Nested generalized binomial sums and new iterated integrals for massive Feynman diagrams

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LHCPhenoNet Annual Meeting
December 3, 2013

Outline

1. Introduction
2. Integral representations
   - Mellin transform
   - Iterated integrals and their properties
3. Mellin convolution via differential equations
   - Construct ODEs by parametric integration
   - Solve ODEs in terms of basis of iterated integrals
   - General patterns in convolution integrals

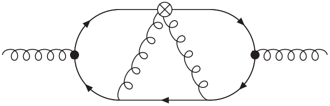
Introduction

Deep-inelastic scattering

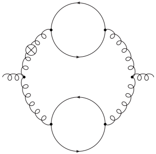
Heavy flavor Wilson coefficients

3-loop contributions in the asymptotic region by massive operator matrix elements at general N

2-loop: [Buza, Matiounine, Smith, Migneron, Neerven hep-ph/9601302]



[Ablinger, Blümlein, De Freitas, Hasselhuhn, Klein, CGR, Round, Schneider, Wißbrock 1212.6823]



[Ablinger, Blümlein, De Freitas, Hasselhuhn, Klein, Schneider, Wißbrock 1212.5950]

Main steps

1. Dimensional regularization
2. ε-expansion via symbolic summation tools
3. Analytic continuation to complex N

[Ablinger, Blümlein, Round, Schneider 1210.1685]

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Harmonic sums: [Vermaseren hep-ph/9806280]

$$a(i) = \frac{(\pm 1)^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized harmonic sums (S-sums): [Moch, Uwer, Weinzierl hep-ph/0110083]

$$a(i) = \frac{c^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized (inverse) binomial sums:

$$a(i) = \frac{c^i}{i^m}\binom{2i}{i}, \qquad a(i) = \frac{c^i}{i^m}, \qquad a(i) = \frac{c^i}{i^m}\binom{2i}{i}^{-1}$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} =$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} =$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} =$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2\binom{2i}{i}} = 2\int_0^1 \frac{x^N-1}{x-1}\operatorname{arctanh}\left(\sqrt{1-x}\right)\,dx$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} = -\frac{1}{\pi}\int_0^1 \frac{(4x)^N-1}{x-\frac{1}{4}}\left(\frac{\arccos(2x-1)^3}{6} - \zeta_2\arccos(2x-1)\right)dx$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = \frac{1}{2}\int_0^1 \frac{(-x)^N-1}{x+1}\frac{x\,\mathrm{H}^*_{(0,(-\frac{1}{4})),0,0}(x)}{\sqrt{x+\frac{1}{4}}}\,dx - {}_5F_4(const)\int_0^1 \frac{\left(\frac{x}{4}\right)^N-1}{x-4}\frac{x}{\sqrt{1-x}}\,dx$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} = 2 \int_0^1 \frac{x^N - 1}{x - 1} \operatorname{arctanh}\left(\sqrt{1-x}\right) dx$$

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$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = \frac{1}{2} \int_0^1 \frac{(-x)^N - 1}{x+1} \frac{x \, \mathrm{H}^*_{(0,(-\frac{1}{4})),0,0}(x)}{\sqrt{x + \frac{1}{4}}} dx - {}_5F_4(const) \int_0^1 \frac{(\frac{x}{4})^N - 1}{x - 4} \frac{x}{\sqrt{1-x}} dx$$

Nested sums and their asymptotic expansion

Examples

$$\sum_{i=1}^{N}\frac{4^i}{i^2\binom{2i}{i}} = 3\zeta_2+\sqrt{\frac{\pi}{N}}\left(-2+\frac{5}{12N}-\frac{21}{320N^2}+\dots\right)$$

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} = \frac{4^N}{\sqrt{\pi N}}\left(\frac{4\zeta_2}{3N}+\frac{3\zeta_2-8}{6N^2}+\frac{363\zeta_2-80}{288N^3}+\dots\right) -\frac{2}{3}\zeta_3$$

$$\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = const+(-1)^N\left(\frac{1}{5N^4}-\frac{16}{25N^5}+\frac{67}{125N^6}+\dots\right) -\frac{{}_5F_4(const)}{4^N}\sqrt{\frac{\pi}{N}}\left(-\frac{1}{3}+\frac{25}{72N}-\frac{683}{1152N^2}+\dots\right)$$

