# Analytic Methods for Precise Predictions

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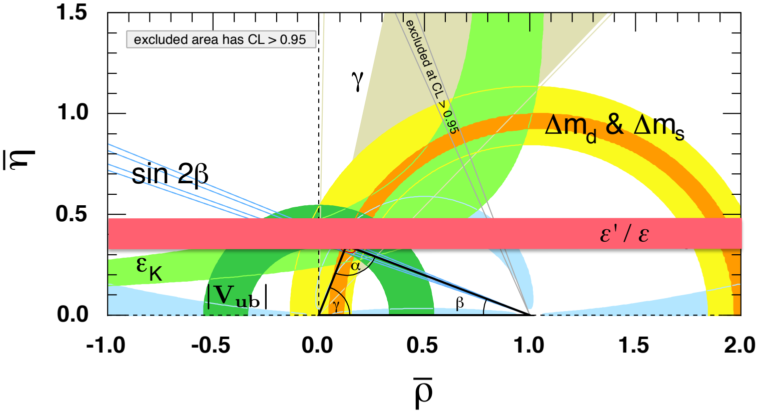
October 1, 2014 – Lattice Meets Continuum, Siegen



excluded area has CL > 0.95
γ
excluded at CL > 0.95
Δm_d & Δm_s
sin 2β
ε' / ε
ε_K
|V_ub|
α
γ
β
1.5
1.0
0.5
0.0
-1.0
-0.5
0.0
0.5
1.0
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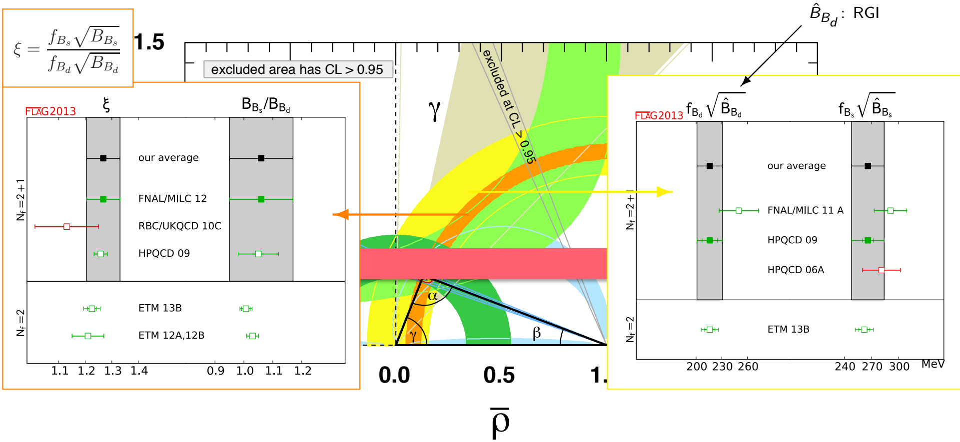
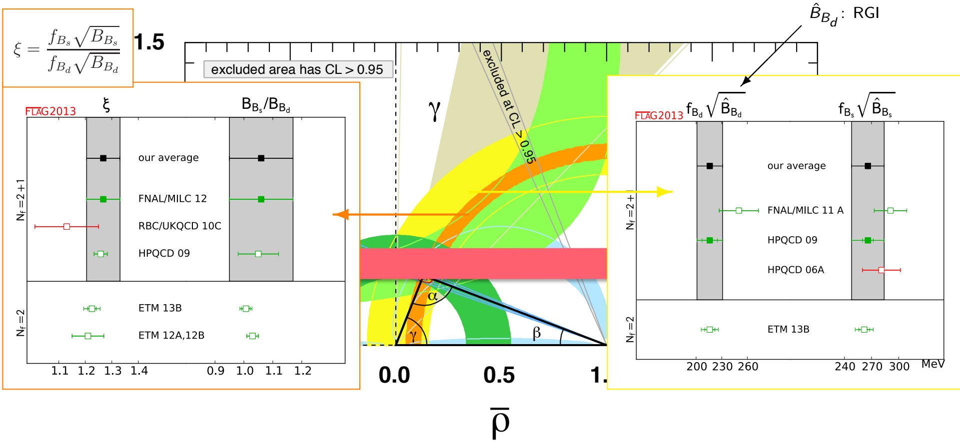
2.0
η̄
ρ̄



