# A Computer Algebra Approach to Calculating 2-bubble type 3-loop Contributions to Heavy Flavour Wilson Coefficients

Mark Round
(In collaboration with J. Blümlein, C. Schneider and F. Wißbrock)

DESY (formerly RISC)

December 2013

# Outline

1. Outline of Physical Problem
2. Mathematical Results
3. Example Diagrams
4. Summary

# Heavy Flavour Wilson Coefficients

To produce parton distribution functions (PDFs) one compares a measurement,

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2}{xyQ^2}\left[(1+(1-y)^2)F_2(x) - y^2 F_L(x)\right],$$

to a product of the (unknown) PDF, $f_j$, and a perturbative piece; the Wilson coefficients,

$$F_i(x) = \sum_j \int \frac{dz}{z} C_{i,j}\left(\frac{x}{z}\right) f_j(z),$$

$i \in \{2, L\}$ and $j$ runs over all (anti-)quarks and the gluon.
Light flavour contributions are known to 3-loops, heavy contributions to 2-loops.
Computing heavy 3-loop diagrams will lead to more accurate PDFs.

# Heavy Flavour Wilson Coefficients

Note that the PDF is introduced as a Mellin convolution,

$$F_i(x) = \sum_j \int \frac{dz}{z} C_{i,j}\left(\frac{x}{z}\right) f_j(z) = \sum_j C_{i,j} * f_j.$$

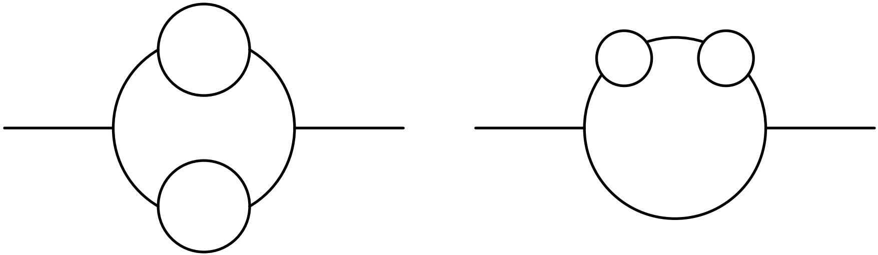
Thus by taking the Mellin transform,

$$\mathcal{M}[g](N) = \int_0^1 dx x^{N-1} g(x) \quad N \in \mathbb{Z}^+$$

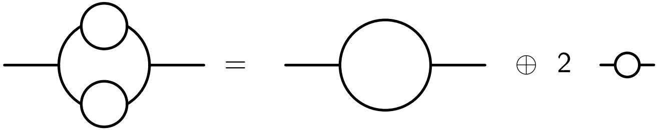
things simplify greatly.
In the DIS formalism (omitted!) one has an operator insertion which will be present throughout the talk.
Today I will discuss 3-loop diagrams that contribute to the Wilson coefficients in the limit that $Q^2 \gg m^2$ (virtuality much larger than heavy quark mass).

# 2-Bubble Type Diagrams

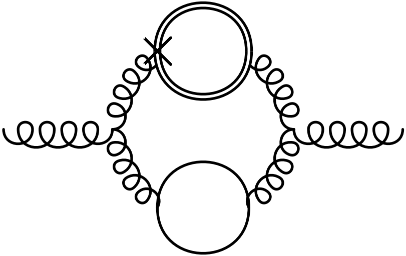
Consider the following class of diagrams that contribute to the heavy Wilson coefficients,



Restrict to one massive fermion and an operator insertion (not shown.)



# 2-Bubble Type Diagrams



The double lines denote a heavy quark and the cross denotes an operator insertion. Single lines for fermions and curly lines for gluons.

In total there are 1,142 unique diagrams to compute for the 1-mass heavy Wilson coefficients. The 2-bubble type diagrams account for 149 of those ($\sim 13\%$).