Integral representations

Integral representations

General form

$$\sum_{i=1}^{N} a(i) = const + \sum_{j=1}^{k} c_j^N \int_0^1 x^N f_j(x)\, dx$$

Integral representations

General form

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Mellin transform

$$\mathbf{M}\left(f(x)\right)(N) := \int_0^1 x^N f(x)\,dx$$

Mellin transform

Basic properties

$$\begin{aligned}
\mathbf{M}\left(c_1 f_1(x) + c_2 f_2(x)\right)(N) &= c_1 \mathbf{M}\left(f_1(x)\right)(N) + c_2 \mathbf{M}\left(f_2(x)\right)(N) \\
\mathbf{M}\left(f(x)\right)(N+k) &= \mathbf{M}\left(x^k f(x)\right)(N) \\
\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) &= \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t}) f_2(t)}{t}\,dt\right)(N)
\end{aligned}$$

Mellin transform

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\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) &= \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t}) f_2(t)}{t}\,dt\right)(N)
\end{aligned}$$

Summation

$$\sum_{k=1}^{N} c^k \mathbf{M}\left(f(x)\right)(k) = c^N \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(N) - \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(0)$$

Mellin transform (cont.)

Basic transforms

$$\frac{1}{N} = \mathbf{M}\left(\frac{1}{x}\right)(N)$$

$$\binom{2N}{N} = \frac{4^N}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(N)$$

$$\frac{1}{N\binom{2N}{N}} = \frac{1}{4^N}\mathbf{M}\left(\frac{1}{x\sqrt{1-x}}\right)(N)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)$$

Example

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{1}{x}\right)(i) \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)
\end{aligned}$$

Example

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} &= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)\\
&= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{1}{x}\right)(i)\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)
\end{aligned}$$

Example

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{1}{x}\right)(i) \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i) \\
&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)
\end{aligned}
$$

Example

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j) \\
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&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\zeta_2 \frac{\arccos(2x-1)}{x} - \frac{\arccos(2x-1)^3}{6x}\right)(i)
\end{aligned}$$

Example

$$
\begin{aligned}
\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} &= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)\\
&= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\tfrac{1}{x}\right)(i)\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)\\
&= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\,\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)\\
&= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\zeta_2\frac{\arccos(2x-1)}{x}-\frac{\arccos(2x-1)^3}{6x}\right)(i)\\
&= \frac{4^N}{\pi}\mathbf{M}\left(\zeta_2\frac{\arccos(2x-1)}{x-1/4}-\frac{\arccos(2x-1)^3}{6(x-1/4)}\right)(N)-\frac{2}{3}\zeta_3
\end{aligned}
$$

Iterated integrals

General form

$$\mathrm{H}_{a_1,\dots,a_k}(x) := \int_0^x f_{a_1}(t_1) \int_0^{t_1} f_{a_2}(t_2) \dots \int_0^{t_{k-1}} f_{a_k}(t_k)\, dt_k \dots dt_1$$

Harmonic polylogarithms

[Remiddi, Vermaseren hep-ph/9905237]

$$f_1(x) := \frac{1}{1-x}, \qquad f_0(x) := \frac{1}{x}, \qquad f_{-1}(x) := \frac{1}{1+x}$$

Iterated integrals

General form

$$\mathrm{H}_{a_1,\dots,a_k}(x) := \int_0^x f_{a_1}(t_1) \int_0^{t_1} f_{a_2}(t_2) \dots \int_0^{t_{k-1}} f_{a_k}(t_k)\, dt_k \dots dt_1$$

Multiple polylogarithms

[Kummer 1840, Lappo-Danilevsky 1953, Goncharov 1998]

$$f_a(x) := \frac{1}{|a| - \mathrm{sign}(a)x}, \qquad f_0(x) := \frac{1}{x}$$

Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \ldots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \ldots dt_1$$

Multiple polylogarithms

[Kummer 1840, Lappo-Danilevsky 1953, Goncharov 1998]

$$f_a(x) := \frac{1}{|a| - \mathrm{sign}(a)x}, \qquad f_0(x) := \frac{1}{x}$$

Generalization

[Hermite 1883, Aglietti, Bonciani hep-ph/0401193]

$$f_a(x) := \frac{\mathrm{sign}(1-a)}{x-a}, \qquad f_1(x) := \frac{1}{1-x},$$

$$f_{(a,(b))}(x) := f_a(x)\sqrt{f_b(x)}, \qquad f_{(a,b)}(x) := \sqrt{f_a(x)f_b(x)},$$

$$f_{(a,(b,c))}(x) := f_a(x)\sqrt{f_b(x)f_c(x)}$$

Examples

Depth 1

- $\mathrm{H}_0^*(x) = \int_x^1 \frac{1}{t}\,dt = -\ln(x)$
- $\mathrm{H}_{(0,1)}^*(x) = \int_x^1 \frac{1}{\sqrt{t(1-t)}}\,dt = \arccos(2x-1)$
- $\mathrm{H}_{(0,(1))}^*(x) = \int_x^1 \frac{1}{t\sqrt{1-t}}\,dt = 2\,\mathrm{arctanh}(\sqrt{1-x})$

