# Perturbative Contributions to Rare $B$-meson Decays


**Mikołaj Misiak**

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1. $B_{s(d)} \to \ell^+\ell^-$
   - **a.** Sensitivity to BSM physics
   - **b.** Electroweak-scale matching at 3 loops in QCD and 2 loops in EW interactions
   - **c.** Updated SM prediction

2. $\bar{B} \to X_s\gamma$ at $\mathcal{O}(\alpha_s^2)$
   - **a.** Knowns and unknowns
   - **b.** $Q_{1,2}$-$Q_7$ interference at $m_c = 0$
   - **c.** Expectations for the upcoming SM prediction update

3. Summary

## $B_s \to \mu^+\mu^-$ — the flavour physics highlight of the LHC

- It is a strongly suppressed, loop-generated process in the SM. Its average time-integrated branching ratio (with final-state photon bremsstrahlung included) reads:

$$\overline{\mathcal{B}}_{\rm SM} = (3.65 \pm 0.23) \times 10^{-9}$$

[ C. Bobeth, M. Gorbahn, T. Hermann, MM, E. Stamou and M. Steinhauser, Phys. Rev. Lett. 112 (2014) 101801 ]

- It is very sensitive to new physics even in models with Minimal Flavour Violation (MFV). Enhancements by orders of magnitude are possible even when constraints from all the other measurements are taken into account.

- Recently measured branching ratios

$$\overline{\mathcal{B}}_{\rm exp} = \begin{cases} \left(2.9\,^{+1.1}_{-1.0}\right) \times 10^{-9}, & \text{LHCb} \quad \text{[Phys. Rev. Lett. 111 (2013) 101805]} \\ \left(3.0\,^{+1.0}_{-0.9}\right) \times 10^{-9}, & \text{CMS} \quad \text{[Phys. Rev. Lett. 111 (2013) 101804]} \end{cases}$$

**Combined:** $\overline{\mathcal{B}}_{\rm exp} = \left(2.8\,^{+0.7}_{-0.6}\right) \times 10^{-9}$

[ F. Archilli, talk at CKM2014, September 10th, 2014 ]

- ATLAS: $\overline{\mathcal{B}}_{\rm exp} < 1.5 \times 10^{-8}$ @ 95%C.L.

## $B$-meson or Kaon decays occur at low energies, at scales $\mu \ll M_W$.

**We pass from the full theory of electroweak interactions to an effective theory by removing the high-energy degrees of freedom, i.e. integrating out the $W$-boson and all the other particles with $m \sim M_W$.**

$$\mathcal{L}_{\text{(full EW}\times\text{QCD)}} \longrightarrow \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED}\times\text{QCD}}\left(\begin{array}{l}\text{quarks} \neq t\\ \text{\& leptons}\end{array}\right) + N \sum_n C_n(\mu)\, Q_n$$

**$Q_n$ – local interaction terms (operators), $C_n$ – coupling constants (Wilson coefficients)**

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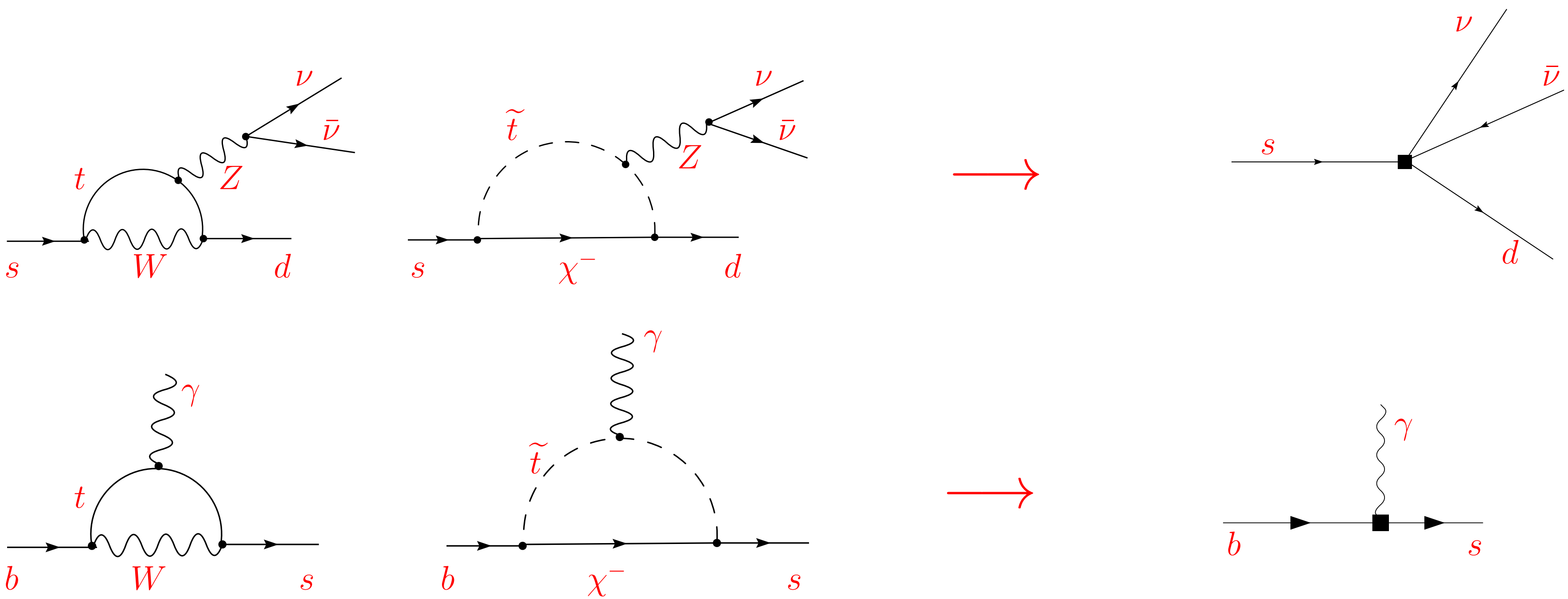
**$Q_n$ – local interaction terms (operators), $\quad$ $C_n$ – coupling constants (Wilson coefficients)**

**Information on the electroweak-scale physics is encoded in the values of $C_i(\mu)$ , e.g.,**




**This is a modern version of the Fermi theory for weak interactions. It is "nonrenormalizable" in the traditional sense but actually renormalizable. It is also predictive because all the $C_i$ are calculable, and only a finite number of them is necessary at each given order in the (external momenta)/$M_W$ expansion.**

**Advantages:** **Resummation of** $\left(\alpha_s \ln \frac{M_W^2}{\mu^2}\right)^n$ **using renormalization group, easier account for symmetries.**

**Operators (dim 6) that matter for $B_s \to \mu^+\mu^-$ read**

$$Q_A = (\bar{b}\gamma^\alpha\gamma_5 s)\,(\bar{\mu}\gamma_\alpha\gamma_5\mu)$$ – **the only relevant one in the SM**

$$Q_{S(P)} = (\bar{b}\gamma_5 s)\,(\bar{\mu}(\gamma_5)\mu) \;=\; \frac{i(\bar{b}\gamma^\alpha\gamma_5 s)\partial_\alpha(\bar{\mu}(\gamma_5)\mu)}{m_b+m_s} + \underbrace{\boxed{E}}_{\text{vanishing by EOM}} + \underbrace{\boxed{T}}_{\text{total derivative}}$$

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$E$: vanishing by EOM

$T$: total derivative

Necessary non-perturbative input: $$\langle 0|\bar{b}\gamma^\alpha\gamma_5 s|B_s(p)\rangle \;=\; ip^\alpha f_{B_s}$$

Recent lattice determinations of the $B_s$-meson decay constant:

$$f_{B_s} = \begin{cases} 225.0(4.0)\text{ MeV}, & \text{HPQCD (r)}, \quad \text{arXiv:1110.4510} \\ 224.0(5.0)\text{ MeV}, & \text{HPQCD (nr)}, \quad \text{arXiv:1302.2644} \\ 234.0(6.0)\text{ MeV}, & \text{ROME}, \quad \text{arXiv:1212.0301} \\ 242.0(9.5)\text{ MeV}, & \text{FNAL/MILC}, \quad \text{arXiv:1112.3051} \\ 232.0(10)\text{ MeV}, & \text{ETM}, \quad \text{arXiv:1107.1441} \\ 219.0(12)\text{ MeV}, & \text{ALPHA}, \quad \text{arXiv:1210.6524} \\ 235.4(12)\text{ MeV}, & \text{RBC/UKQCD}, \quad \text{arXiv:1404.4670} \\ 224.0(14)\text{ MeV}, & \text{ALPHA}, \quad \text{arXiv:1404.3590} \end{cases}$$

Flavour Lattice Averaging Group (FLAG), arXiv:1310.8555 gives

$$f_{B_s} = 227.7(4.5)\text{ MeV}.$$

**Average time-integrated branching ratio:**

$$\overline{\mathcal{B}}(B_s \to \mu^+\mu^-) = \frac{|N|^2 M_{B_s}^3 f_{B_s}^2}{8\pi\Gamma_H^s}\beta\left(\left|rC_A - uC_P\right|^2 F_P + \left|u\beta C_S\right|^2 F_S\right) + \mathcal{O}(\alpha_{em}),$$

where $N = \frac{V_{tb}^* V_{ts} G_F^2 M_W^2}{\pi^2}$, $r = \frac{2m_\mu}{M_{B_s}}$, $\beta = \sqrt{1-r^2}$, $u = \frac{M_{B_s}}{m_b+m_s}$ ,

$$F_P = 1 - \frac{\Delta\Gamma^s}{\Gamma_L^s}\sin^2\left[\tfrac{1}{2}\phi_s^{\mathrm{NP}} + \arg(rC_A - uC_P)\right] \xrightarrow{\text{SM CP}} 1 ,$$

$$F_S = 1 - \frac{\Delta\Gamma^s}{\Gamma_L^s}\cos^2\left[\tfrac{1}{2}\phi_s^{\mathrm{NP}} + \arg C_S\right] \xrightarrow{\text{SM CP}} \frac{\Gamma_H^s}{\Gamma_L^s}$$

derived following [ K. de Bruyn *et al.*, Phys. Rev. Lett. 109 (2012) 041801]

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In the limit of no CP-violation, mass eigenstates are CP eigenstates:

Heavier, CP-odd: $B_s^H = \frac{1}{\sqrt{2}}(B_s + \bar{B}_s)$, annihilated by $\bar{b}\gamma_5 s + \bar{s}\gamma_5 b$, $\quad (\tau_H = 1.615(21)\text{ ps})$

Lighter, CP-even: $B_s^L = \frac{1}{\sqrt{2}}(B_s - \bar{B}_s)$, annihilated by $\bar{b}\gamma_5 s - \bar{s}\gamma_5 b$, $\quad (\tau_L = 1.516(11)\text{ ps})$

Our interactions in this limit are all CP-even:

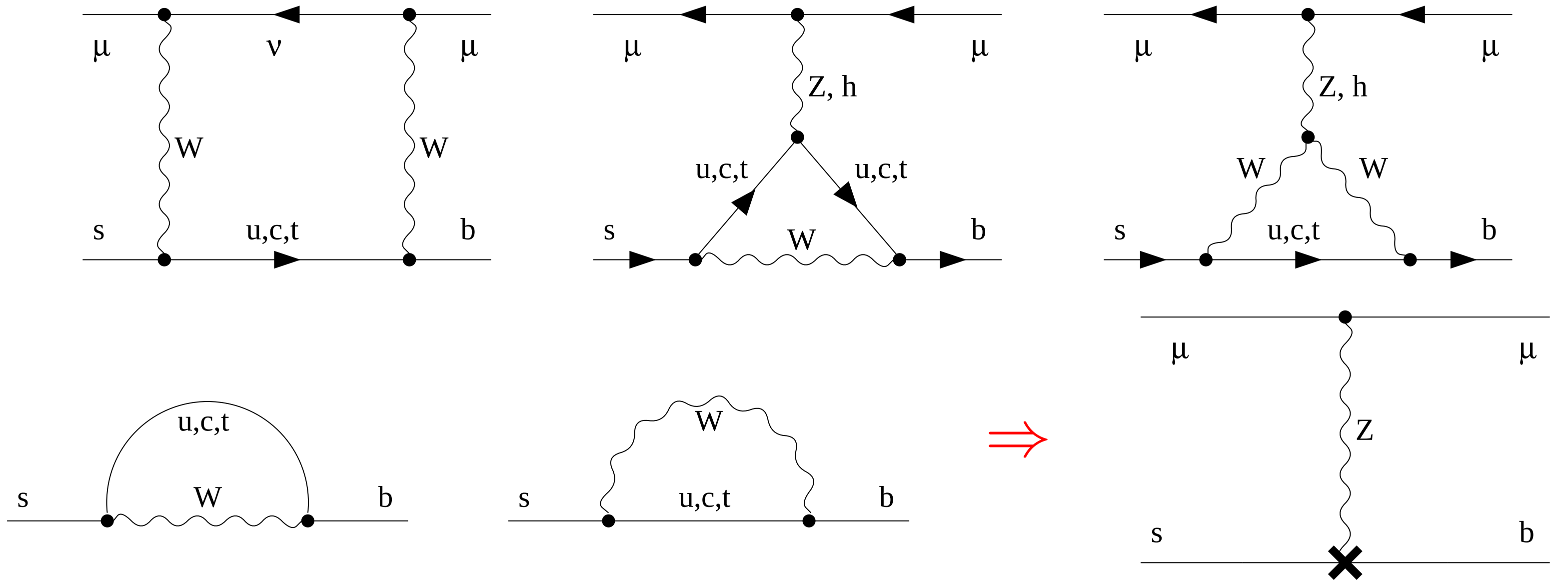
$$\left.\begin{aligned} Q_A + Q_A^\dagger &= \left[(\bar{b}\gamma^\alpha\gamma_5 s) + (\bar{s}\gamma^\alpha\gamma_5 b)\right](\bar{\mu}\gamma_\alpha\gamma_5\mu) \\ Q_P + Q_P^\dagger &= \left[(\bar{b}\gamma_5 s) + (\bar{s}\gamma_5 b)\right](\bar{\mu}\gamma_5\mu) \end{aligned}\right\} \text{annihilate } B_s^H\text{, produce CP-odd dimuons}$$

$$\left. Q_S + Q_S^\dagger = \left[(\bar{b}\gamma_5 s) - (\bar{s}\gamma_5 b)\right](\bar{\mu}\mu) \right\} \text{annihilates } B_s^L\text{, produces CP-even dimuons}$$

With SM-like CP-violation – still $Q_{A,P}$ annihilate $B_s^H$ and $Q_S$ annihilates $B_s^L$.

Beyond SM – interesting time-dependent observables, see arXiv:1303.3820, 1407.2771.

# Evaluation of the LO Wilson coefficients in the SM:




$\bar{b}_L \not{Z} s_L$

$$C_A^{(0)} = \frac{1}{2} Y_0\left(m_t^2/M_W^2\right), \qquad Y_0(x) = \frac{3x^2}{8(x-1)^2}\ln x + \frac{x^2-4x}{8(x-1)},$$

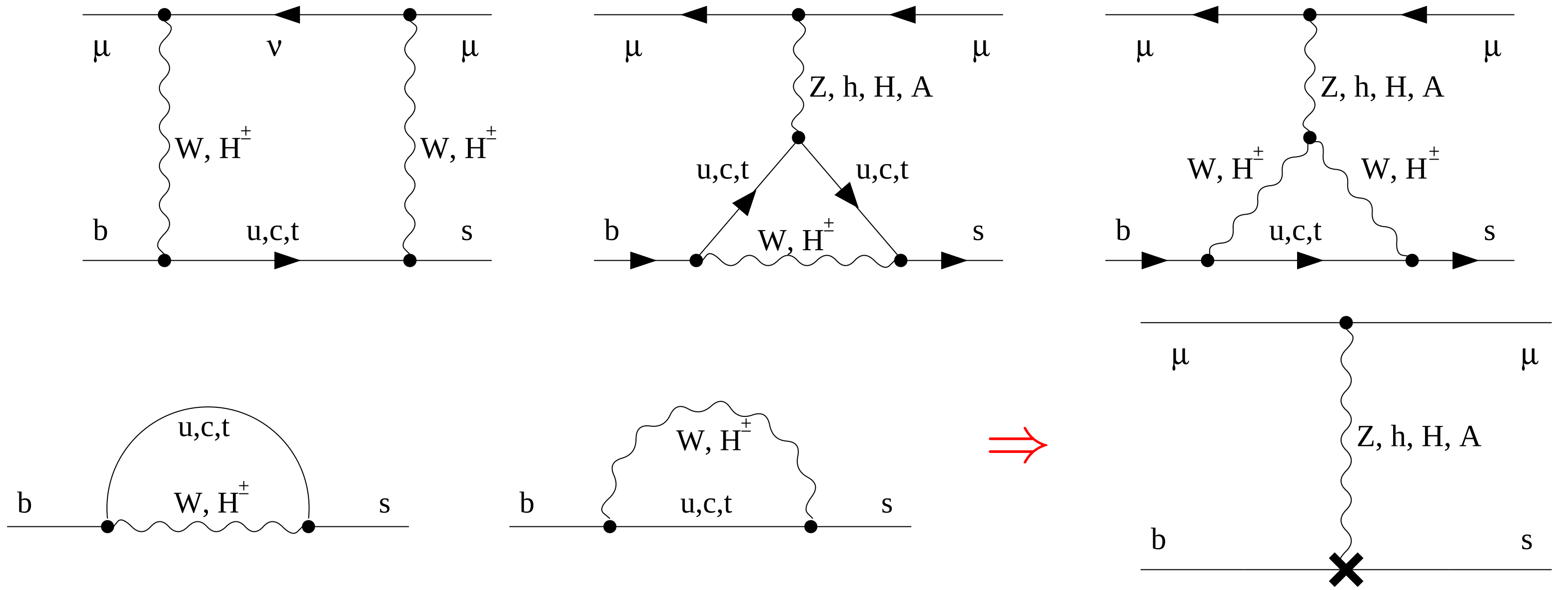
$$C_{S,P} = \mathcal{O}\left(\frac{m_\mu}{M_W}\right).$$

Effects of $C_{S,P}$ on the branching ratio are suppressed by $M_{B_s}^2/M_W^2$ $\Rightarrow$ negligible.

**Thus, only $C_A$ matters in the SM.**

# Evaluation of the Wilson coefficients beyond the SM.

## Example 1: the Two-Higgs-Doublet Model II




$$\tan\beta = v_2/v_1, \qquad z = M_{H^\pm}^2/m_t^2,$$

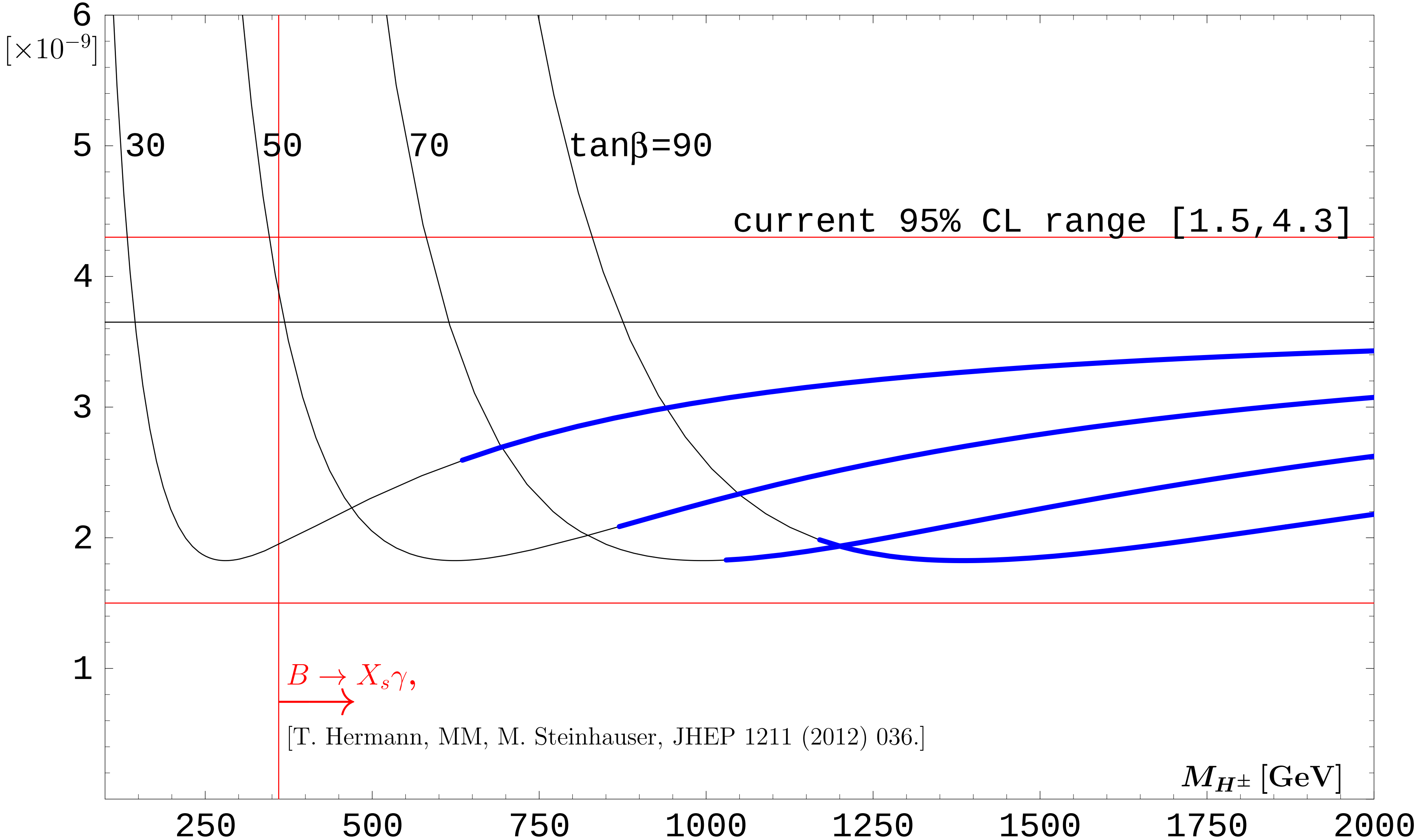
$$C_S \;\simeq\; C_P \;\simeq\; \frac{m_\mu m_b \tan^2\beta}{4M_W^2}\,\frac{\ln z}{z-1} > 0,$$

