of double poles, either use Cauchy theorem, either IBP´s

-Bierenbaum, Buchta ,Draggiotis,M.I. Rodrigo, JHEP 03(2013)025

Singularities of the loop integrands

- Motivation: Calculate (numerically) amplitudes

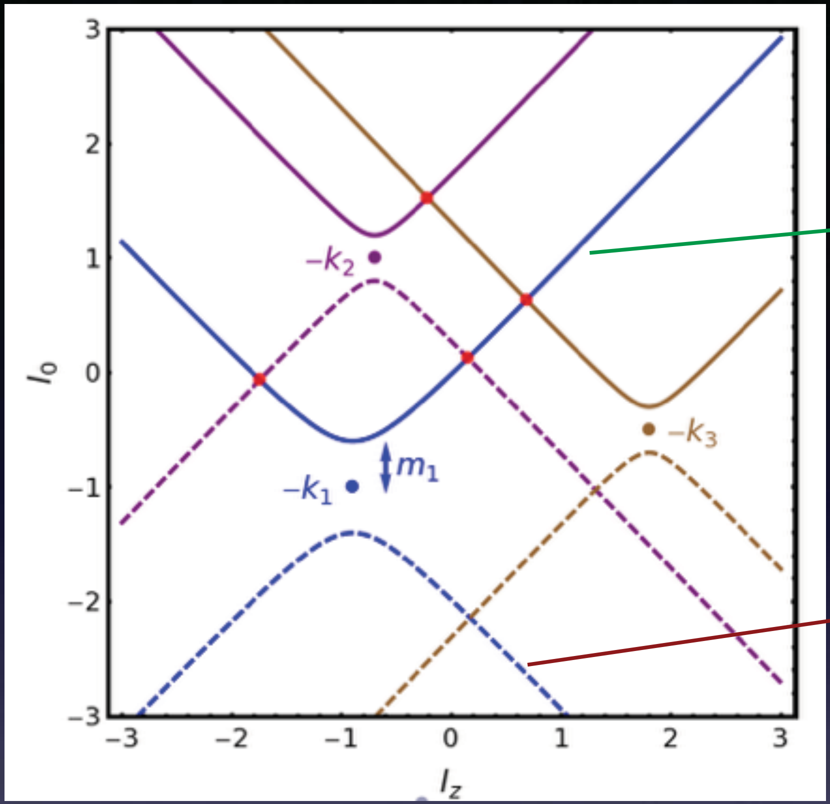
    - Need to identify singular contributions

- Assume for the moment that UV divergencies have been subtracted

- Duality helps us identify IR contributions that cancel each other (Virtual-Real)

Loop integrals can be viewed as Phase-Space integrals (slightly modified P-S)

- Threshold singularities are integrable but can lead to numerical instabilities



Positive energy solution for a vanishing propagator (Forward light-cone)

Negative energy solution (Backward light-cone)

- The hyperboloids above are the lines where: $G_F^{-1}(q_i) = q_i^2 - m_i^2 + i0 = 0$
- Duality means integrate along the positive lines (for every contribution one positive line)
- At the intersection points more than one propagators become zero → singularities
- Dual integrals are positive inside the light cone and negative outside

Study of the different types of intersections

- To make the study of intersections of propagators easier, notice that dual propagators can be written in the following form :

$$\tilde{\delta}(q_i)\, G_D(q_i; q_j) = i\, 2\pi \frac{\delta(q_{i,0} - q_{i,0}^{(+)})}{2 q_{i,0}^{(+)}} \frac{1}{(q_{i,0}^{(+)} + k_{ji,0})^2 - (q_{j,0}^{(+)})^2}$$

with $q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$ (after some simple algebra)

- The intersection is now explicit and happens when one of the following condition is fulfilled

$$q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0} = 0\,,$$

$$q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = 0\,.$$

Forward-Forward intersection (first condition)

Forward of -k_i with Backward of -k_j (second condition)