# 2-Bubble Type Diagrams

1. Most of the diagrams in the 2-bubble class are easy and could be solved with numerous methods.
2. The key challenge here is to work algorithmically enough so as to solve all the diagrams in the class.
3. To do that here a special function approach is emphasised.

# Computer Algebra Approach

One tries to find a sum representation of the Feynman parameter integrals which can then be solved by powerful computer algebra summation packages.

The most naïve way of doing that is to binomially expand as required and recognise only $B$-integrals (example later). In particular the operator insertion gives a sum directly in the Feynman rules (FRs) for example, after momentum integration,

$$f(x_1, x_2, \ldots) \sum_{i=0}^{N-3} (1 - x_1)^i$$

Such an approach led to very high-fold sums. For example a triple sum from a 5-point vertex easily leads to 4-fold sums. Tough!

# Special Function Approach

Try to express the problem in terms of several steps that can be automated with computer algebra packages.

1. Perform momentum integrals by introducing Feynman parameters and using textbook formulae plus recognise FR all sums.
2. Recognise the Feynman parameter integrals as special functions; we can even deliver sum free results
3. Convert special functions to a sum form that can be expanded in the DREG parameter, $\epsilon$.

These steps are easy to automate for the 2-bubble class.

# Mathematical Results

For the Feynman parameter integrals in the 2-bubble class we will need just a few classical results about special functions,

1. The $B$-function,

$$B(x,y) = \int_0^1 (1-t)^{y-1} t^{x-1} dt$$

2. The $\Gamma$-function,

$$\Gamma[x] = \int_0^\infty t^{x-1} e^{-t} dt$$

   which is related to the $B$-function by,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

3. The hypergeometric function, or ${}_2F_1$,

$$ {}_2F_1(a,b,c;z) = \frac{1}{B(b,c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx $$

4. The Pfaff transformations which allow one to change the ${}_2F_1$ argument,

$$ {}_2F_1(a,b,c;z) = (1-z)^{-b} {}_2F_1\left(b, c-a, c; \frac{z}{z-1}\right) $$

5. Euler's transformation,

$$ {}_{p+1}F_{q+1}\left[\begin{array}{c} a_1,\ldots,a_p,c \\ b_1,\ldots,b_q,d \end{array}; z\right] = $$

$$ \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} {}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}; tz\right] dt $$

which defines the generalised hypergeometric functions. Such as the ${}_3F_2$.

6. The generalised hypergeometric functions admit a sum representation,

$$ {}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z\right] = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_p)_i}{(b_1)_i\cdots(b_q)_i}\frac{z^i}{i!} $$

which makes use of the Pochhammer symbol,

$$ x_n = \frac{\Gamma(x+n)}{\Gamma(x)}. $$

# Mathematical Results

Sums of this type can be solved by, for example Sigma(Schneider) and $\rho$Sum (Schneider & MR) based around algorithms in Sigma.
The output of such packages is in terms of rational functions, $\zeta$-values,

$$\zeta_n = \sum_{i=1}^{\infty} \frac{1}{i^n}$$

and harmonic sums,

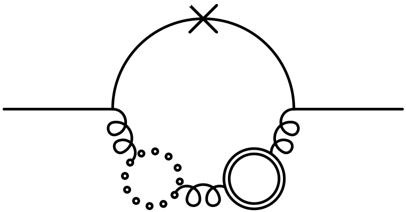
$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_a(N) = \sum_{i_1=1}^{N} \frac{[\mathrm{sign}(a)]^i}{i^{|a|}}$$

$$S_{a_1,a_2,\ldots,a_k}(N) = \sum_{i_1=1}^{N} \frac{[\mathrm{sign}(a_1)]^{i_1}}{i_1^{|a_1|}} \sum_{i_2=1}^{i_1} \frac{[\mathrm{sign}(a_2)]^{i_2}}{i_2^{|a_2|}} \ldots \sum_{i_k=1}^{i_{k-1}} \frac{[\mathrm{sign}(a_k)]^{i_k}}{i_k^{|a_k|}}$$

The talk will focus more on methods.