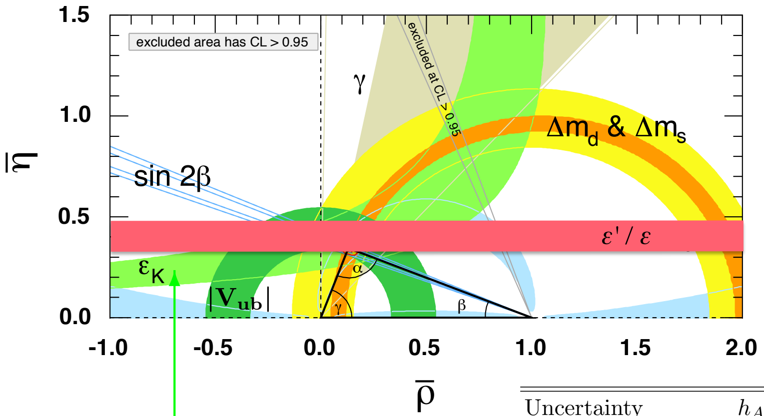

Experimental error on $\Delta m_d$, $\Delta m_s \approx 0.5\%$

$$\xi = \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}}$$




**Experimental error on $\Delta m_d$, $\Delta m_s \approx 0.5\%$**

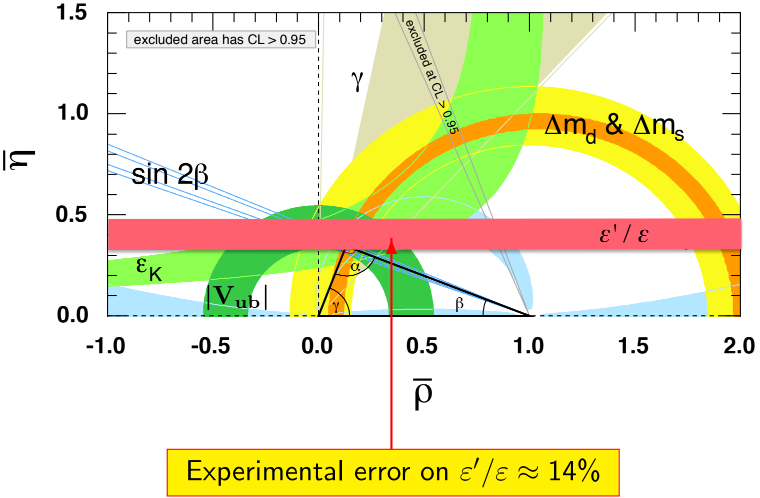
| Source of error | $\xi$ FNAL/MILC12 | HPQCD09 | ETM 13 | $f_{B_d}\sqrt{\hat{B}_d}$ FNAL/MILC11 | HPQCD09 | ETM 13 |
|---|---|---|---|---|---|---|
| Statistics and Chiral | 4.9% | 2.0% | 1.3% | 7% | 4.1% | 3.8% |
| Matching to Cont. | 0.5% | 0.7% | ... | 4% | 4.0% | 0.2% |
| Discretization | 0.5% | 0.5% | 1.1% | 2% | 3.2% | 2.3% |
| Total | 5.0% | 2.6% | 2.5% | 10% | 7.1% | 4.5% |




Error on $\varepsilon_K$ dominated by $|V_{cb}|^4$

For example (Kronfeld's talk): arXiv:1403.0635

| Uncertainty | $h_{A_1}(1)$ |
|---|---|
| Statistics | 0.4% |
| Scale ($r_1$) error | 0.1% |
| $\chi$PT fits | 0.5% |
| $g_{D^*D\pi}$ | 0.3% |
| Discretization errors | 1.0% |
| Perturbation theory | 0.4% |
| Isospin | 0.1% |
| Total | 1.4% |



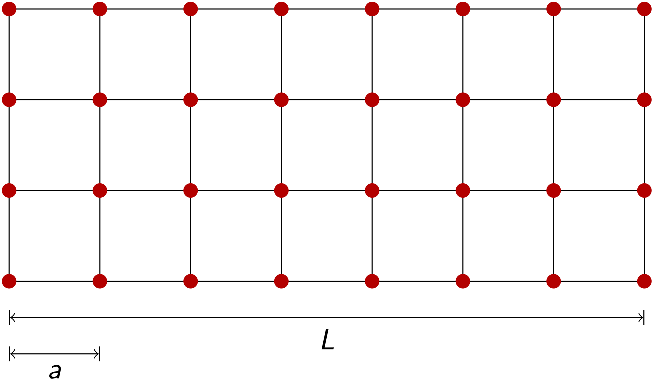

RBC/UKQCD 12 $I = 2$, RBC/UKQCD 11* $I = 0$

On top of matching: analytic control of finite-volume effects (also $\Delta M_K$)