Notation here: $k_{ji,\mu} = (q_j - q_i)_\mu.$

Cancellation of threshold singularities

- Imagine for example the intersection of two propagators when $q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = 0$

  (Forward-Forward intersection)

- Two relevant contributions from the two dual integrals

- One intersection point- the two contributions have a different sign coming from crossing in an opposite way the intersection point where dual propagators change sign

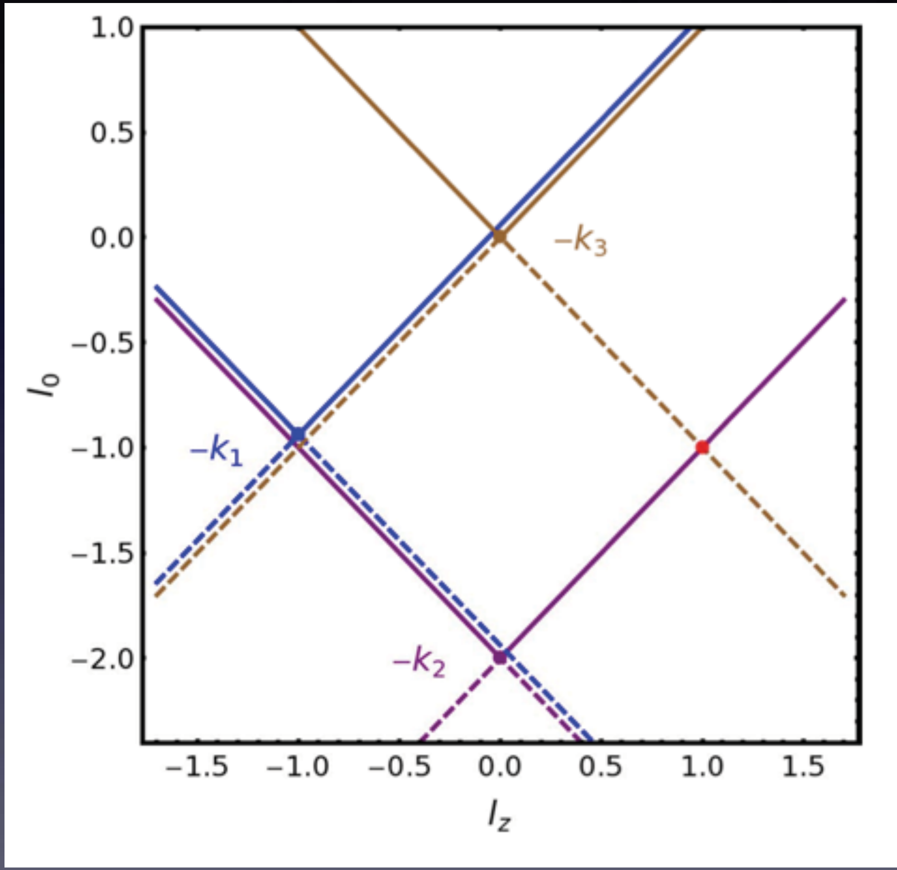
- Setting : $x = q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0}.$ and taking the limit $x \to 0$

  one can prove that the singularity cancels

$$\lim_{x\to 0} \left(\tilde{\delta}\left(q_i\right) G_D(q_i;q_j) + (i \leftrightarrow j)\right) = i\,2\pi \left(\frac{1}{x} - \frac{1}{x}\right) \frac{1}{2q_{i,0}^{(+)}} \frac{1}{2q_{j,0}^{(+)}} \delta(q_{i,0} - q_{i,0}^{(+)}) + \mathcal{O}(x^0)\,,$$