# Diagram 1



The diagram consists of 101 terms all of the form,

$$C(\text{Color Factors, Couplings}, \epsilon) B\left(3+\frac{\epsilon}{2}, -1-\frac{\epsilon}{2}\right)^2 B\left(3+\frac{\epsilon}{2}, -\frac{3\epsilon}{2}\right)$$
$$\times \frac{\Gamma(3-\epsilon)}{\Gamma\left(-1-\frac{\epsilon}{2}\right)\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3}(1-x_4)^{1-\epsilon}x_4^{1-\epsilon}(1-x_6)^{1+\frac{\epsilon}{2}}x_6^{1+\frac{\epsilon}{2}}$$
$$\times (1-x_8)^{N-1-\frac{\epsilon}{2}}x_8(1-x_9)^{1-\frac{\epsilon}{2}}x_9^{\epsilon-2} + \cdots$$

The $B$-functions and $\Gamma$-functions are from the momentum integration.

# Diagram 1

Notice that all integrals are manifestly $B$-functions! That is the simplest possible diagran in the 2-bubble class.
Therefore the integration is immediate and gives,

$$
\begin{aligned}
&C(\text{Color Factors, Couplings}, \epsilon) B\left(3+\frac{\epsilon}{2},-1-\frac{\epsilon}{2}\right)^{2} B\left(3+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right) \\
&\times \frac{\Gamma(3-\epsilon)}{\Gamma\left(-1-\frac{\epsilon}{2}\right)\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^{3}} B(2-\epsilon, 2-\epsilon) B(2+\frac{\epsilon}{2}, 2+\frac{\epsilon}{2}) \\
&\times B(N-\frac{\epsilon}{2}, 2) B(2-\frac{\epsilon}{2}, \epsilon-1)+\cdots
\end{aligned}
$$

To expand the diagram in $\epsilon$ one can use standard formulae, or most CAS's,

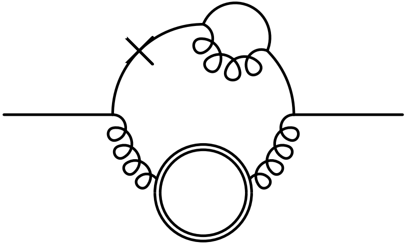
$$
B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
$$

$$
\Gamma(x+\epsilon)=\Gamma[x]+\left[-\gamma_E+S_1(x)-\frac{1}{x}\right]\Gamma[x]\epsilon+\ldots
$$

# Diagram 1

The total result is,

$$
\frac{4\left(N^2+N-2\right)}{27N(N+1)}\frac{1}{\epsilon^3}-\frac{2\left(N^4+2N^3+17N^2+4N-6\right)}{81N^2(N+1)^2}\frac{1}{\epsilon^2}
+\left[\frac{29N^6+87N^5+17N^4-79N^3-40N^2-2N-6}{81N^3(N+1)^3}+\frac{N^2+N-2}{18N(N+1)}\zeta_2\right]\frac{1}{\epsilon}
+\frac{355N^8+1420N^7-146N^6-4148N^5-3755N^4-404N^3+378N^2}{1458N^4(N+1)^4}
+\frac{72N+54}{1458N^4(N+1)^4}+\frac{-N^4-2N^3-17N^2-4N+6}{108N^2(N+1)^2}\zeta_2-\frac{7\left(N^2+N-2\right)}{54N(N+1)}\zeta_3
$$



This diagram consists of 216 terms all of the form,

$$C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)B\left(3+\frac{\epsilon}{2},-1-\frac{\epsilon}{2}\right)B\left(4+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)$$
$$\times\frac{\Gamma(4-\epsilon)}{\Gamma(-1-\frac{\epsilon}{2})\Gamma(3-\frac{\epsilon}{2})\Gamma\left(2+\frac{\epsilon}{2}\right)^3}(1-x_4)^{\frac{\epsilon}{2}}x_4^{1+\frac{\epsilon}{2}}(1-x_6)^{1-ep}x_6^{1-ep}$$
$$\times(1-x_8)^{1+\frac{\epsilon}{2}}x_8(1+x_9)^{\frac{\epsilon}{2}}$$
$$\times(x_8+(1-x_8)x_9)^{N-1}$$

This diagram represents the second simplest diagram in the class.