Depth 2

- $\mathrm{H}_{1,0}^*(x) = \int_x^1 \frac{1}{1-t}\,\mathrm{H}_0^*(t)\;dt = \mathrm{Li}_2(1-x)$
- $\mathrm{H}_{(2,(0,8)),1}^*(x) = \int_x^1 \frac{\ln(1-t)}{(2-t)\sqrt{t(8-t)}}\,dt$

Algebraic relations

Shuffle relations

$$
\begin{aligned}
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1}}(x) \\
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x) \\
&\vdots
\end{aligned}
$$

Algebraic relations

Shuffle relations

$$
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\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x)\\
&\ \vdots
\end{aligned}
$$

Theorem

The functions $\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)$ do not satisfy any algebraic relations apart from the shuffle relations.

Mellin transforms

Theorem

Let $h(x) = \frac{rat(x)}{\sqrt{poly(x)}}$ and let $\mathbf{a_1}, \dots, \mathbf{a_k}$ be some generalized letters as above. If for all $i \in \{0, \dots, k\}$ the product

$$h(x) f_{\mathbf{a_1}}(x) \dots f_{\mathbf{a_i}}(x)$$

has at most one squareroot singularity different from $x = 0$, then

$$\mathbf{M}\left(h(x)\,\mathrm{H}^{*}_{\mathbf{a_1},\dots,\mathbf{a_k}}(x)\right)(N)$$

is expressible in terms of generalized (inverse) binomial sums.

Examples

Basic examples

- $$\mathbf{M}\left(\frac{1}{\sqrt{x-a}}\right)(N) =$$
$$\frac{(4a)^N}{(N+1/2)\binom{2N}{N}}\left(\sqrt{1-a}-\sqrt{-a}+\sqrt{1-a}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{(4a)^i}\right)$$
- $$\mathbf{M}\left(\frac{1}{\sqrt{x(x-a)}}\right)(N) =$$
$$\left(\frac{a}{4}\right)^N\binom{2N}{N}\left(2\operatorname{arcsinh}\left(\frac{1}{\sqrt{-a}}\right)+\sqrt{1-a}\sum_{i=1}^{N}\frac{(4/a)^i}{i\binom{2i}{i}}\right)$$

Mellin convolution via differential equations

Mellin convolution

Recall

$$\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) = \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt\right)(N)$$

Approach

Compute

$$F(x) = \int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt$$

as solution of a differential equation

$$c_m(x)F^{(m)}(x) + \cdots + c_0(x)F(x) = r(x)$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

1. Find $g(x,t)$ and $c_0(x), \ldots, c_m(x)$ s.t.

$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

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2. Transfer this to a relation of corresponding integrals

$$c_m(x) \int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \dots + c_0(x) \int_{a(x)}^{b(x)} f(x,t)\,dt =$$

Construction of the ODEs

Parametric integration

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2. Transfer this to a relation of corresponding integrals

$$c_m(x) \int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \cdots + c_0(x) \int_{a(x)}^{b(x)} f(x,t)\,dt = \\ = g(x,b(x)) - g(x,a(x))$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

1. Find $g(x,t)$ and $c_0(x), \dots, c_m(x)$ s.t.
$$c_m(x) D_x^m f(x,t) + \dots + c_0(x) f(x,t) = D_t g(x,t)$$
2. Transfer this to a relation of corresponding integrals
$$c_m(x) \int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \dots + c_0(x) \int_{a(x)}^{b(x)} f(x,t)\,dt = \\ = g(x, b(x)) - g(x, a(x))$$
3. Rewrite as ODE for $F(x)$
$$c_m(x) F^{(m)}(x) + \dots + c_0(x) F(x) = r(x)$$