H.E. Logan and U. Nierste,
NPB 586 (2000) 39
($\mathcal{O}(\tan\beta)$ neglected)

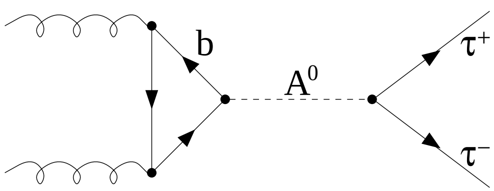
$$\mathcal{B}(B_s \to \mu^+\mu^-) \simeq (\text{const.})\left[\left|\frac{2m_\mu}{M_{B_s}}C_A - C_P\right|^2 + |C_S|^2\right]$$

$$\underset{\text{}}{C_A} = \underbrace{C_A^{\text{SM}}}_{\text{positive}} + \underbrace{\Delta C_A}_{\text{small}} \qquad \Rightarrow \left\{\begin{array}{c}\text{suppression for moderate } C_{S,P}\\ \text{enhancement for huge } \tan\beta \text{ only}\end{array}\right.$$

# $\mathcal{B}(B_s \to \mu^+\mu^-)$ in the Two-Higgs-Doublet Model II

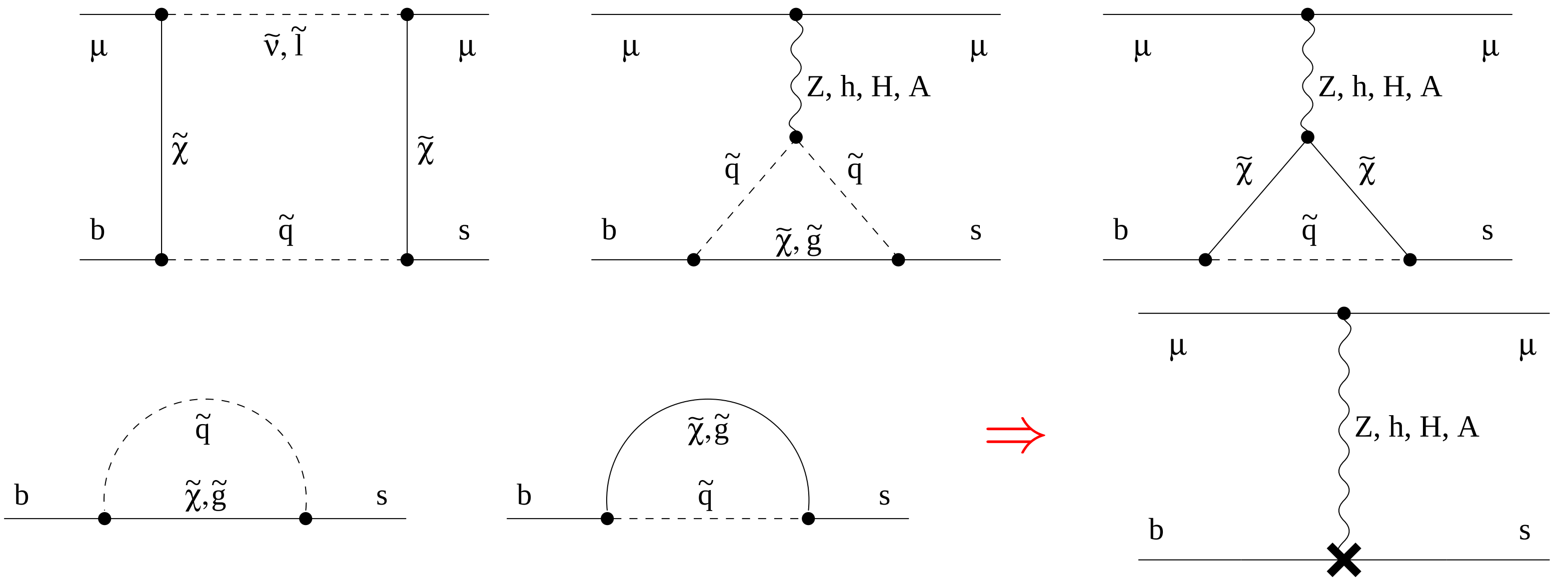



Blue lines — still allowed for $M_{H^\pm} = \sqrt{M_A^2 + M_W^2}$ after taking into account the LHC searches for $pp \to A^0 \to \tau^+\tau^-$ [CMS arXiv:1408.3316, ATLAS arXiv:1408.3316].



# Evaluation of the Wilson coefficients beyond the SM.

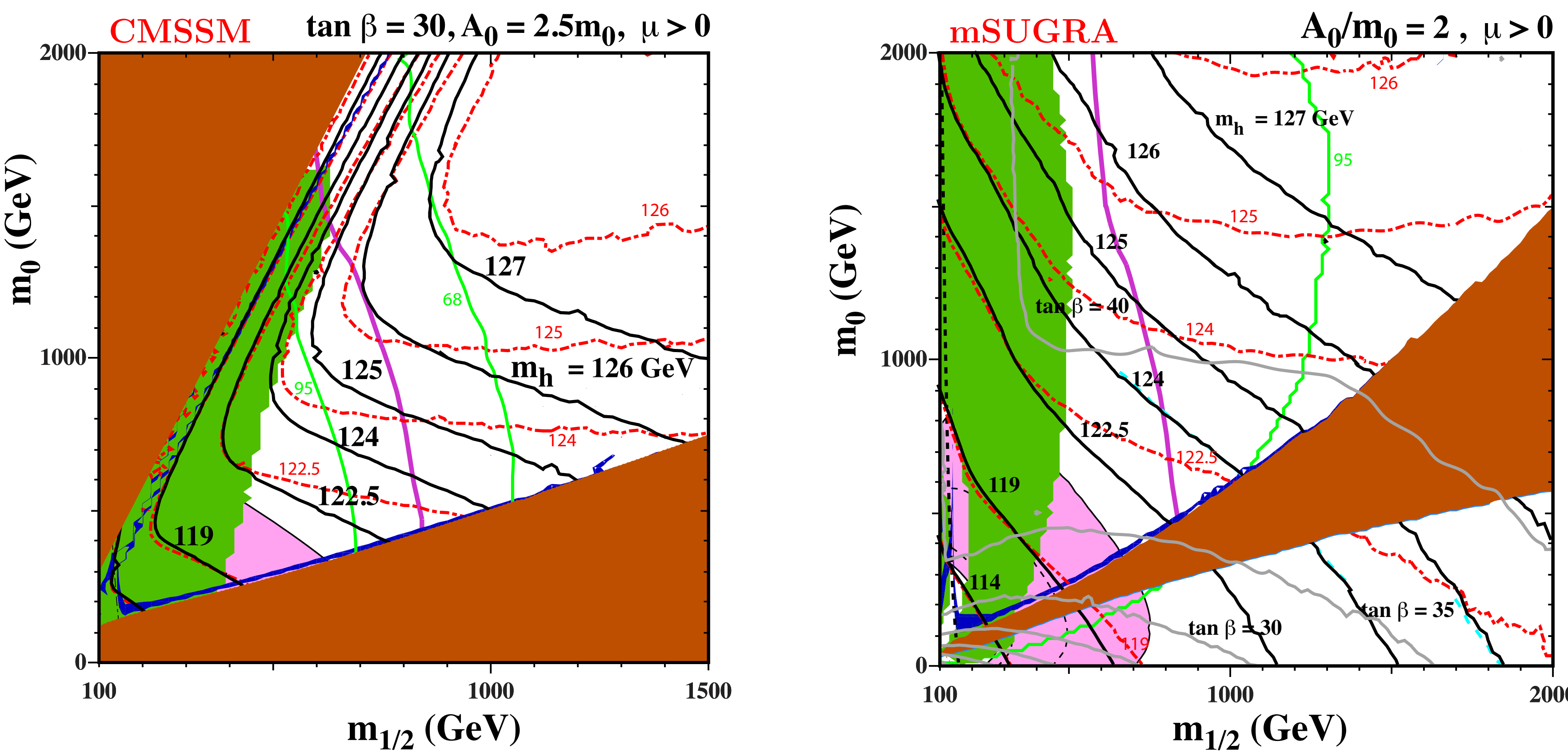
## Example 2: the MSSM.




**For large $\tan\beta$:** $$\mathcal{B}\left(B_s \to \mu^+\mu^-\right) \sim \frac{m_b^2 m_\mu^2}{M_A^4} \tan^6\beta$$

K. S. Babu and C. F. Kolda, Phys. Rev. Lett. 84 (2000) 228.

# Examples of constraints on the MSSM parameter space:



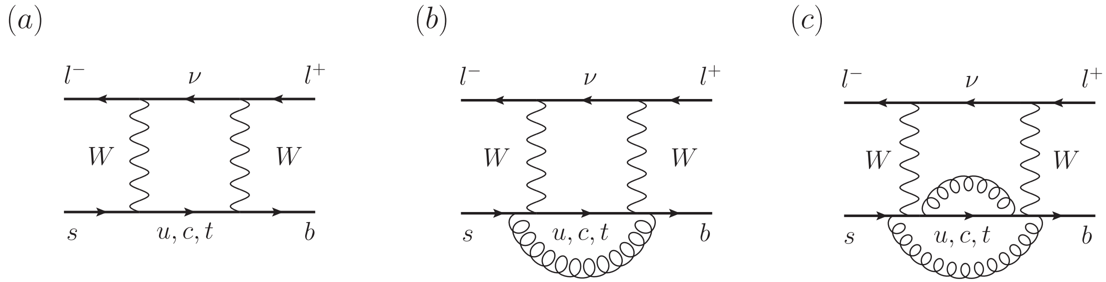

Figs. 1 and 7 from arXiv:1312.5426 by John Ellis.

| | |
|---|---|
| green lines | – bounds from $B_s \to \mu^+\mu-$ (CMS & LHCb 2013, exclusion to the left) |
| purple lines | – ATLAS 95%CL bounds from $\not{E}_T+$ jets |
| green shaded | – excluded by $b \to s\gamma$ |
| brown shaded | – charged LSP |
| pink shaded | – SUSY helps with $g-2$ |
| blue strips | – favoured by $\Omega_{\mathrm{DM}}$ |

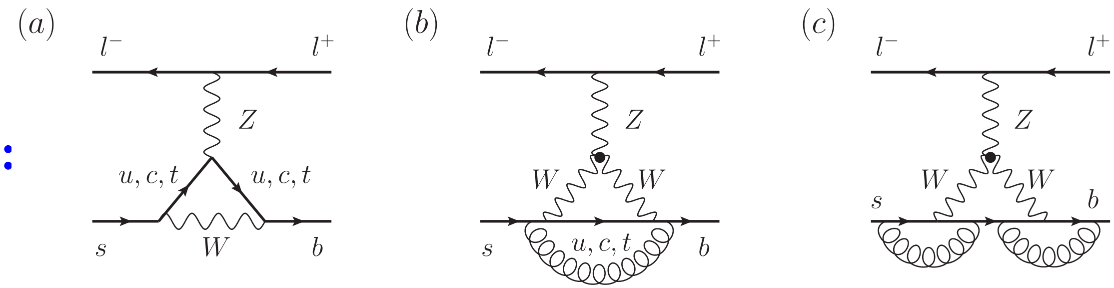
# Evaluation of the NNLO QCD matching corrections in the SM