# Diagram 2

One can immediately recognise B-functions,

$$
\begin{aligned}
&C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)B\left(3+\frac{\epsilon}{2},-1-\frac{\epsilon}{2}\right)B\left(4+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)\\
&\times\frac{\Gamma(4-\epsilon)}{\Gamma(-1-\frac{\epsilon}{2})\Gamma(3-\frac{\epsilon}{2})\Gamma\left(2+\frac{\epsilon}{2}\right)^3}B\left(2-\epsilon,2-\epsilon\right)B\left(2+\frac{\epsilon}{2},1+\frac{\epsilon}{2}\right)\\
&\times(1-x_8)^{1+\frac{\epsilon}{2}}x_8(1+x_9)^{\frac{\epsilon}{2}}\\
&\times(x_8+(1-x_8)x_9)^{N-1}
\end{aligned}
$$

In addition one can recognise a hypergeometric function.

# Diagram 2

Recognising a ${}_2F_1$ one has,

$$C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)B\left(3+\frac{\epsilon}{2},-1-\frac{\epsilon}{2}\right)B\left(4+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)$$

$$\times\frac{\Gamma(4-\epsilon)}{\Gamma(-1-\frac{\epsilon}{2})\Gamma(3-\frac{\epsilon}{2})\Gamma\left(2+\frac{\epsilon}{2}\right)^3}B\left(2-\epsilon,2-\epsilon\right)B\left(2+\frac{\epsilon}{2},1+\frac{\epsilon}{2}\right)$$

$$\times(1-x_9)^{2-\frac{\epsilon}{2}}x_9^{N-3+\epsilon}B\left(2,2+\frac{\epsilon}{2}\right){}_2F_1\left(1-N,2,4+\frac{\epsilon}{2};\frac{x_9-1}{x_9}\right)$$

Then one can perform a Pfaff transformation,

$${}_2F_1\left(1-N,2,4+\frac{\epsilon}{2};\frac{x_9-1}{x_9}\right)=(1-x_9)^{1-N}{}_2F_1\left(1-N,2+\frac{\epsilon}{2},4+\frac{\epsilon}{2};x_9\right)$$

# Diagram 2

Then immediately one can apply Euler's transformation,

$$C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)B\left(3+\frac{\epsilon}{2},-1-\frac{\epsilon}{2}\right)B\left(4+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)$$
$$\times\frac{\Gamma(4-\epsilon)}{\Gamma(-1-\frac{\epsilon}{2})\Gamma(3-\frac{\epsilon}{2})\Gamma\left(2+\frac{\epsilon}{2}\right)^3}B\left(2-\epsilon,2-\epsilon\right)B\left(2+\frac{\epsilon}{2},1+\frac{\epsilon}{2}\right)$$
$$B\left(2,2+\frac{\epsilon}{2}\right)\Gamma\left(3-\frac{\epsilon}{2}\right)\frac{\Gamma\left(\epsilon-1\right)}{\Gamma\left(2+\frac{\epsilon}{2}\right)}{}_3F_2\left[\begin{array}{c}1-N,2+\frac{\epsilon}{2},3-\frac{\epsilon}{2}\\4+\frac{\epsilon}{2},2+\frac{\epsilon}{2}\end{array};1\right]$$