## Outline

- Control of lattice discretization errors
- Matching non-perturbative and perturbative results
- Control of finite volume effects

# Control of lattice discretization errors




Competing constraints $am_h \ll 1$ and $1 \ll m_\pi L$ (in particular for physical $m_\pi$)

## Solutions

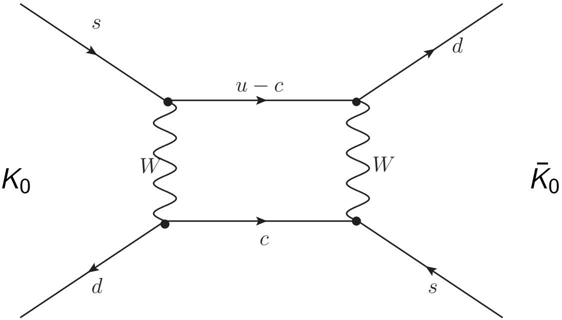
1. Extrapolation / Interpolation in $am_h$: ETM, HPQCD, RBC/UKQCD (in progress), ...
2. Effective Field Theories: HQET, NRQCD, Fermilab/RHQ, OK
3. Direct simulation at $am_h \ll 1$: Frozen Topology?

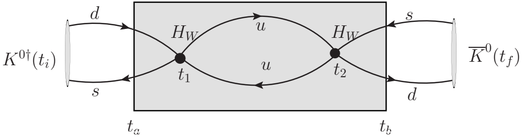
Analytic methods: removal of dominant $am_h$ errors in 1) and 3), tuning of parameters in 2)

## Methodology

- Lattice Perturbation Theory (discussed later in detail)
- (Partly) non-perturbative: tune parameters of EFTs by, e.g.,
  - matching of long-distance observables to experiment (RBC/UKQCD RHQ)
  - matching of short-distance observables to a QCD simulation in smaller volume and lattice spacing (Alpha NP HQET)

An entire different type of lattice artifact in long-distance contributions Christ et al. 2012:




GIM cancellation difference: $\Delta M_K$ versus long-distance $\varepsilon_K$

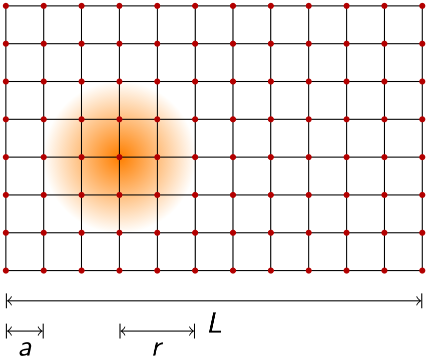
# Matching non-perturbative and perturbative results

## Scheme matching methodology

- LPT+CPT
- NP+CPT — See talk by Christian Sturm
- NP+LPT+CPT — Takeda et al. 2003, Rubio et al. 2012, Constantinou et al. 2013

## Gradient Flow Renormalization

History on next slide




Smear fields and thus composite operators via a differential equation in flow-time t with $r = \sqrt{8t}$.
For gluons with $B_\mu(t = 0, x) = A_\mu(x)$: $\partial_t B_\mu = D_\nu G_{\nu\mu}$, $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$, $D_\mu = \partial_\mu + [B_\mu, \cdot]$

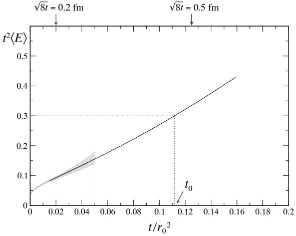
A renormalized operator at $t > 0$ can then be expressed as

$$\mathcal{O}^r(t,x) = \sum_{n,m,i_{n+m}} O^{(n+m,i_{n+m})}(x) C^{(n+m,i_{n+m})} \sqrt{t}^n a^m$$

with a local operator $O^{(d,i)}(x)$ of dimension $d$. Simple mixing

# History

- First mentioned in lattice context (Wilson loops): Narayanan and Neuberger 2006
- Systematically discussed for gluons: Luscher 2010 (including proof of finiteness and a definition of a renormalized coupling constant)




- Discussion of perturbative treatment: Luscher and Weisz 2011
- Including fermions: Luscher 2013
- Renormalization of Energy-Momentum Tensor: Suzuki 2013

RI schemes — Martinelli et al. 1995

RI renormalization condition on gauge-fixed off-shell amplitudes: details will be explained by Christian in the next talk.