[T. Hermann, MM, M. Steinhauser, JHEP 1312 (2013) 097]

**$W$-boxes:**
(1LPI)



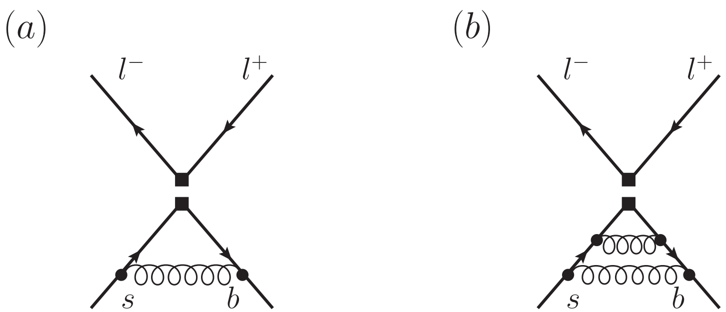

**$Z$-penguins:**
(1LPI)



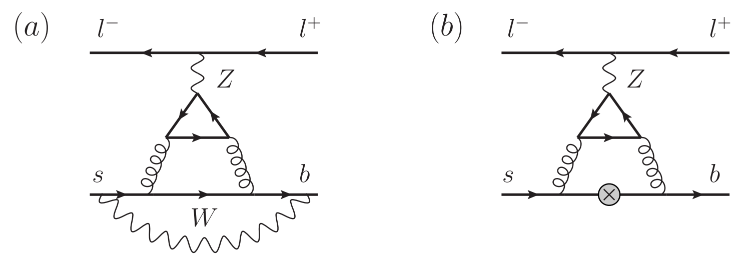

**Subtleties:** (i) counterterms with finite parts $\sim \bar{b}_L \not{D} s_L$

(ii) evanescent operators: $E_B = (\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 s)(\bar{\mu}\gamma^\sigma\gamma^\rho\gamma^\nu\gamma_5\mu) - 4(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$

$E_T = \mathrm{Tr}\,(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5)(\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma s)(\bar{\mu}\gamma_\alpha\gamma_5\mu) + 24(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$




Renormalization of $E_B$

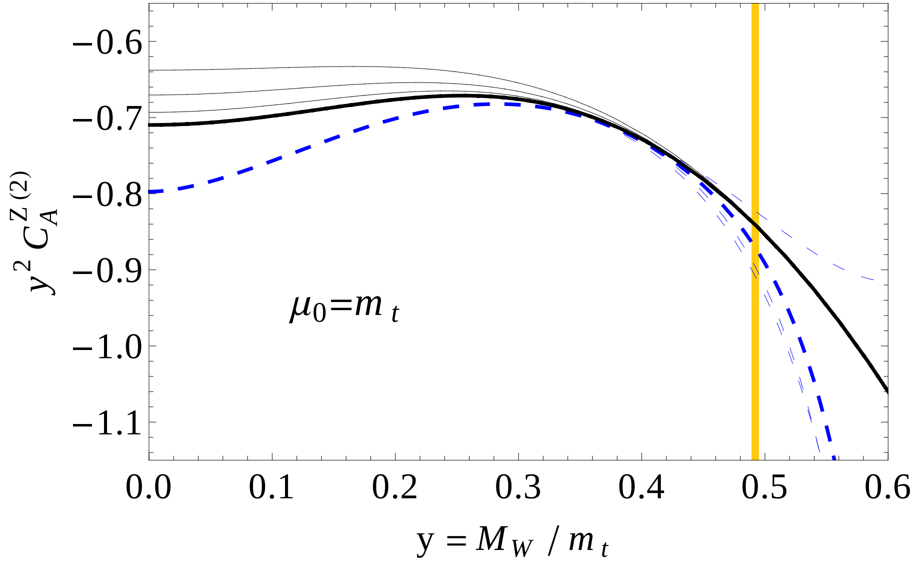



Diagrams generating $E_T$

# Perturbative series for the Wilson coefficient at $\mu = \mu_0 \sim m_t, M_W$:

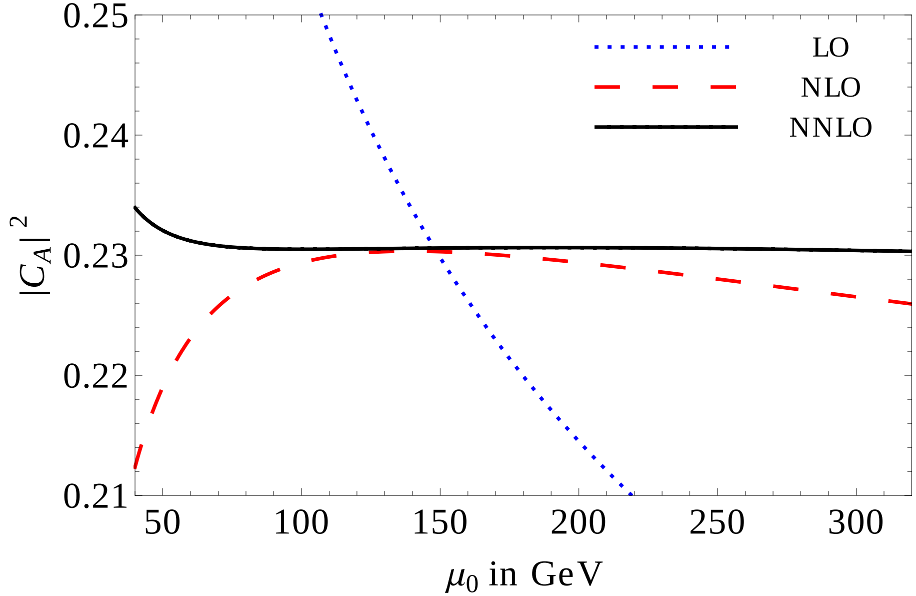
$$C_A(\mu_0) = C_A^{(0)}(\mu_0) + \frac{\alpha_s}{4\pi} C_A^{(1)}(\mu_0) + \left(\frac{\alpha_s}{4\pi}\right)^2 C_A^{(2)}(\mu_0) + \frac{\alpha_{em}}{4\pi} \Delta_{\rm EW} C_A(\mu_0) + \ldots$$

The top quark mass is $\overline{\rm MS}$-renormalized at $\mu_0$ with respect to QCD, and on shell with respect to the EW interactions. Both $\alpha_s$ and $\alpha_{em}$ are $\overline{\rm MS}$-renormalized at $\mu_0$ in the effective theory.




$$C_A^{(n)} = C_A^{W,(n)} + C_A^{Z,(n)}$$

To deal with single-scale tadpole integrals, we expand around $y = 1$ (solid lines) and around $y = 0$ (dashed lines), where $y = M_W/m_t$. The expansions reach $(1 - y^2)^{16}$ and $y^{12}$, respectively. The blue band indicates the physical region.




Matching scale dependence of $|C_A|^2$ gets significantly reduced. The plot corresponds to $\Delta_{\rm EW} C_A(\mu_0) = 0$. However, with our conventions for $m_t$ and the global normalization, $\mu_0$-dependence is due to QCD only.

## NNLO fit (with $\Delta_{\rm EW} C_A(\mu_0) = 0$):

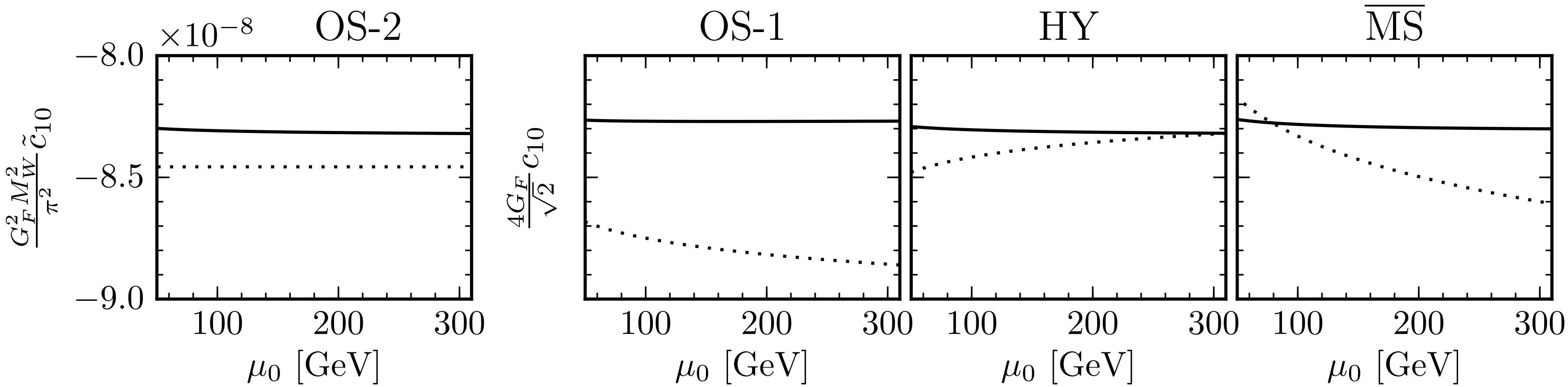
$$C_A = 0.4802 \left(\frac{M_t}{173.1}\right)^{1.52} \left(\frac{\alpha_s(M_Z)}{0.1184}\right)^{-0.09} + \mathcal{O}(\alpha_{em})$$

# Evaluation of the NLO EW matching corrections in the SM

[C. Bobeth, M. Gorbahn, E. Stamou, Phys. Rev. D 89 (2014) 034023]

**Method: similar to the NNLO QCD case. Two-loop integrals with three mass scales are present.**

**Dependence of the final result on $\mu_0$ in various renormalization schemes (dotted – LO, solid – NLO):**




**In all the four plots: no QCD corrections to $C_A$ included, $m_t(m_t)$ w.r.t. QCD used.**

**OS-2 scheme:** **Global normalization factor in $\mathcal{L}_{\rm eff}$ set to $N = V_{tb}^* V_{ts}\, G_F^2 M_W^2/\pi^2$**

**Masses at the LO renormalized on-shell w.r.t. EW interactions (including $M_W$ in $N$)**

**Plotted quantity: $-2C_A\, G_F^2 M_W^2/\pi^2$ in GeV$^{-2}$**

**NLO EW matching correction to the BR: $-3.7\%$**

**other schemes:** **Global normalization factor in $\mathcal{L}_{\rm eff}$ set to $4V_{tb}^* V_{ts}\, G_F/\sqrt{2}$**

**At the LO, $\alpha_{em}(\mu_0)$ used**

**$\overline{\mathrm{MS}}$: Masses and $\sin^2\theta_W$ renormalized at $\mu_0$**

**OS-1: Masses as in OS-2, $\sin^2\theta_W$ on-shell**

**HY (hybrid): Masses as in OS-2, $\sin^2\theta_W$ as in $\overline{\mathrm{MS}}$.**

## SM predictions for all the branching ratios $\overline{\mathcal{B}}_{q\ell} \equiv \overline{\mathcal{B}}(B_q \to \ell^+\ell^-)$