Computational approach

Problem

Given: C a field, V a C-vector space, $f(x,t),\dots,D_x^m f(x,t) \in V$

Find: $g(x,t) \in V$ and $c_0(x),\dots,c_m(x) \in C$ s.t.

$$c_m(x)D_x^m f(x,t) + \dots + c_0(x)f(x,t) = D_t g(x,t)$$

Differential fields

[Risch 1969, Singer, Saunders, Caviness 1985, Bronstein 1990/97, CGR 2012]

- (V, D_t) a differential field containing $f(x,t),\dots,D_x^m f(x,t)$
- $C = \mathrm{Const}_{D_t}(V)$ its constant field

Ore algebras

[Almkvist, Zeilberger 1990, Chyzak 2000, Chyzak, Kauers, Salvy 2009, Koutschan 2009]

- V a $C(t)[D_x, D_t]$-module containing $f(x,t)$
- C the field of rational functions in x

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

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$$F^{(3)}(x) - \frac{3(4x-3)}{2x(1-x)} F''(x) - \frac{7x-3}{x^2(1-x)} F'(x) - \frac{1}{x^2(1-x)} F(x) = \frac{1}{x^2(1-x)\sqrt{x(1-x)}}$$

Solution of the ODEs

Observations

- ODEs obtained factor completely into first-order ODEs

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Consequence

Solutions can be written as linear combinations of expressions of the form

$$\frac{r(x)}{\sqrt{p(x)}}\,\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)$$

where r is a rational function and p is a squarefree polynomial

Example

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k^2}}{j2^j}}{i\binom{2i}{i}} = \sum_{i=1}^{N} \frac{1}{i\binom{2i}{i}} \left(\frac{1}{2^i} \mathbf{M} \left(\frac{\mathrm{H}^*_{1,0}(x) - \zeta_2}{2-x} \right)(i) + \frac{5}{8}\zeta_3 \right)$$

involves convolution $F(x) = \int_x^1 \frac{\mathrm{H}^*_{1,0}(t) - \zeta_2}{x(2-t)\sqrt{1-\frac{x}{t}}}\,dt$ satisfying

$$F^{(5)}(x) + \cdots + \frac{15}{4x^3(1-x)(2-x)} F(x) = -\frac{3}{4x^4(1-x)^{7/2}(2-x)}$$

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Solution is given by

$$F(x) = \frac{1}{x}\left(\mathrm{H}^*_{0,0,(0,(1))}(x) - \mathrm{H}^*_{(0,(1)),(0,(1)),(0,(1))}(x) + \zeta_2\,\mathrm{H}^*_{(0,(1))}(x)\right) - \frac{2}{x\sqrt{2-x}}\left(\mathrm{H}^*_{(0,(2)),0,(0,(1))}(x) - \mathrm{H}^*_{(0,(1,2)),(0,(1)),(0,(1))}(x) + \zeta_2\,\mathrm{H}^*_{(0,(1,2))}(x)\right)$$

Patterns

Many patterns emerge from convolutions arising in the absorption of simple prefactors, for example

Theorem

$$\binom{2N}{N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_1,\ldots,a_k}(x)}{x-a_0}\right)(N)=\frac{4^N}{\pi}\mathbf{M}\left(\frac{\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{\sqrt{x(x-a_0)}}\right)(N)$$

where $\mathbf{b_i}=(a_i,a_{i+1})$ for $i\in\{0,\ldots,k-1\}$ and $\mathbf{b_k}=(1,a_k)$

Theorem

$$\frac{1}{N\binom{2N}{N}}\mathbf{M}\left(\frac{x\,\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{(x-c)\sqrt{x(x-a_0)}}\right)(N)=$$
$$\frac{\pi}{4^N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_0,\ldots,a_k}(x)}{x}+c\frac{\mathrm{H}^*_{\mathbf{w},a_1,\ldots,a_k}(x)}{x\sqrt{x-c}}\right)(N)$$

where $\mathbf{w}=(a_0,(c))$ and $\mathbf{b_0},\ldots,\mathbf{b_k}$ are as above

Conclusion

Summary

- Massive operator matrix elements can lead to nested generalized (inverse) binomial sums
- Mellin-type integral representations of these sums in terms of new iterated integrals involving squareroots

Computational methods

- Conversion between sum and integral representations
- Asymptotic expansion of the sums
- ODE for Mellin convolution by parametric integration

Theoretical results

- Algebraic relations of new iterated integrals
- Conditions for Mellin transform in terms of nested sums
- Patterns in Mellin convolutions