SF schemes — Luscher et al. 1992

Schroedinger Functional (transition amplitude between specific field configurations at $t = 0$ and $t = T$ Euclidean times) allows for the definition of renormalized quantities (boundary fields serve as source terms).

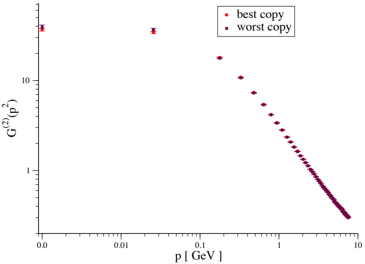
This yields finite-volume schemes in which $1/T$ is the renormalization scale. No gauge fixing is necessary for the non-perturbative computation.

## RI schemes – Gribov copies

The Landau gauge fixing procedure on the lattice minimizes a function whose minima satisfy the gauge fixing condition. Only one of the minima corresponds to the continuum Gribov copy.

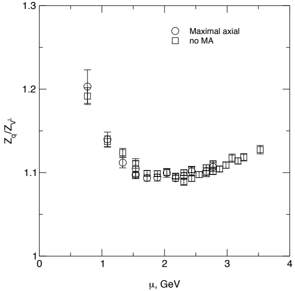
### Jamie Hudspith 2013

Spread of 150 random gauge transformations prior to Landau-GF. Result for gluon propagator:




### Yuri Zhestkov 2001

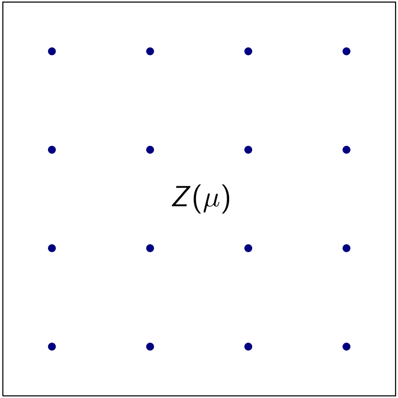
Effect of first fixing to maximal axial gauge:



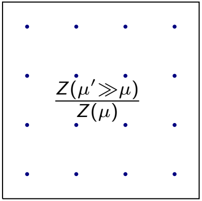

Also: as $\alpha_s \to 0$ all links are naturally close to the unit link

# Step scaling

SF: Luscher et al. 1993
RI: Arthur et al. 2010



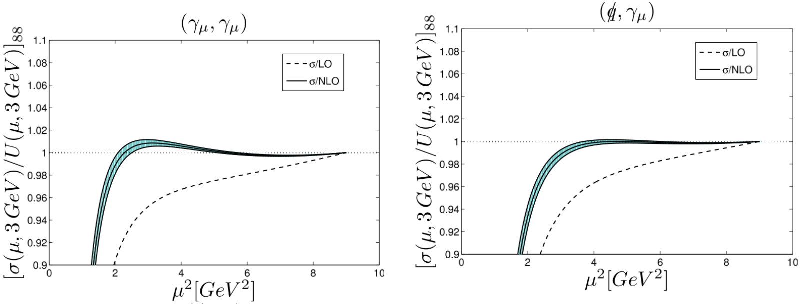

$a\mu$ small, $\alpha_s(\mu)$ non-perturbative




$\alpha_s(\mu')$ smaller

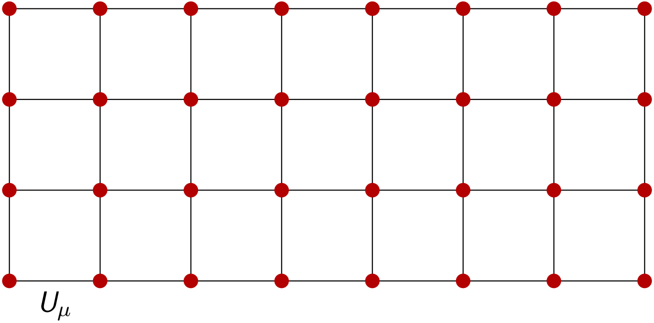
SF: tuning of volume

# Step scaling – Example $\Delta S = 1$ operator




(Blum et al. 2012): $\sigma/U$ is ratio of non-perturbative vs. perturbative running (Sturm/CL) from $\mu$ to 3 GeV for operator $Q_8 = \bar{s}_a\gamma_\mu(1-\gamma_5)d_b \sum_{q=u,d,s} \bar{q}_b\gamma_\mu(1+\gamma_5)q_a(3e_q/2)$; results for two RI-SMOM operator schemes with $\gamma$ wave-function scheme

# Lattice Perturbation Theory

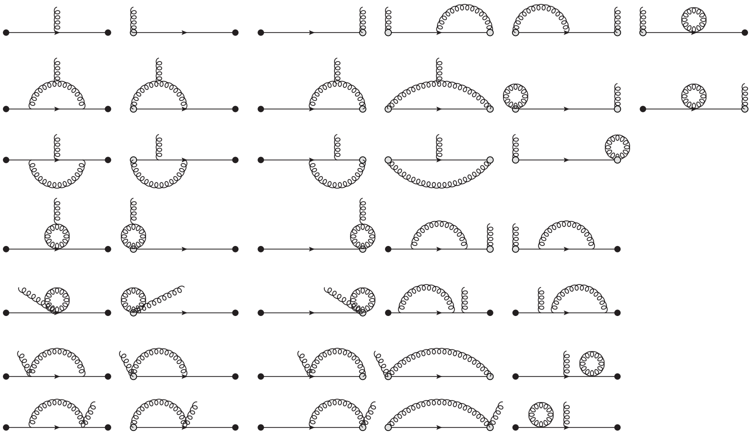



Express links in terms of algebra

$$U_\mu(x) = e^{igaA_\mu(x)}$$

and expand in $g$

Increased complexity (1-loop quark-gluon vertex for RHQ lattice action):



Also: each vertex is very complicated (see P. Lepage's talk)

# Automation of LPT

Luscher-Weisz 1986: simple CAS-based approach has prohibitive complexity, propose specific data structure and algorithm for automation

- Hart et al. 2005: extension to include Fermions
- Hart et al. 2009: extension to allow for complex smearing
- Takeda 2009 and Hesse et al. 2011: further refinement for various Fermion actions

Advantages of CAS-based approach:

Analytic simplifications (CSE, recurrence relations), Flexibility, Same footing for lattice and continuum regulator

# A **Phy**sics **Sy**stem based on **H**ierarchical **C**omputer **Al**gebra

- Completely automates perturbative computations for a wide class of regulators (lattice/continuum) (Lehner 2012)

http://physyhcal.lhnr.de/

# PhySyHCAl

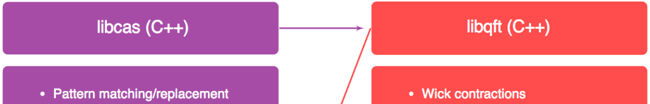
## A **Phy**sics **Sy**stem based on **H**ierarchical **C**omputer **Al**gebra

Overview | Tutorial | libcas | libqft | libint | libint-python | Repository | Wiki

### Introduction

PhySyHCAl is a framework for automated perturbation theory for a wide range of regulators both in the continuum and using a lattice. It is based on a new hierarchical computer algebra system. The current implementation includes support for Wilson-type fermions, DWF, and the Schroedinger Functional. Apart from perturbative calculations, it can also generate contractions for non-perturbative lattice computations.

The four main libraries are described schematically below



## Different classes of regulators

- 4-dimensional momentum space: Wilson, RHQ, Gauge, Continuum ($d$-dimensional)
- 4-dimensional momentum space plus one extra dimension: Domain Wall Fermions (algebraic or numerical treatment of 5d)
- 3-dimensional momentum space plus temporal dimension in position space: Schroedinger Functional implementations
- Continuum NDR

A few applications

- B physics with RHQ bottom and DWF light quarks by Kawanai (BNL), Meinel (MIT), and Witzel (BU),
- B physics with Oktay-Kronfeld bottom quarks (1-loop tuning) by Jang (Seoul National University),
- the study of improved Brillouin heavy quarks by Cho and Hashimoto (KEK),
- the determination of $m_b$ using NRQCD (2-loop matching) by Lehner (BNL) and Monahan (College of William & Mary),
- the control of discretization and matching uncertainties for static quarks by Ishikawa, Izubuchi, and Lehner (BNL),
- the $(g-2)_\mu$ light-by-light computation by Blum (UConn), Hayakawa (Nagoya), Izubuchi (BNL), and Lehner (BNL).