[ C. Bobeth, M. Gorbahn, T. Hermann, MM, E. Stamou, M. Steinhauser, PRL 112 (2014) 101801]

$$\begin{aligned}
\overline{\mathcal{B}}_{se} \times 10^{14} &= (8.54 \pm 0.13)\, R_{t\alpha}\, R_s = 8.54 \pm 0.55, \\
\overline{\mathcal{B}}_{s\mu} \times 10^{9} &= (3.65 \pm 0.06)\, R_{t\alpha}\, R_s = 3.65 \pm 0.23, && (\text{LHCb \& CMS}:\ 2.8^{+0.7}_{-0.6}) \\
\overline{\mathcal{B}}_{s\tau} \times 10^{7} &= (7.73 \pm 0.12)\, R_{t\alpha}\, R_s = 7.73 \pm 0.49, \\
\overline{\mathcal{B}}_{de} \times 10^{15} &= (2.48 \pm 0.04)\, R_{t\alpha}\, R_d = 2.48 \pm 0.21, \\
\overline{\mathcal{B}}_{d\mu} \times 10^{10} &= (1.06 \pm 0.02)\, R_{t\alpha}\, R_d = 1.06 \pm 0.09, && (\text{LHCb \& CMS}:\ 3.9^{+1.6}_{-1.4}) \\
\overline{\mathcal{B}}_{d\tau} \times 10^{8} &= (2.22 \pm 0.04)\, R_{t\alpha}\, R_d = 2.22 \pm 0.19,
\end{aligned}$$

where

$$R_{t\alpha} = \left(\frac{M_t}{173.1\ \text{GeV}}\right)^{3.06} \left(\frac{\alpha_s(M_Z)}{0.1184}\right)^{-0.18},$$

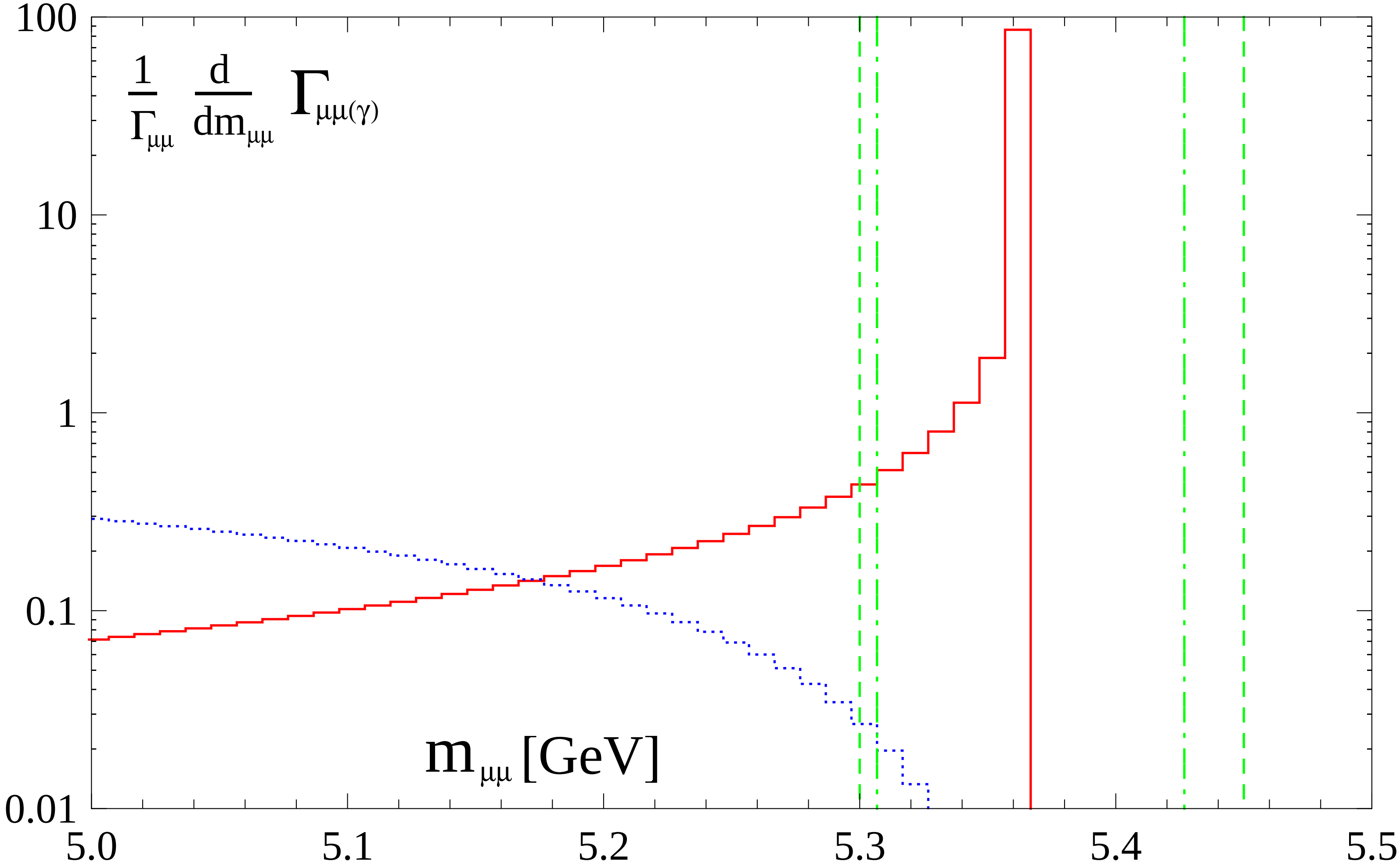
$$R_s = \left(\frac{f_{B_s}[\text{MeV}]}{227.7}\right)^2 \left(\frac{|V_{cb}|}{0.0424}\right)^2 \left(\frac{|V_{tb}^\star V_{ts}/V_{cb}|}{0.980}\right)^2 \frac{\tau_H^s\,[\text{ps}]}{1.615},$$

$$R_d = \left(\frac{f_{B_d}[\text{MeV}]}{190.5}\right)^2 \left(\frac{|V_{tb}^\star V_{td}|}{0.0088}\right)^2 \frac{\tau_d^{\text{av}}\,[\text{ps}]}{1.519}.$$

| Sources of uncertainties | $f_{B_q}$ | CKM | $\tau_H^q$ | $M_t$ | $\alpha_s$ | other parametric | non-parametric | $\sum$ | |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{\mathcal{B}}_{s\ell}$ | 4.0% | 4.3% | 1.3% | 1.6% | 0.1% | $< 0.1\%$ | 1.5% | 6.4% | $\longrightarrow$ 4.7% (?) |
| $\overline{\mathcal{B}}_{d\ell}$ | 4.5% | 6.9% | 0.5% | 1.6% | 0.1% | $< 0.1\%$ | 1.5% | 8.5% | |

In the case of $\overline{\mathcal{B}}_{s\ell}$, the main uncertainty (4.2%) originates from $|V_{cb}| = 0.0424(9)$ that comes from a recent fit to the inclusive semileptonic data [P. Gambino and C. Schwanda, arXiv:1307.4551 ].

# Radiative tail in the dimuon invariant mass spectrum




Ggreen vertical lines – experimental windows (→ MC)

Red line – no real photon and/or radiation only from the muons. It vanishes when $m_\mu \to 0$.

Blue line – remainder due to radiation from the quarks. IR-safe because $B_s$ is neutral.
Phase-space suppressed but survives in the $m_\mu \to 0$ limit.

Interference between the two contributions is negligible – suppressed both by phase-space and $m_\mu^2/M_{B_s}^2$.

# Inclusive weak radiative $B$-meson decay

SM estimate [hep-ph/0609232]:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm SM}_{E_\gamma > 1.6\ {\rm GeV}} = (3.15 \pm 0.23) \times 10^{-4}$$

Contributions to the total TH uncertainty (summed in quadrature):

5% non-perturbative, 3% from the interpolation in $m_c$

3% higher order $\mathcal{O}(\alpha_s^3)$, 3% parametric

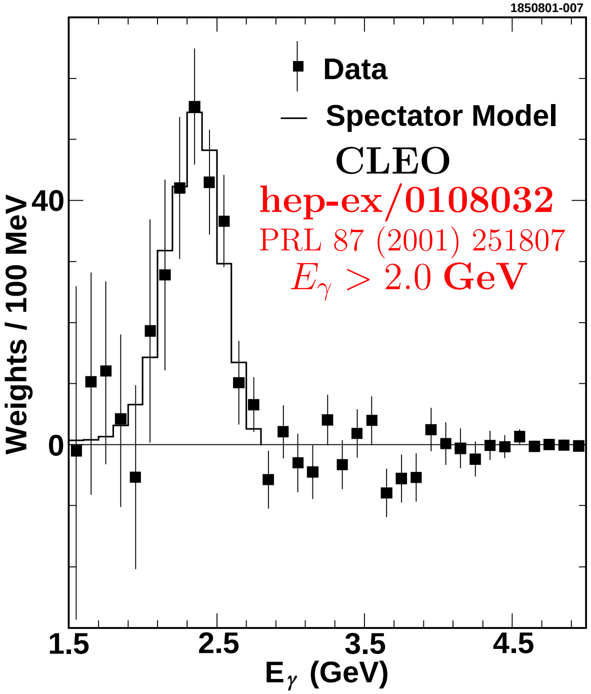
Experimental world average (HFAG, 2.08.2012):

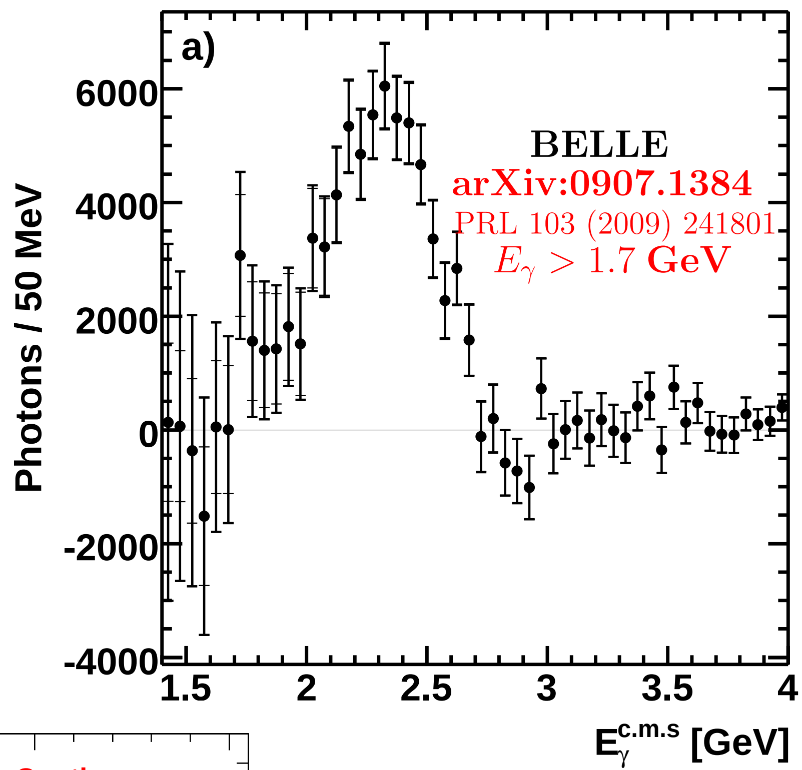
$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm EXP}_{E_\gamma > 1.6\ {\rm GeV}} = (3.43 \pm 0.21 \pm 0.07) \times 10^{-4}$$