- The same is true for intersection of 3 or more propagators

Massless cases-IR divergencies

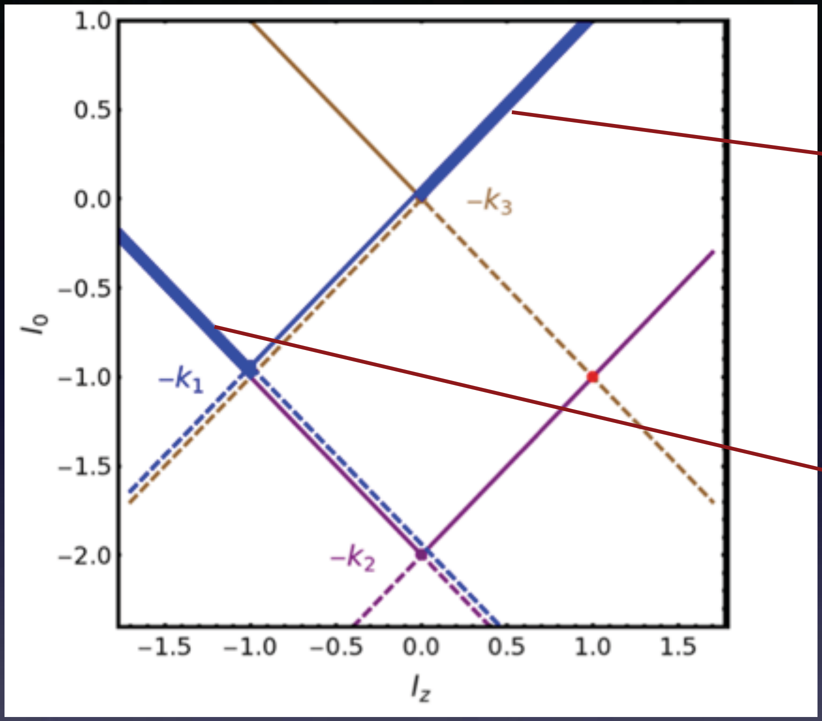


-Interested in cases with massless internal lines and external momenta on-shell

- The intersections are tangental

-Collinear divergencies, intersections lines

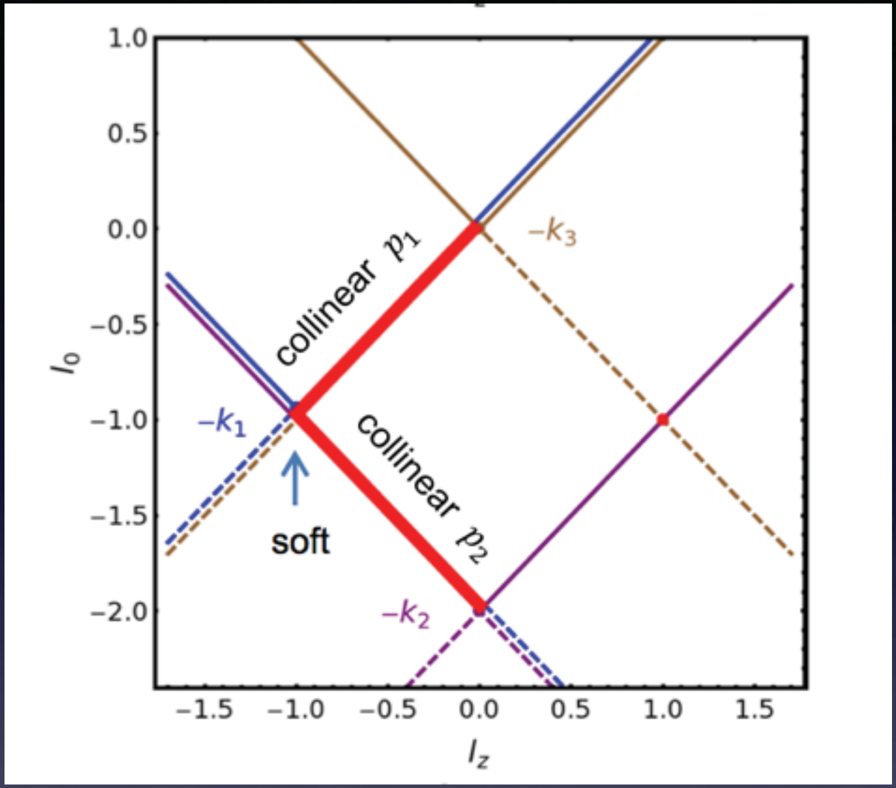
-Soft divergencies, intersection points



Forward-Forward intersection of
Dual propagators 1 and 3-Cancels

F-F of 1 and 2, Cancels

- Forward-Backward singularities survive but…

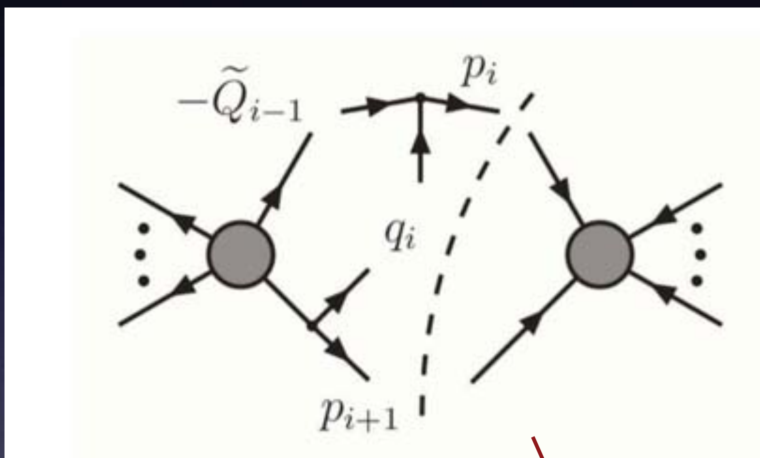


Now they are restricted to a finite region

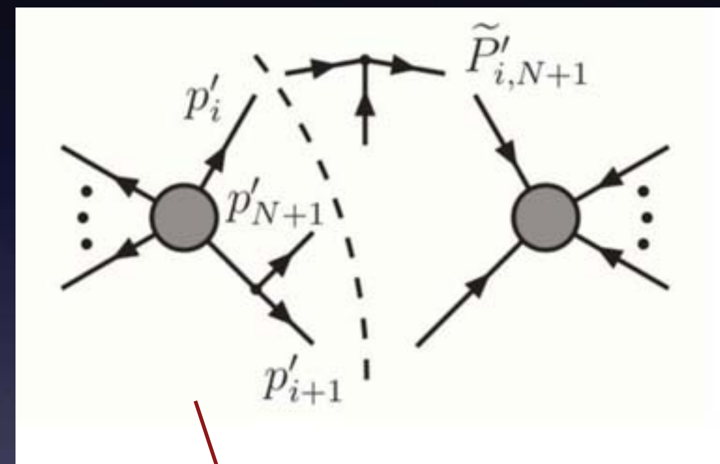
and they can be mapped to some phase-space contribution

Cancellation of collinear divergencies-Example