In progress: Two-loop methodology such as color-twisted boundary conditions

Brillouin action used by JLQCD (Cho et al. 2013):

$$D^{IB} = \sum_\mu \gamma_\mu (1 - \frac{a^2}{12}\triangle^{bri})\nabla_\mu^{iso}(1 - \frac{a^2}{12}\triangle^{bri}) + c_{IB}a^3\left(\triangle^{bri}\right)^2 + ma$$

$$c_{IB} = 1/8$$

$$a\triangle^{bri}(n,m)\psi_m = \frac{1}{64}\sum_\mu D_\mu^+\psi_n''' - \frac{15}{4}\psi_n$$

$$\psi_n''' \equiv 8\psi_n + \frac{1}{2}\sum_{\nu\neq\mu} D_\nu^+\psi_n''$$

$$\psi_n'' \equiv 4\psi_n + \frac{1}{3}\sum_{\rho\neq\mu,\nu} D_\rho^+\psi_n'$$

$$\psi_n' \equiv 2\psi_n + \frac{1}{4}\sum_{\sigma\neq\mu,\nu,\rho} D_\sigma^+\psi_n$$

$$\nabla_x^{iso}(n,m)\psi_m = \frac{1}{432}\left(D_x^-\xi_n''' + \frac{1}{2}\sum_{\nu\neq x} D_\nu^+\eta_n'''\right)$$

$$\xi_n''' \equiv 64\psi_n + \frac{1}{2}\sum_{\nu\neq x} D_\nu^+\xi_n'' \qquad \eta_n''' \equiv D_x^-\xi_n'' + \frac{1}{3}\sum_{\rho\neq x,\nu} D_\rho^+\eta_n''$$

$$\xi_n'' \equiv 16\psi_n + \frac{1}{3}\sum_{\rho\neq x,\nu} D_\rho^+\xi_n' \qquad \eta_n'' \equiv D_x^-\xi_n' + \frac{1}{4}\sum_{\sigma\neq x,\nu,\rho} D_\sigma^+\eta_n'$$

$$\xi_n' \equiv 4\psi_n + \frac{1}{4}\sum_{\sigma\neq x,\nu,\rho} D_\sigma^+\psi_n \qquad \eta_n' \equiv D_x^-\psi_n$$

$$D_\mu^\pm = U_\mu(n)\,\psi_{n+\hat\mu}''' \pm U_\mu^\dagger(n-\hat\mu)\,\psi_{n-\hat\mu}'''$$

## Recurrence relations

- Becher and Melnikov 2002:
  Asymptotic expansion of lattice loop integrals around the continuum limit. Expand propagators in $a$, add analytic regulator to define leading order. Significant complication arises from tensor decomposition with reduced symmetry. Then use recurrence relations to reduce to master integrals.

- Becher et al. 2003 Mass renormalization in Asqtad action at 1 loop

- Becher et al. 2005 $\Delta S = 2$ operator staggered renormalization factors 1 loop

## Numerical Stochastic Perturbation Theory

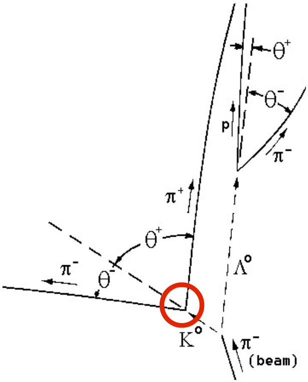
Parisi and Wu 1981, Renzo et al. 1994

Stochastic quantization (similar to WF equations with a noise term) allows for derivation of coupled differential equations for each order in the bare coupling. Averages taken in the long "time" limit should correspond to the proper quantum average.

- Hasegawa et al. 2012: 3-loop renormalization constants for quark bilinears using Wilson fermions (RI'/MOM)
- Bali et al. 2014: expansion of the plaquette to $O(\alpha^{35})$ evidence for renormalon. Color-twisted boundary conditions to regulate IR.