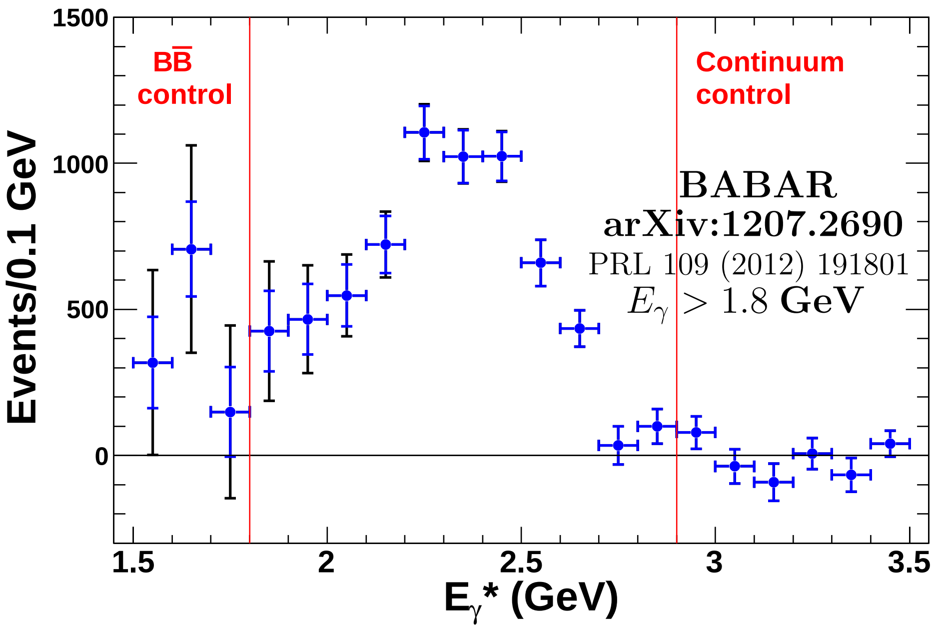
Experiment agrees with the SM at better than $\sim 1\sigma$ level.

Uncertainties: TH $\sim 7\%$, EXP $\sim 6.5\%$.

# The “raw” photon energy spectra in the inclusive measurements




The peaks are centred around

$$\frac{1}{2} m_b \simeq 2.35 \text{ GeV}$$

which corresponds to a two-body $b \to s\gamma$ decay.

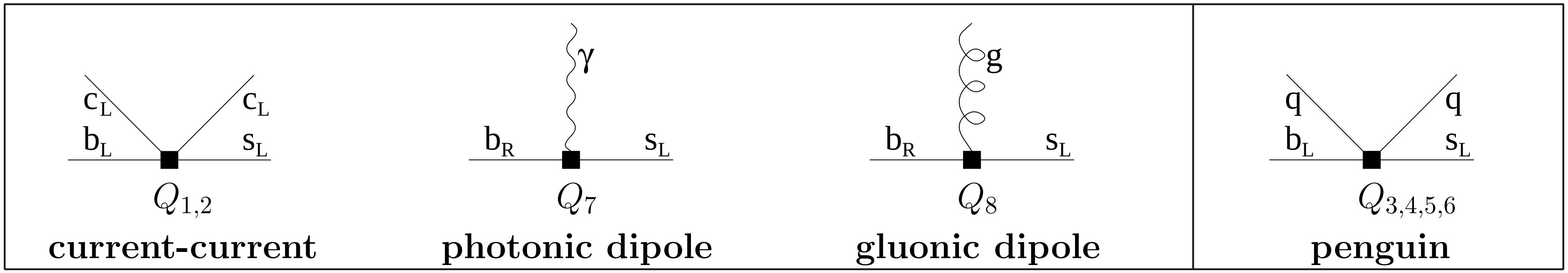
Broadening is due to (mainly):

- perturbative gluon bremsstrahlung,
- motion of the $b$ quark inside the $\bar{B}$ meson,
- motion of the $\bar{B}$ meson in the $\Upsilon(4S)$ frame.

## Decoupling of $W$, $Z$, $t$, $H^0$ $\Rightarrow$ effective weak interaction Lagrangian:

$$L_{\text{weak}} \sim \sum C_i(\mu_b)\, Q_i$$

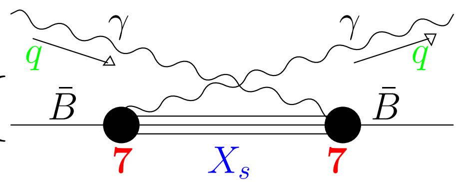
8 operators matter in the SM when the higher-order EW and/or CKM-suppressed effects are neglected:



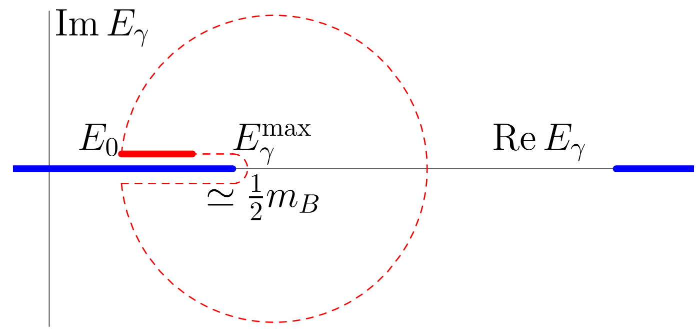

$$\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = |C_7|^2\, \Gamma_{77}(E_0) + (\text{other})$$

Optical theorem:

$$\frac{d\Gamma_{77}}{dE_\gamma} \sim \text{Im}\{\ \cdots\ \} \equiv \text{Im}\, A$$




Integrating the amplitude $A$ over $E_\gamma$:




OPE on the ring $\Rightarrow$ Non-perturbative corrections to $\Gamma_{77}(E_0)$ form a series in $\frac{\Lambda_{\text{QCD}}}{m_b}$ and $\alpha_s$ that begins with

$$\frac{\mu_\pi^2}{m_b^2},\ \frac{\mu_G^2}{m_b^2},\ \frac{\rho_D^3}{m_b^3},\ \frac{\rho_{LS}^3}{m_b^3}, \ldots;\ \frac{\alpha_s\mu_\pi^2}{(m_b-2E_0)^2},\ \frac{\alpha_s\mu_G^2}{m_b(m_b-2E_0)}; \ldots,$$

where $\mu_\pi, \mu_G, \rho_D, \rho_{LS} = \mathcal{O}(\Lambda_{\text{QCD}})$ are extracted from the semileptonic $\bar{B} \to X_c e\bar{\nu}$ spectra and the $B$–$B^\star$ mass difference.

# NNLO QCD corrections to $\bar{B} \to X_s \gamma$

**The relevant perturbative quantity:**

$$\frac{\Gamma[b \to X_s \gamma]_{E_\gamma > E_0}}{|V_{cb}/V_{ub}|^2 \, \Gamma[b \to X_u e \bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\rm em}}{\pi} \underbrace{\sum_{i,j} C_i \, C_j \, K_{ij}}_{P(E_0)}$$

**Expansions of the Wilson coefficients and $K_{ij}$:**

$$C_i(\mu_b) = C_i^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i^{(1)}(\mu_b) + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 C_i^{(2)}(\mu_b) + \ldots$$

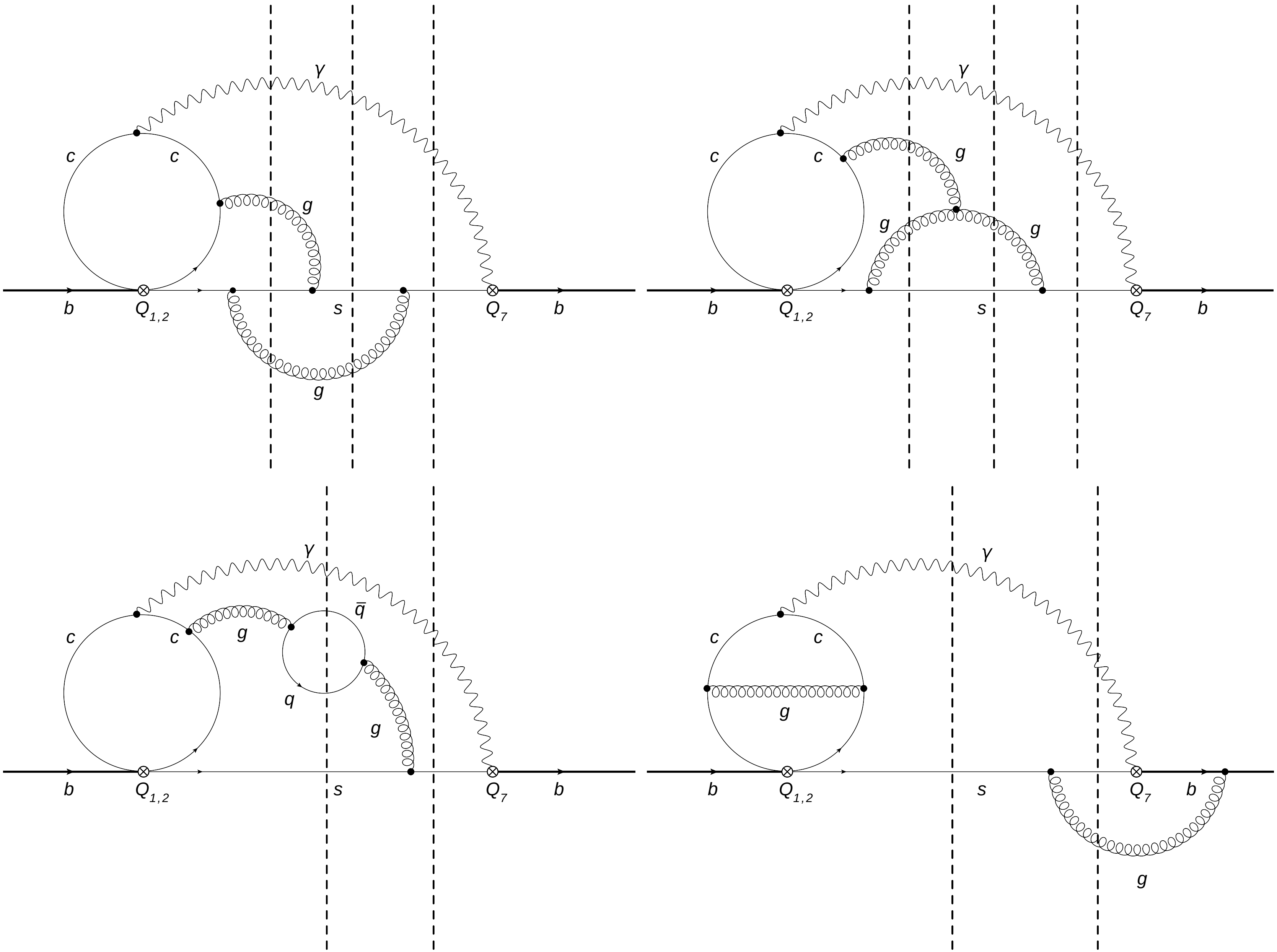
$$K_{ij} = K_{ij}^{(0)} + \frac{\alpha_s(\mu_b)}{4\pi} K_{ij}^{(1)} + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 K_{ij}^{(2)} + \ldots \qquad\qquad \mu_b \sim \frac{m_b}{2}$$