For splitting function also look at -Catani, de Florian, Rodrigo, PLB586(2004), JHEP 07(2012)026 and G.Sborlini´s talk



Virtual



Real

$$I_i^{(1)} = -2\mathrm{Re}\left(\prod_{i=2}^{N}\int_{p_i}\tilde{\delta}\left(p_i\right)\right)\delta^{(d)}\left(\sum_{i=1}^{N}p_i\right)$$
$$\times\int_{\ell}\tilde{\delta}\left(q_i\right)\theta(p_{i,0}-q_{i,0}^{(+)})\left\langle\mathcal{M}_N^{(0)}(\ldots,p_i,p_{i+1},\ldots)\right|\mathcal{M}_{N+2}^{(0)}(\ldots,p_i,-q_i,q_i,p_{i+1},\ldots)\rangle$$

$$I_{ab}^{(0)} = 2\mathrm{Re}\left(\prod_{i=1}^{N+1}\int_{p'_i}\tilde{\delta}\left(p'_i\right)\right)\delta^{(d)}\left(\sum_{i=2}^{N+1}p'_i\right)\left\langle\mathcal{M}_{N+1}^{(0),a}(p'_1,\ldots,p'_{N+1})\right|\mathcal{M}_{N+1}^{(0),b}(p'_1,\ldots,p'_{N+1})\rangle$$