# Diagram 2

To compute a Laurent expansion the Γ-functions are no problem but there is no simple and general formula to expand a ${}_3F_2$,

$$ {}_3F_2\left[\begin{array}{c} P_1(\epsilon), P_2(\epsilon), P_3(\epsilon) \\ P_4(\epsilon), P_5(\epsilon) \end{array};1\right] $$

for $P_i(\epsilon)$ a degree-1 polynomial in $\epsilon$.
However one can use the sum form,

$$ {}_3F_2\left[\begin{array}{c} 1-N, 2+\frac{\epsilon}{2}, 3-\frac{\epsilon}{2} \\ 4+\frac{\epsilon}{2}, 4+\frac{\epsilon}{2} \end{array};1\right] = \sum_{i=0}^{\infty} \frac{(1-N)_i \left(2+\frac{\epsilon}{2}\right)_i \left(3-\frac{\epsilon}{2}\right)_i}{\left(4+\frac{\epsilon}{2}\right)_i \left(4+\frac{\epsilon}{2}\right)_i} \frac{1}{i!} $$

however the sum is ill-posed!
Notice that $(1-N)_r$ vanishes when $r > N-1$. E.g. $(-3)_2 = 6$ while $(-3)_4 = 0$. Thus the upper bound of the sum must be corrected,

$$ {}_3F_2\left[\begin{array}{c} 1-N, 2+\frac{\epsilon}{2}, 3-\frac{\epsilon}{2} \\ 4+\frac{\epsilon}{2}, 4+\frac{\epsilon}{2} \end{array};1\right] = \sum_{i=0}^{N-1} \frac{(1-N)_i \left(2+\frac{\epsilon}{2}\right)_i \left(3-\frac{\epsilon}{2}\right)_i}{\left(4+\frac{\epsilon}{2}\right)_i \left(4+\frac{\epsilon}{2}\right)_i} \frac{1}{i!} $$

# Diagram 2

The sum is now precisely of the form solved by $\rho$Sum and indeed it returns that,

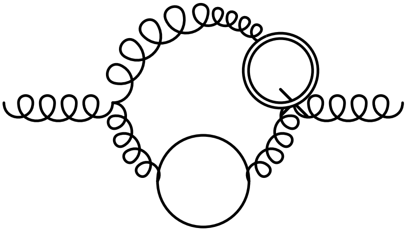
$$\sum_{i=0}^{N-1} \frac{(1-N)_i \left(2+\frac{\epsilon}{2}\right)_i \left(3-\frac{\epsilon}{2}\right)_i}{\left(4+\frac{\epsilon}{2}\right)_i \left(4+\frac{\epsilon}{2}\right)_i} \frac{1}{i!} = \frac{6}{N(1+N)(2+N)}$$

to constant order in $\epsilon$. Taking the product of the above Laurent series with the relevant ones for the $\Gamma$-functions gives the correct expansion for the diagram.

Alternatively one could have introduced a binomial expansion for $(x_8 + (1-x_8)x_9)^{N-1}$ which would give precisely the same sum.

The technologies discussed solve all but about 10 diagrams which are more difficult. Importantly one can automate the steps.

# Diagram 3



This diagram consists of 168 terms mostly of the forms seen before ($B$-functions and ${}_3F_2$'s). However there is also a new object,

$$C(\cdots)\frac{B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)^2 B\left(3+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)\Gamma(3-\epsilon)}{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3\Gamma\left(-\frac{\epsilon}{2}\right)}$$
$$\times(1-x_4)^{\frac{\epsilon}{2}}x_4^{1+\frac{\epsilon}{2}}(1-x_6)^{-\epsilon}x_6^{2-\epsilon}(1-x_8)^{1+\frac{\epsilon}{2}}(1-x_9)^{1-\frac{\epsilon}{2}}x_9^{\epsilon-1}$$
$$\times\frac{(1-x_6+x_6x_8+x_6(1-x_8)x_9)^{N-1}}{(-x_8-x_9+x_8x_9)}$$