**Most important at the NNLO:** $K_{77}^{(2)}$, $K_{27}^{(2)}$ **and** $K_{17}^{(2)}$.

**They depend on** $\frac{\mu_b}{m_b}$, $\frac{E_0}{m_b}$ **and** $r = \frac{m_c}{m_b}$.

# Evaluation of $K_{27}^{(2)}$ and $K_{17}^{(2)}$ for $m_c = 0$:

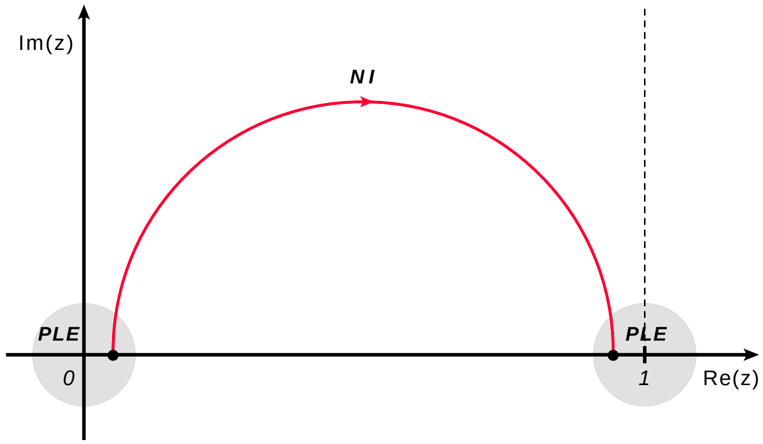
[M. Czakon, P. Fiedler, T. Huber, MM, T. Schutzmeier, M. Steinhauser, to be published]



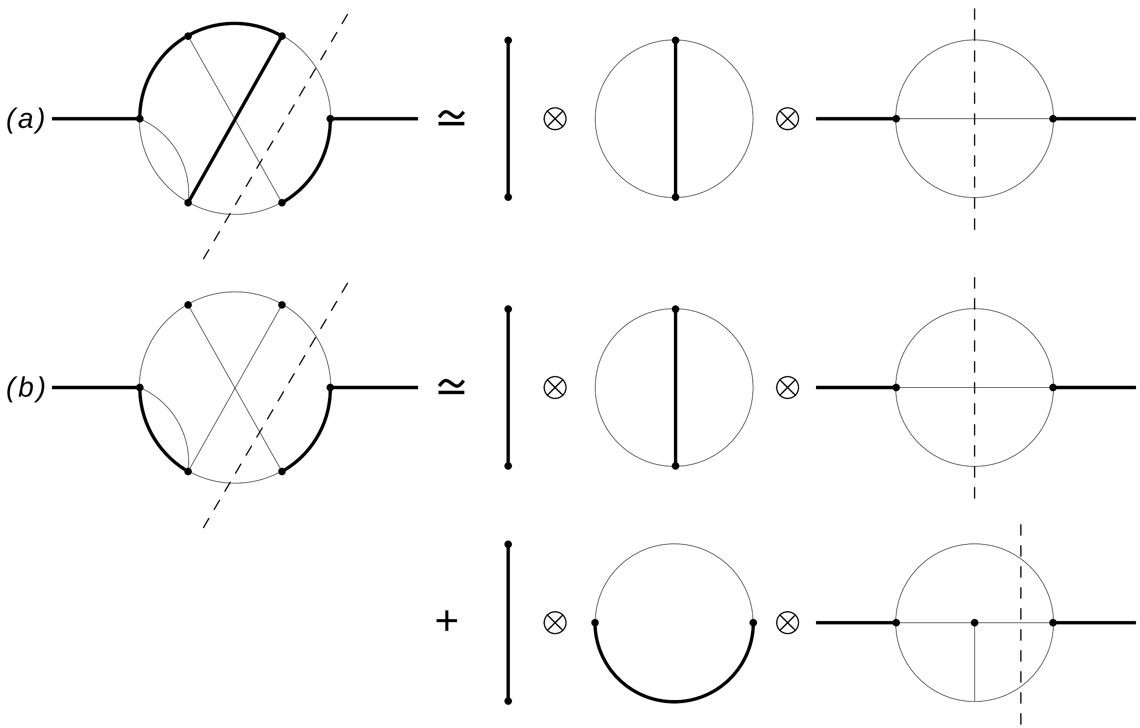
# Master integrals and differential equations:

| | $n_D$ | $n_{OS}$ | $n_{eff}$ | $n_{massless}$ |
|---|---|---|---|---|
| **2-particle cuts** | **292** | **92** | **143** | **9** |
| **3-particle cuts** | **267** | **54** | **110** | **11** |
| **4-particle cuts** | **292** | **17** | **37** | **7** |

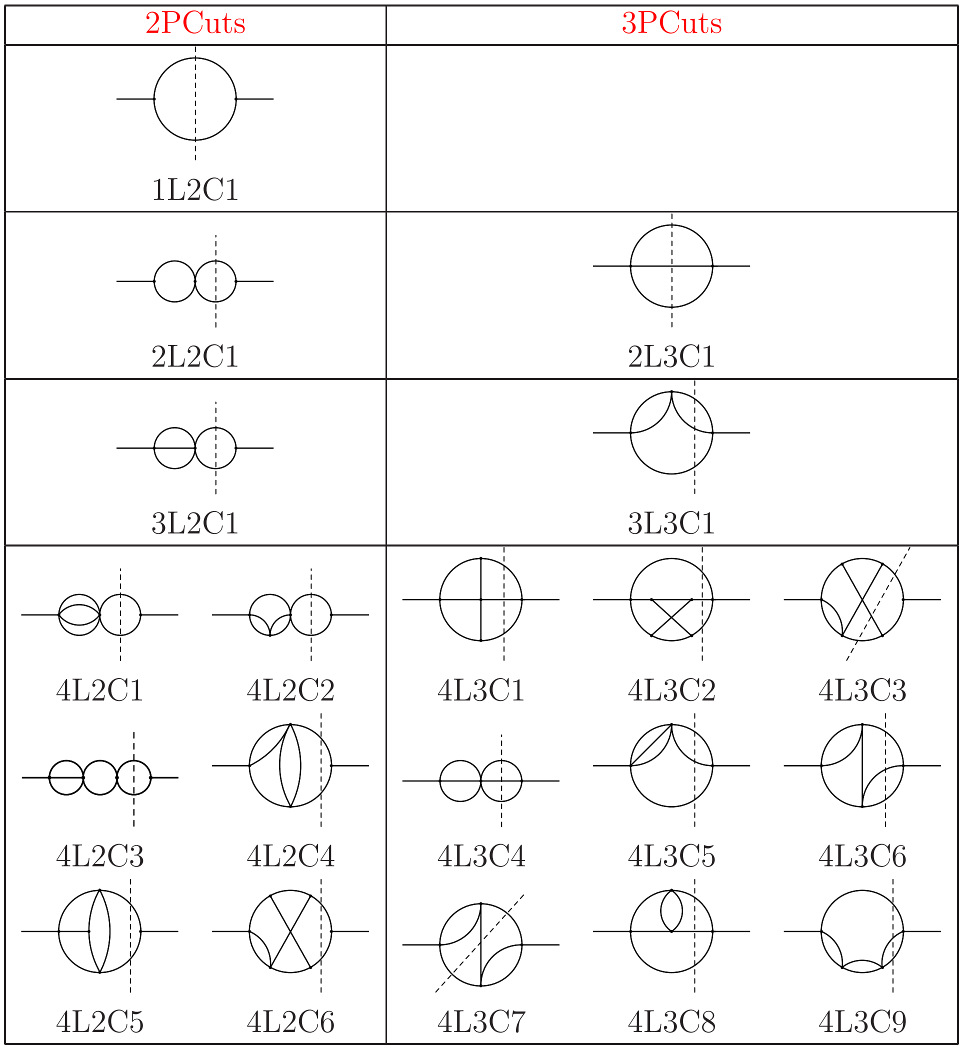
$$\frac{d}{dz} I_i(z) = \sum_j R_{ij}(z) I_j(z), \qquad z = \frac{p^2}{m_b^2}.$$

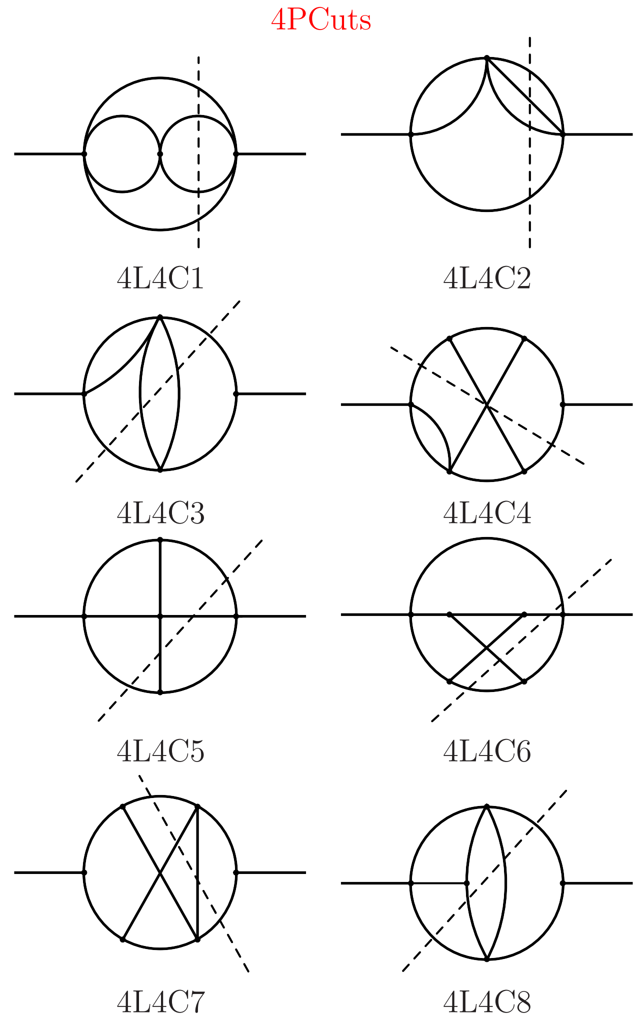



# Boundary conditions in the vicinity of $z = 0$:



# Massless integrals for the boundary conditions:



## Results for the NNLO corrections:

$$
\begin{aligned}
K_{27}^{(2)}(r,E_0) \;=\;& A_2 + F_2(r,E_0) + 3 f_q(r,E_0) + f_b(r) + f_c(r) + \frac{8}{3}\phi_{27}^{(1)}(r,E_0)\ln r \\
&+ \left[ (4L_c - x_m)\, r\frac{d}{dr} \;+\; x_m E_0 \frac{d}{dE_0} \right] f_{NLO}(r,E_0) + \frac{416}{81} x_m \\
&+ \left( \frac{10}{3} K_{27}^{(1)} - \frac{2}{3} K_{47}^{(1)} - \frac{208}{81} K_{77}^{(1)} - \frac{35}{27} K_{78}^{(1)} - \frac{254}{81} \right) L_b - \frac{5948}{729} L_b^2,
\end{aligned}
$$