- In the virtual part, Duality has been used to open the loop to tree

$$\left|M_N^{(1)}(p_1, \dots, p_N)\right\rangle \rightarrow \left|M_{N+2}^{(0)}(\dots, p_i, -q_i, q_i, \dots)\right\rangle$$

- Phase-Space different-need some mapping to show the cancellation

- When p_i and q_i become collinear

$$\begin{aligned} |\mathcal{M}_{N+2}^{(0)}(\dots, p_i, -q_i, q_i, p_{i+1}, \dots)\rangle &= \boldsymbol{Sp}^{(0)}(p_i, -q_i; -\widetilde{Q}_{i-1}) \\ &\times |\overline{\mathcal{M}}_{N+1}^{(0)}(\dots, p_{i-1}, -\widetilde{Q}_{i-1}, q_i, p_{i+1}, \dots)\rangle + \mathcal{O}(q_{i-1}^2) \end{aligned}$$

where

$$\widetilde{Q}_{i-1}^{\mu} = q_{i-1}^{\mu} - \frac{q_{i-1}^2 \, n^{\mu}}{2nq_{i-1}}$$

- Similarly

$$\langle \mathcal{M}_{N+1}^{(0),a}(p'_1, \ldots, p'_{N+1})| = \langle \overline{\mathcal{M}}_N^{(0)}(\ldots, p'_{i-1}, \widetilde{P}'_{i,N+1}, p'_{i+1}, \ldots)| \, \boldsymbol{Sp}^{(0)\dagger}(p'_i, p'_{N+1}; \widetilde{P}'_{i,N+1}) + \mathcal{O}(s'_{i,N+1})$$

where $s'_{i,N+1} = (p'_i + p'_{N+1})^2$, and

$$\widetilde{P}'^{\mu}_{i,N+1} = (p'_i + p'_{N+1})^{\mu} - \frac{s'_{i,N+1}\, n^{\mu}}{2n(p'_i + p'_{N+1})}$$

in the collinear limit of p´_i and p´_N+1

- From the two graphs we can see that the mapping should be the following

$$\begin{aligned} p_i &= \widetilde{P}'_{i,N+1} \\ p_j &= p'_j \qquad j \neq i \\ -\widetilde{Q}_{i-1} &= p'_i \\ q_i &= p'_{N+1} \end{aligned}$$

- Under this mapping some of the momenta and the matrix elements match completely

- Difference in the propagators of splitting functions and some integration measurements

From
$$\frac{1}{(q_i-p_i)^2} \to \frac{1}{(p'_{N+1}-\widetilde{P}'_{i,N+1})^2} = -\frac{n\widetilde{P}'_{i,N+1}}{np'_i}\frac{1}{(p'_i+p'_{N+1})^2}$$

we get
$$\boldsymbol{Sp}^{(0)\dagger}(p'_i,p'_{N+1};\widetilde{P}'_{i,N+1}) = -\frac{np'_i}{n\widetilde{P}'_{i,N+1}}\boldsymbol{Sp}^{(0)}(\widetilde{P}'_{i,N+1},-p'_{N+1};p'_i) .$$

- We get the cancellation if

$$\int d\Phi_N(p'_{j\neq i},\cdots,\widetilde{P}'_{i,N+1})\tilde{\delta}\,(p'_{N+1})\,\frac{1}{(p'_{N+1}-\widetilde{P}'_{i,N+1})^2} = -\int d\Phi_{N+1}(p'_i,\cdots,p'_{N+1})\frac{1}{(p'_i+p'_{N+1})^2}$$

in the collinear limit
$$p'_i+p'_{N+1} = \widetilde{P}'_{i,N+1} + \mathcal{O}(s_{i,N+1})$$