# Diagram 3

Here we can recognise a $B$-function and a ${}_2F_1$,

$$C(\cdots)\frac{B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)^2 B\left(3+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)\Gamma(3-\epsilon)}{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3\Gamma\left(-\frac{\epsilon}{2}\right)}$$
$$\times B\left(1+\frac{\epsilon}{2},2+\frac{\epsilon}{2}\right)(1-x_8)^{1+\frac{\epsilon}{2}}(1-x_9)^{1-\frac{\epsilon}{2}}x_9^{\epsilon-1}$$
$$\times B\left(3-\epsilon,1-\epsilon\right){}_2F_1(1-N,3-\epsilon,4-2\epsilon;(1-x_8)(1-x_9))$$
$$\times\frac{1}{(-x_8-x_9+x_8x_9)}$$

There is no useful transformation and one is forced to write out the hypergeometric function as a (finite) sum.

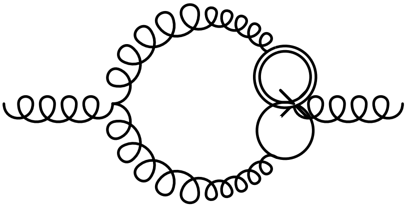
# Diagram 3

Inserting a sum representation,

$$C(\cdots)\frac{B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)^2 B\left(3+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)\Gamma(3-\epsilon)}{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3\Gamma\left(-\frac{\epsilon}{2}\right)}$$
$$\times B\left(1+\frac{\epsilon}{2},2+\frac{\epsilon}{2}\right)(1-x_8)^{1+\frac{\epsilon}{2}}(1-x_9)^{1-\frac{\epsilon}{2}}x_9^{\epsilon-1}$$
$$\times B(3-\epsilon,1-\epsilon)\sum_{i=0}^{N-1}\frac{(1-N)_i(3-\epsilon)_i}{(4-2\epsilon)_i}\frac{(1-x_8)^i(1-x_9)^i}{i!}$$
$$\times\frac{1}{(-x_8-x_9+x_8x_9)}$$

It is safe to commute the finite sum with the $x_8$ and $x_9$ integrals and one immediately finds a ${}_3F_2$. Thus one obtains a double-sum representation for the diagram which can be expanded as before.

# Diagram 4



This diagram consists of 48 terms all of which have already been seen. Similar to diagram 3 we can recognise and expand ${}_2F_1$'s. This time that happens in two parameters, $x_4$ and $x_6$ corresponding to both 'merged bubbles'.

# Diagram 4

In some terms one can recognise an integral in $x_4$,

$$C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)^2 B(\epsilon,1)\frac{\Gamma(2-\epsilon)}{\Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3\Gamma\left(-\frac{\epsilon}{2}\right)}$$
$$\times(1-x_4)^{N-2-\epsilon}x_4^{1-\epsilon}(1-x_6)^{\frac{1+\epsilon}{2}}x_6^{\frac{\epsilon}{2}-1}(1-x_8)^{\epsilon}x_8^{N-\frac{\epsilon}{2}}$$
$$\times\frac{1}{-x_6-x_8+x_6x_8}\left(1+x_4\frac{x_8}{x_6(1-x_8)}\right)$$

# Diagram 4

others an integral in $x_6$,

$$C(\cdots)B\left(2+\frac{\epsilon}{2},-\frac{3\epsilon}{2}\right)B\left(2+\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)^2\frac{\Gamma\left(2-\epsilon\right)}{\Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(2+\frac{\epsilon}{2}\right)^3\Gamma\left(-\frac{\epsilon}{2}\right)}$$
$$\times\, x_9^{\epsilon-1}(1-x_4)^{-\epsilon}x_4^{N-1-\epsilon}(1-x_6)^{\frac{\epsilon}{2}-1}x_6^{\frac{\epsilon}{2}}(1-x_8)^{\epsilon-2}x_8^{N-\frac{\epsilon}{2}}$$
$$\times\,(1-x_8+x_4x_8)\left(1+x_6\frac{1-x_8}{x_4x_8}\right)$$

The diagram has the features of diagram 3 ‘twice’.