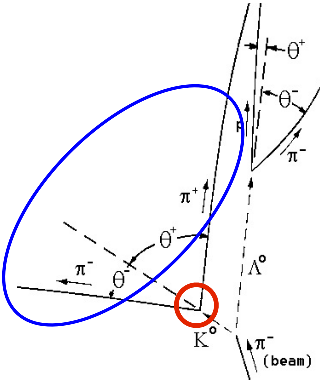
# Control of finite volume effects

# Finite volume as a boon




Weak phase shift

# Finite volume as a boon

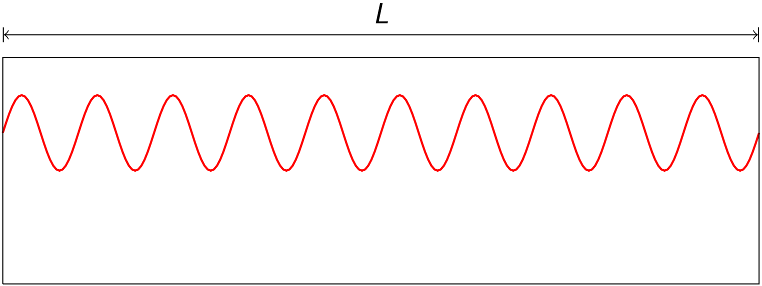



Weak phase shift

Strong phase shift

# Finite volume as a boon

Scattering phases accessible in finite volume



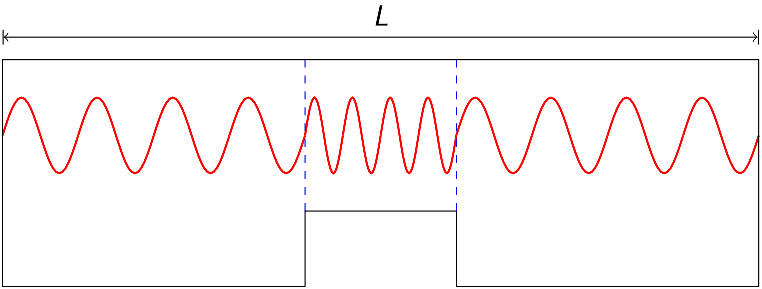

Free case:

Periodic wave-function $\Rightarrow$ quantization condition $p = (2\pi/L)\mathbb{Z}$

# Finite volume as a boon

(Lüscher 1991)

Scattering phases accessible in finite volume




Interacting case:

Periodic wave-function $\Rightarrow$ quantization condition for scattering phase-shift $\delta(p)$ $\Rightarrow$ measured finite-volume energy yields $\delta(p)$

# Finite volume as a boon (Lellouch and Lüscher 2001)

- Consider weak Hamiltonian as perturbation

- Study of phase shift in perturbed theory yields a relation between finite-volume and infinite-volume $K \to \pi\pi$ amplitude

## Beyond $K \to \pi\pi$

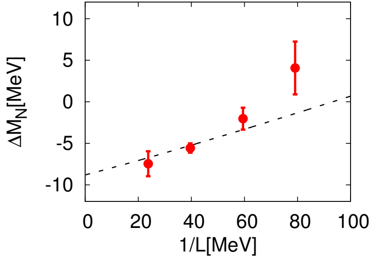
- Hansen and Sharpe 2012: multiple strongly coupled two scalar particles ($\pi\pi$,$KK$)
- Briceno et al. 2014: extend to allow for arbitrary momentum injection
- Horgan et al. 2014 (see also Wingate's talk): $B^0 \to K^{*0}\ell^+\ell^-$ ($\pi K$, $\eta K$ included)

A different extension of Lellouch–Lüscher for long-distance contributions (such as $\Delta M_K$ and the LD part of $\varepsilon_K$) have been discussed in Christ et al. 2014.

## QCD+QED

Precision requirements mandate the inclusion of QED in future computations. QED acts on larger distances and yields substantial finite-volume effects.

Successful correction using analytic methods (finite-volume QED): BMW 2014




Alternative NP method suggested at Lattice 2014 CL and Izubuchi

# Conclusion

Lattice QCD is entering the age of precision. Simulations are performed at physical light quark masses and QED and other isospin breaking effects are started to be included.

In order to keep up with these improvements, systematic uncertainties of which many can be addressed with analytical methods need to be controlled better.

The expertise of continuum theorists should be applied to these problems!