We start from the left hand side and perform some trivial delta integrations

$$\int \tilde{\delta}\left(\widetilde{P}'_{i,N+1}\right) \tilde{\delta}\left(p'_{N+1}\right) \frac{1}{(p'_{N+1} - \widetilde{P}'_{i,N+1})^2} = \frac{1}{4|\vec{p'}_{N+1}||\vec{\widetilde{P'}}_{i,N+1}|} \frac{-1}{2(|\vec{p'}_{N+1}||\vec{\widetilde{P'}}_{i,N+1}| - \vec{p'}_{N+1}\vec{\widetilde{P'}}_{i,N+1})}$$

focusing only on the terms that don´t match

- We take now the collinear limit : $$\vec{p'}_{N+1} = x\vec{\widetilde{P'}}_{i,N+1} - \vec{l}_T$$

with $0 < x < 1, \widetilde{P'}_{i,N+1} \cdot \vec{l}_T = 0$ and $\vec{l}_T \to 0$

- We get (after expansion) :

$$\int \tilde{\delta}\left(\widetilde{P}'_{i,N+1}\right) \tilde{\delta}\left(p'_{N+1}\right) \frac{1}{(p'_{N+1} - \widetilde{P}'_{i,N+1})^2} = \frac{-1}{4\vec{l}_T^2 \vec{\widetilde{P'}}^2_{i,N+1}}(1 + \mathcal{O}(\vec{l}_T^{\,2}))$$

- For the right hand side we need to perform a Phase-Space decomposition first

$$d\Phi_{N+1}(p'_i, \cdots, p'_{N+1}) = d\Phi_N(p'_{i,N+1}, p'_{j\neq i}, \cdots, p'_N)d\Phi_2(p'_{i,N+1}; p'_i, p'_{N+1})ds'_{i,N+1}$$

- Performing again some delta integrations we are able to show

$$\int d\Phi_{N+1}(p'_i, \cdots, p'_{N+1})\frac{1}{(p'_i + p'_{N+1})^2} = \int \delta^4(P - p'_{i,N+1} - \sum_{j\neq i}^{N} p'_j)\left(\prod_{i\neq j}^{N}\frac{d^3\vec{p}'_j}{2|\vec{p'}_j|}\right)$$
$$\frac{d^3\vec{p'}_{N+1}}{2|\vec{p'}_{N+1}|}\frac{d^3\vec{p'}_{i,N+1}}{2|\vec{p'}_{i,N+1} - \vec{p'}_{N+1}|}\frac{1}{(|\vec{p'}_{i,N+1} - \vec{p'}_{N+1}| + |\vec{p'}_{N+1}|)^2 - (\vec{p'}_{i,N+1})^2}$$

- We take the collinear limit as before $\vec{p'}_{N+1} = x\vec{P'}_{i,N+1} - \vec{l}_T$

and we get

$$\frac{1}{4|\vec{p'}_{N+1}||\vec{p'}_{i,N+1} - \vec{p'}_{N+1}|}\frac{1}{(|\vec{p'}_{i,N+1} - \vec{p'}_{N+1}| + |\vec{p'}_{N+1}|)^2 - (\vec{p'}_{i,N+1})^2} = \frac{1}{4\vec{l}_T^2\vec{P'}^2_{i,N+1}}(1 + \mathcal{O}(\vec{l_T}^2))$$

- The two terms cancel exactly
- In a similar way, all possible collinear divergences cancel
- Cancellation of the soft divergences (omitted in the presentation)
- Loop-tree duality ⟺ Virtual-Real duality

Conclusions and Future

- Duality is a method for calculating loop Amplitudes and has interesting properties
- The method is extended to higher loops
- Threshold and IR singularities among dual integrals cancel when the intersections happen at the Forward-Forward light cone
- The remaining singularities are restricted in a finite region of the loop momentum

- Numerical implementation
- UV divergences
- Singularities in higher orders

THANK YOU