# Diagram 4

The result is much too big to show here; the $\epsilon^{-3}$ coefficient is,

$$\frac{4\left((-1)^N+1\right)\left(3N^2+17N+24\right)}{3(N+1)(N+2)^3}$$
$$+\frac{4\left((-1)^N+1\right)\left(N^3-10N^2-19N+8\right)}{3N(N+1)^2(N+2)^2}S_1(N)$$
$$-\frac{2\left((-1)^N+1\right)(N-8)}{3N(N+1)(N+2)}[S_1(N)]^2$$
$$-\frac{2\left((-1)^N+1\right)(7N+8)}{3N(N+1)(N+2)}S_2(N)$$
$$-\frac{16\left((-1)^N+1\right)}{3(N+1)(N+2)}S_{-2}(N)$$

# Diagram 4

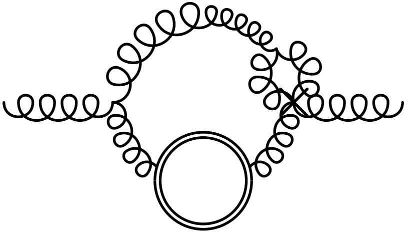
In the constant term, $\epsilon^0$, there are harmonic numbers of weight 5,

$$S_5(N) \quad S_{-2,1,1,1}(N) \quad S_{2,1,1,1}(N)$$

and both $\zeta_2$ and $\zeta_3$. In intermediate steps some of the sums have $\zeta_5$ contributions.

This is the probably most difficult diagram in the class and required minor improvements to the summation algorithm.

# Diagram 5



This is nothing more than diagram 3 with different flavour assignments.
There are 10,606 terms in the diagram making it also the largest 2-bubble diagram in the class.
Here the approach was unique. The Feynman rules for the operator insertion vertex give a double sum.
However in this diagram one gets denominators that give infinite sums,

$$C(\cdots)f(x_4, x_6, x_8, x_9)\sum_{j_1=0}^{N-4}\sum_{j_2=0}^{j_1} x_6^{j_2}(1-x_8)^{j2}(x_6+x_8-x_6x_8)^{j_1-j_2}$$

# Diagram 5

Recognising the sums as geometric series one finds amongst the many terms,

$$C(\cdots)f(x_4, x_6, x_8, x_9)\frac{x_6x_8(-x_6x_8 + x_6 + x_8)^{N-2}}{(1-x_6)(1-x_8)^2(x_6 - x_6x_8 - 1)}$$

after applying a partial fraction decomposition one would obtain an infinite and a finite sum. Note the growth of terms,

$$\text{FR Sums } 1 \to 23$$
$$\text{Decomposition } 1 \to 4$$

# Diagram 5

In practice it turned out that the methodology was so indirect that the summation representation 'missed the structure' of the problem and was tough to solve.
Not performing the 'Feynman rule' sums and binomally expanding,

$$C(\cdots)f(x_4,x_6,x_8,x_9)\sum_{j_1=0}^{N-4}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_1-j_2}\binom{j_1-j_2}{j_3}x_6^{j_2}(1-x_8)^{j2}(x_6(1-x_8))^{j_3}x_8^{j_1-j_2-j_3}$$

recognise all integrals as $B$-functions to get a triple-sum representation.

This happened to be a much easier sum representation that could be solved. Of all the 2-bubble diagrams this was the only such example.

# Summary

1. The 2-bubble class of diagrams contributing to the 3-loop heavy Wilson coefficients was introduced.
2. A special function approach was shown to give a sum representation for nearly all the diagrams which could be solved by computer algebra.
3. The more difficult 'merged bubble' diagrams could be solved with mild extensions of the algorithms.
4. All 2-bubble diagrams, about 13% of the heavy Wilson coefficients, were computed.

Special acknowledgement to Thorsten Ohl for help with FeynMP.