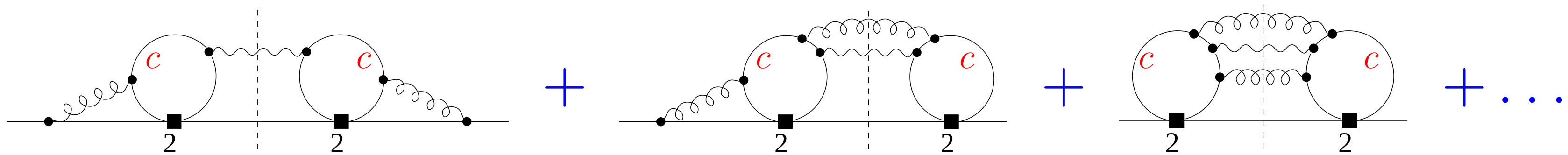
$$
K_{17}^{(2)}(r,E_0) \;=\; -\frac{1}{6} K_{27}^{(2)}(r,E_0) + A_1 + F_1(r,E_0) + \left( \frac{94}{81} - \frac{3}{2} K_{27}^{(1)} - \frac{3}{4} K_{78}^{(1)} \right) L_b - \frac{34}{27} L_b^2,
$$

**where** $F_i(0,0) \equiv 0$**,** $A_1 \simeq 22.605$**,** $A_2 \simeq -81.179$ **from the present calculation.**

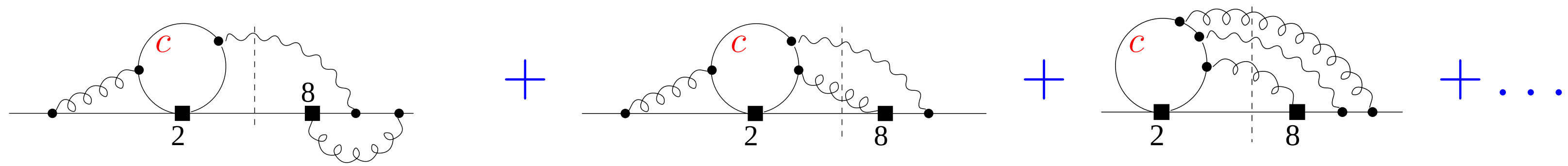
**Next, we interpolate in** $m_c$ **by assuming that** $F_i(r,0)$ **are linear combinations of** $f_q(r,0)$**,** $f_{NLO}(r,0)$**,** $r\frac{d}{dr}f_{NLO}(r,0)$ **and a constant term. The known large-**$r$ **behaviour of** $F_i$ **[hep-ph/0609241] and the condition** $F_i(0,0) \equiv 0$ **fix these linear combinations in a unique manner.**

# Interferences not involving the photonic dipole operator are treated as follows:

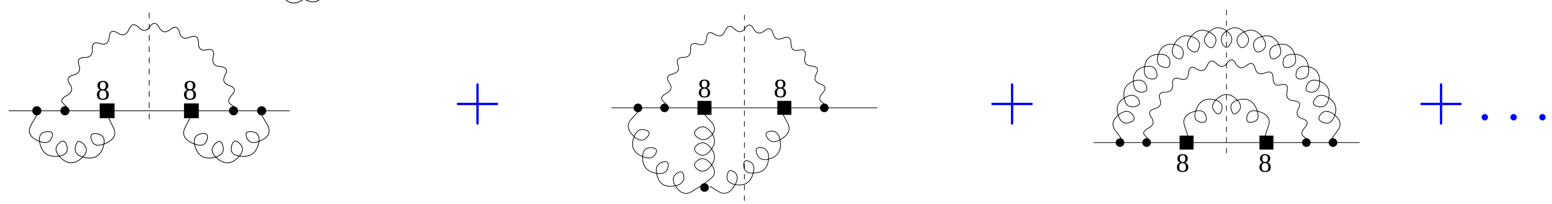
$K_{22}$**:**
(and analogous $K_{11}$ & $K_{12}$)



$K_{28}$**:**
(and analogous $K_{18}$)



$K_{88}$**:**



**Two-particle cuts are known (just $|\mathrm{NLO}|^2$).**

**Three- and four-particle cuts are known in the BLM approximation only. The NLO+(NNLO BLM) corrections are not big $(+3.8\%)$.**

## Incorporating other perturbative contributions evaluated after the previous phenomenological analysis in hep-ph/0609232:

1. Four-loop mixing (current-current) $\to$ (gluonic dipole)
   M. Czakon, U. Haisch, MM, JHEP 0703 (2007) 008 [hep-ph/0612329]
2. Diagrams with massive quark loops on the gluon lines
   R. Boughezal, M. Czakon and T. Schutzmeier, JHEP 0709 (2007) 072 [arXiv:0707.3090]
   H. M. Asatrian, T. Ewerth, H. Gabrielyan and C. Greub, Phys. Lett. B 647 (2007) 173 [hep-ph/0611123]
   T. Ewerth, Phys. Lett. B 669 (2008) 167 [arXiv:0805.3911]
3. Complete interference (photonic dipole)–(gluonic dipole)
   H. M. Asatrian, T. Ewerth, A. Ferroglia, C. Greub and G. Ossola,
   Phys. Rev. D 82 (2010) 074006 [arXiv:1005.5587]
4. New BLM corrections to contributions from 3-body and 4-body final states
   for interferences not involving the photonic dipole
   A. Ferroglia and U. Haisch, Phys. Rev. D 82 (2010) 094012 [arXiv:1009.2144]
   MM and M. Poradziński, Phys. Rev. D 83 (2011) 014024 [arXiv:1009.5685]
5. LO contributions from $b \to s\gamma q\bar{q}$, $(q = u, d, s)$ from 4-quark operators ("penguin" or CKM-suppressed)
   M. Kamiński, MM and M. Poradziński, Phys. Rev. D 86 (2012) 094004 [arXiv:1209.0965]
6. NLO contributions from $b \to s\gamma q\bar{q}$, $(q = u, d, s)$ from interferences of the above operators with $Q_{1,2,7,8}$
   T. Huber, M. Poradziński, J. Virto, in preparation

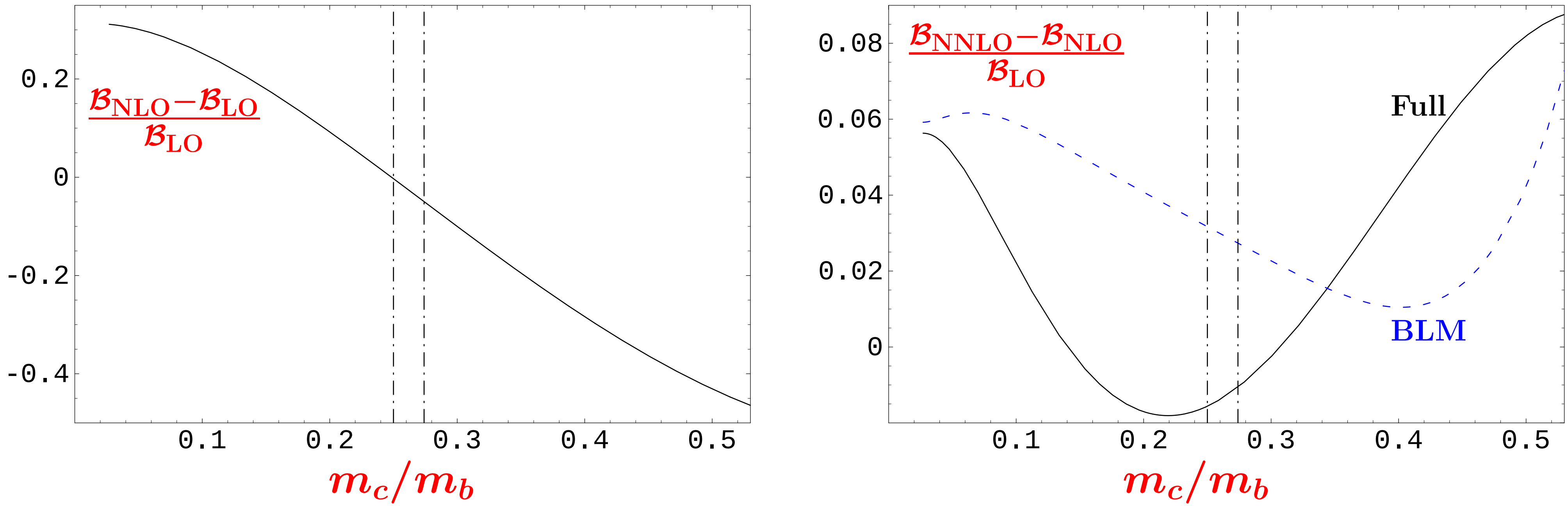
## Taking into account new non-perturbative analyses:

M. Benzke, S. J. Lee, M. Neubert and G. Paz, JHEP 1008 (2010) 099 [arXiv:1003.5012]
T. Ewerth, P. Gambino and S. Nandi, Nucl. Phys. B 830 (2010) 278 [arXiv:0911.2175]

## Updating the parameters (Parametric uncertainties go down to $2.4\%$)

P. Gambino, C. Schwanda, Phys. Rev. D 89 (2014) 014022

# Relative NLO and NNLO QCD corrections to $\mathcal{B}(\bar{B} \to X_s \gamma)$ and their dependence on $m_c/m_b$



# Summary

- **Combining the recently calculated NNLO QCD and NLO EW corrections to $\overline{\mathcal{B}}(B_s \to \mu^+\mu^-)$, we find a significant reduction of the non-parametric theoretical uncertainties ($\sim 8\% \to \sim 1.5\%$).**

- **The current SM result $\overline{\mathcal{B}}(B_s \to \mu^+\mu^-) = (3.65 \pm 0.23) \times 10^{-9}$ is consistent with the measured value of $(2.9 \pm 0.7) \times 10^{-9}$. The main theory uncertainties are parametric ($|V_{cb}|,\ f_{B_s}, \ldots$).**

- **Dominant NNLO corrections to $\mathcal{B}(\bar{B} \to X_s\gamma)$ will soon be known not only in the large $m_c$ limit, but also at $m_c = 0$. If the current result survives, no reduction of uncertainties with respect to the 2006 estimate is expected, except for the parametric one.**